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On a Chain of Generalized Laplace Transform Involving the H-Function. (**)

1. - Introduction.

We write

$$f(p) \doteqdot h(x) ,$$

when

(1.1) $$f(p) = p \int_0^\infty e^{-px} h(x) \, dx$$

and

$$\varphi(p) \underset{\eta,\ k,\ r}{\overset{W}{=\!=}} h(x) ,$$

when

(1.2) $$\varphi(p) = p \int_0^\infty (px)^{-\eta-\frac{1}{2}} e^{-\frac{1}{2}px} W_{k+\frac{1}{2},r}(px) \, h(x) \, dx .$$

(1.2) is a generalization of (1.1) as given by MAINRA ([5], p. 24) and reduces to (1.1) when $\eta = k = -r$ on account of the well-known identity

$$W_{\frac{1}{2}-r,r}(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}-r} .$$

In (1.1) $f(p)$ is called the image of $h(x)$, which is known as the original.

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In this Note first we generalize two results of these proceedings ([7], p. 288; [6], p. 285), by establishing a theorem concerning MAINRA transform. Later on this theorem is utilized in obtaining a chain of relations for MAINRA transform. The chain obtained by us is also quite general and includes as its particular case a similar chain given recently by VARMA ([8], p. 287).

The following abbreviations will be used:

$$(a \pm b,\ c) = (a + b,\ c),\ (a - b,\ c); \qquad \{(a_p\,,\ \alpha_p)\} = (a_1\,,\ \alpha_1)\,, \ldots, (a_p\,,\ \alpha_p)\,.$$

2. - The *H*-function.

The H-function introduced by FOX ([2], p. 408) will be represented and defined as by GUPTA ([3], p. 2) in the following manner:

$$(2.1)\qquad \begin{cases} H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_1\,,\ \alpha_1), \ldots,\ (a_p\,,\ \alpha_p) \\ (b_1\,,\ \beta_1), \ldots,\ (b_q\,,\ \beta_q) \end{matrix} \right. \right] = \\ = \dfrac{1}{2\pi i} \displaystyle\int_L \dfrac{\prod_{1}^{m}{}_j\, \Gamma(b_j - \beta_j\, \xi) \quad \prod_{1}^{n}{}_j\, \Gamma(1 - a_j + \alpha_j\, \xi)}{\prod_{m+1}^{q}{}_j\, \Gamma(1 - b_j + \beta_j\, \xi) \quad \prod_{n+1}^{p} \Gamma(a_j - \alpha_j\, \xi)}\, x^{\xi}\, \mathrm{d}\xi\,, \end{cases}$$

where x is not equal to zero and an empty product is interpreted as unity; p, q, m, n are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $\alpha_j\ (j = 1, \ldots, p)$, $\beta_j\ (j = 1, \ldots, q)$ are positive numbers and $a_j\ (j = 1, \ldots, p)$, $b_j\ (j = 1, \ldots, q)$ are complex numbers such that no pole of $\Gamma(b_h - \beta_h\, \xi)$ $(h = 1, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_i + \alpha_i\, \xi)$ $(i = 1, \ldots, n)$ i.e.

$$(2.2)\qquad \alpha_i\, (b_h + \nu) \neq (a_i - \eta - 1)\, \beta_h\,,$$

$(\nu,\ \eta = 0, 1, \ldots;\quad h = 1, \ldots, m;\quad i = 1, \ldots, n)$.

Further the contour L runs from $\sigma - i\,\infty$ to $\sigma + i\,\infty$ such that the points:

$$(2.3)\qquad \xi = (b_h + \nu)/\beta_h \qquad (h = 1, \ldots, m;\ \nu = 0, 1, \ldots)$$

which are poles of $\Gamma(b_h - \beta_h\, \xi)$, lie to the right and the points:

$$(2.4)\qquad \xi = (a_i - \eta - 1)/\alpha_i \qquad (i = 1, \ldots, n;\ \eta = 0, 1, \ldots)$$

which are poles of $\Gamma(1 - a_i + \alpha_i\, \xi)$, lie to the left of L. Such a contour is possible on account of (2.2).

3. - Theorem 1. *If,*

$$p^{l}\varphi(p^{\sigma}) \underset{\eta,\ k,\ r}{\overset{W}{=\!=\!=}} h(x)\,, \tag{3.1}$$

$$f(p) \underset{\delta,\ \lambda,\ \mu}{\overset{W}{=\!=\!=}} x^{c}\varphi(x) \tag{3.2}$$

and

$$g(p) \underset{\alpha,\ \beta,\ \gamma}{\overset{W}{=\!=\!=}} x^{a} f(x), \tag{3.3}$$

then

$$g(p) = \sigma p^{c-a-\frac{l}{\sigma}}\,. \tag{3.4}$$

$$\cdot \int_0^\infty \frac{h(x)}{x} H^{2,4}_{5,4}\left[\frac{1}{px^{\sigma}}\left|\begin{matrix}(\eta \pm r,\sigma),\left(\alpha \pm \gamma + c - a - \frac{l}{\sigma},\ 1\right),\left(1 + c - \delta - \lambda - \frac{l}{\sigma},\ 1\right)\\ \left(1 + c - \delta \pm \mu - \frac{l}{\sigma},\ 1\right),\ \left(\alpha + \beta + c - a - \frac{l}{\sigma},\ 1\right),\ (\eta + k,\ \sigma)\end{matrix}\right.\right] dx$$

provided that $R(p) > 0$, $\sigma > 0$, $R[a - \alpha \pm \gamma \pm \mu - \delta + 2] > 0$, $R[1 - l - \eta \pm \pm r + \sigma\cdot(1 + c \pm \mu - \delta)] > 0$ *and the* M a i n r a *transform of* $|h(x)|$, $|x^{c}\varphi(x)|$ *and* $|x^{a} f(x)|$ *exist.*

