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A Theorem on the Absolute Nörlund Summability of a Fourier Series. (**)

1.1. - Definitions. Let $\sum a_n$ be a given infinite series and $\{s_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of constants real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n, \qquad P_{-1} = p_{-1} = 0 .$$

The sequence-to-sequence transformation:

$$t_n = \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu / P_n = \sum_{\nu=0}^{n} P_\nu\, a_{n-\nu} / P_n \qquad (P_n \neq 0), \tag{1.1.1}$$

defines the sequence $\{t_n\}$ of NÖRLUND means (1) of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, $\sum |t_n - t_{n-1}| < \infty$, or symbolically, $\{t_n\} \in BV$.

In the special case in which $p_n = 1/(n+1)$, this method of summability is called *harmonic summability* (2).

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(1) NÖRLUND [3]. Substantially the same definition is also due to WORONOI; see WORONOI [7], which is the English translation by Prof. J. D. TAMARKIN of the original paper of WORONOI: Proc. 11th Congress of Russian Naturalists and Scientists, St. Petersburg (1902), pp. 60-61.

(2) RIESZ [4].

1.2. – Let $f(t)$ be a periodic function with period 2π and integrable in the LEBESGUE sense over $(-\pi, \pi)$. We assume without any loss of generality, that the constant term in the FOURIER series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0 \tag{1.2.1}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_n A_n(t) . \tag{1.2.2.}$$

We use the notations:

$$\Phi(t) = f(x+t) + f(x-t) - 2f(x) ,$$

$$\alpha(t) = \sum_{k=0}^{\infty} p_k \cos kt , \qquad \beta(t) = \sum_{k=0}^{\infty} p_k \sin kt ,$$

$$\alpha_n = \int_0^{\pi} \Phi(t)\, \alpha(t) \cos nt\, dt , \qquad \beta_n = \int_0^{\pi} \Phi(t)\, \beta(t) \sin nt\, dt ,$$

$\omega(\delta)$ = the modulus of continuity of $f(t)$ (3),

$\omega_1(\delta)$ = the modulus of continuity of $\Phi(t)$,

$\tau = [1/t]$, that is the greatest integer contained in $1/t$,

$P_{[1/n]} = P(1/n)$,

K is a positive constant, not necessarily the same at each occurrence.

By $F(t) \in \mathrm{BV}(h, k)$ we mean that $F(t)$ is a function of bounded variation over (h, k).

1.3. – With regard to the absolute harmonic summability of FOURIER series at a point, VARSHNEY proved the following theorem.

(3) Let $f(x)$ be a function defined for $a \leqslant x \leqslant b$; let $\omega(\delta) = \mathrm{Max}\,|f(x_1) - f(x_2)|$ for all x_1, x_2 belonging to (a, b) and such that $|x_1 - x_2| \leqslant \delta$. The function $\omega(\delta)$ is called the *modulus of continuity* of $f(x)$.

Theorem A [4]. If $f(x) \in \mathrm{BV}(0, 2\pi)$ and satisfies the condition

$$| f(x+h) - f(x) | \leqslant K \log^{-1-\varepsilon}(1/h) \qquad (\varepsilon > 0,\ 0 \leqslant x \leqslant 2\pi),$$

then the Fourier series of $f(t)$, at $t = x$, is absolutely harmonic summable.

This theorem has been generalised by Shah in the following form.

Theorem B [5]. If $f(x) \in \mathrm{BV}(0, 2\pi)$ and is also continuous over $(0, 2\pi)$, and the series

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega\left(\frac{1}{n}\right)$$

is convergent, then the Fourier series of $f(t)$, at $t = x$, is absolutely harmonic summable.

In the present paper we establish the following more general theorem:

Theorem. *If* $f(t) \in \mathrm{BV}(0, 2\pi)$,

(i) $$\omega\left(\frac{1}{n}\right) = o(1) \qquad as\ n \to \infty,$$

and

(ii) $$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega\left(\frac{1}{n}\right) < \infty,$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|\mathrm{N}, p_n|$, when the generating sequence $\{p_n\}$ satisfies the conditions

(iii) $$\frac{d}{dt}\alpha(t+\lambda) = O(1/t) \qquad for\ 0 < \lambda < \pi/2\ and\ 0 < t \leqslant \pi,$$

(iv) *$\{p_n\}$ is a positive monotonic non-increasing sequence such that $\{\Delta p_n\}$ is also non-increasing,*

(v) $$\sum_{n=1}^{\infty} 1/P_{2^n}^2 \leqslant K,$$

(4) Varshney [6].
(5) Shah [5].

and

$$\text{(vi)} \qquad \int_0^\delta P^2(1/t)\,dt = O\big(P^2(1/\delta)\cdot\delta\big) \qquad \text{for } 0<\delta\leqslant\pi .$$

2.1. – We require the following lemmas for the proof of our theorem.

Lemma 1. *We have*

$$|\Phi