Proof. In a recent paper [4] I have shown that if

$$p^{l}\varphi(p^{\sigma}) \underset{\eta,\ k,\ r}{\overset{W}{=\!=\!=}} h(x)$$

and

$$f(p) \underset{\delta,\ \lambda,\ \mu}{\overset{W}{=\!=\!=}} x^{c}\varphi(x),$$

then

$$f(p) = \sigma p^{\frac{l}{\sigma}-c} \int_0^\infty \frac{h(x)}{x} H^{2,2}_{3,3}\left[px^{-\sigma}\left|\begin{matrix}(\eta \pm r,\ \sigma),\left(1 + c - \delta - \lambda - \frac{l}{\sigma},\ 1\right)\\ \left(1 + c - \delta \pm \mu - \frac{l}{\sigma},\ 1\right)\ (\eta + k,\ \sigma)\end{matrix}\right.\right] dx\,, \tag{3.5}$$

provided that $R(p) > 0$, $\sigma > 0$, $R[1 - l - \eta \pm r + \sigma\cdot(1 + c \pm \mu - \delta)] > 0$ and the MAINRA transform of $|x^{c}\varphi(x)|$ and $|h(x)|$ exist.

Substituting the value of $f(x)$ from (3.5) in (3.3) and inverting the order of integration, we get:

$$(3.6)\quad \begin{cases} g(p) = \sigma \displaystyle\int_0^\infty \frac{h(t)}{t}\, p \int_0^\infty (p\,x)^{-\alpha-\frac{1}{2}}\, e^{-\frac{1}{2}px}\, W_{\beta+\frac{1}{2},\gamma}(p\,x) \ldots \\[2ex] \ldots x^{a+\frac{l}{\sigma}-c}\, H^{2,2}_{3,3}\left[x\,t^{-\sigma}\left|\begin{matrix}(\eta\pm r,\sigma),\left(1+c-\delta-\lambda-\frac{l}{\sigma},1\right)\\ \left(1+c-\delta\pm\mu-\frac{l}{\sigma},1\right),(\eta+k,\sigma)\end{matrix}\right.\right] dx\; dt\,. \end{cases}$$

In order to justify the inversion of the order of integration in (3.6), we observe that x-integral is absolutely convergent when $\mathrm{R}(p) > 0$, $\sigma > 0$ and $\mathrm{R}(a - \alpha \pm \gamma \pm \mu - \delta + 2) > 0$, the t-integral is absolutely convergent if the MAINRA transform of $|x^c\,\varphi(x)|$ exist, and the repeated integral is absolutely convergent if MAINRA transform of $|x^a\,f(x)|$ exist. Hence the change of order of integration in (3.6) is justified by DE LA VALLÉE POUSSIN's theorem ([**1**], p. 504).

Now evaluating the inner integral in (3.6) with the help of the result ([**3**], p. 13),

$$(3.7)\quad \begin{cases} \displaystyle\int_0^\infty (s\,x)^{-\eta-\frac{1}{2}}\, e^{-\frac{1}{2}sx}\, W_{k+\frac{1}{2},r}(s\,x)\, x^l\, H^{m,n}_{p,q}\left[z\,x^\sigma\left|\begin{matrix}\{(a_p\,,\ \alpha_p)\}\\ \{(b_q\,,\ \beta_q)\}\end{matrix}\right.\right] dx = \\[2ex] = s^{-l-1}\, H^{m,n+2}_{p+2,q+1}\left[z\,s^{-\sigma}\left|\begin{matrix}(\eta-l\pm r,\ \sigma),\ \{(a_p\,,\ \alpha_p)\}\\ \{(b_q\,,\ \beta_q)\},\ (\eta-l+k,\ \sigma)\end{matrix}\right.\right] \end{cases}$$

provided that $\mathrm{R}(s) > 0$, $\sigma > 0$, $\mathrm{R}\left(l - \eta \pm r + 1 + \sigma\cdot\min\dfrac{b_h}{\beta_h}\right) > 0$ $(h = 1, \ldots, m)$,

$$\lambda = \sum_1^n{}_j\,(\alpha_j) - \sum_{n+1}^p{}_j\,(\alpha_j) + \sum_1^m{}_j\,(\beta_j) - \sum_{m+1}^q{}_j\,(\beta_j) > 0$$

and $|\arg z| < \frac{1}{2}\,\lambda\,\pi$, we arrive at the result.

Particular cases:

1) On putting $\sigma = \dfrac{S}{N}$, $l = 1 - \dfrac{S}{N}\sigma + \dfrac{S}{N}$, $c = 0$, $\eta = -r$, $\delta = -\mu$, $\alpha = -\gamma$ and replacing k by $k - \frac{1}{2}$, λ by $\lambda - \frac{1}{2}$ and β by $\beta - \frac{1}{2}$ in the above theorem, we arrive at a result obtained earlier by SAXENA ([**7**], p. 288).

2) Again if we take $\eta = k = -r$, $c = 0$, $\delta = -\mu$, $\alpha = -\gamma$, $l = 1 - \varrho + \sigma$ and replace λ by $\lambda - \frac{1}{2}$ and β by $\beta - \frac{1}{2}$, we arrive at a result due to NARAIN ([6], p. 285).

4. – Theorem 2. *If*

$$p^l \varphi(p^\sigma) \underset{\eta,\ k,\ r}{\overset{W}{=\!=\!=}} h(x)\,, \tag{4.1}$$

$$f(p) \underset{\delta,\ \lambda,\ \mu}{\overset{W}{=\!=\!=}} x^c \varphi(x), \tag{4.2}$$

$$\lambda_1(p) \underset{\alpha_1,\ \beta_1,\ \gamma_1}{\overset{W}{=\!=\!=}} x^{a_1} f(x), \tag{4.3}$$

$$\lambda_2(p) \underset{\alpha_2,\ \beta_2,\ \gamma_2}{\overset{W}{=\!=\!=}} x^{a_2} \lambda_1(1/x)\,, \tag{4.4}$$

$$\lambda_3(p) \underset{\alpha_3,\ \beta_3,\ \gamma_3}{\overset{W}{=\!=\!=}} x^{a_3} \lambda_2(1/x)\,, \tag{4.5}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\lambda_n(p) \underset{\alpha_n,\ \beta_n,\ \gamma_n}{\overset{W}{=\!=\!=}} x^{a_n} \lambda_{n-1}(1/x)\,, \tag{4.6}$$

then

$$\lambda_n(p) = \sigma\, p^{A-A_n}. \tag{4.7}$$

$$\cdot \int_0^\infty \frac{h(x)}{x} H^{2,2n+2}_{2n+3,n+3}\left[\frac{1}{p x^\sigma}\left|\begin{matrix}(\eta \pm r,\sigma),\ \{(\alpha_n \pm \gamma_n + A - A_n, 1)\},\ \left(1+c-\delta-\lambda-\dfrac{l}{\sigma},\ 1\right)\\ \left(1+c-\delta\pm\mu-\dfrac{l}{\sigma},\ 1\right),\ \{(\alpha_n+\beta_n+A-A_n, 1)\},\ (\eta+k,\sigma)\end{matrix}\right.\right] dx\,,$$

where $A = c - \dfrac{l}{\sigma}$, $A_i = \sum\limits_1^i a_i$ $(i = 1, \ldots, n)$; $\mathrm{R}(p) > 0$, $\sigma > 0$, $\mathrm{R}[1 - l - \eta \pm$ $\pm r + \sigma \cdot (1 + c \pm \mu - \delta)] > 0$, $\mathrm{R}(A_i - \alpha_i \pm \gamma_i \pm \mu + 2 - \delta) > 0$ $(i = 1, \ldots, n)$ *and the Mainra transform of* $|h(x)|$, $|x^c \varphi(x)|$, $|x^{a_1} f(x)|$ *and* $|x^{a_j} \lambda_{j-1}(1/x)|$ $(j = 2, \ldots, n)$ *exist.*

Proof. Substituting the value of $\lambda_1(1/x)$ from (3.4) in (4.4), interchanging the order of integration and evaluating the later integral with the help of (3.7), we obtain

(4.8) $$\lambda_2(p) = \sigma\, p^{A-A_2} \cdot$$

$$\cdot \int_0^\infty \frac{h(x)}{x} H_{7,5}^{2,6}\left[\frac{1}{p\,x^\sigma}\left|\begin{matrix}(\eta \pm r,\, \sigma),\ \{(\alpha_2 \pm \gamma_2 + A - A_2\,,\ 1)\},\ \left(1 + c - \delta - \lambda - \frac{l}{\sigma},\ 1\right)\\ \left(1 + c - \delta \pm \mu - \frac{l}{\sigma},\ 1\right),\ \{(\alpha_2 + \beta_2 + A - A_2\,,\ 1)\},\ (\eta + k,\, \sigma)\end{matrix}\right.\right] dx\,.$$

The change of order of integration here can be shown to be justified as given in the proof of Theorem 1.

Repeating this process successively with the correspondences (4.5), ..., we arrive at the result stated above.

A particular case. On taking $l = 0$, $c = 0$, $a_1 = \varrho - 2$, $\eta = k$, $\delta = \lambda = -\mu$, $\sigma = 1$, $\alpha_1 = \beta_1 = -\gamma_1$ and $\alpha_j = \beta_j$ $(j = 2, \ldots, n)$, we get a known result given recently by VARMA ([8], p. 287).

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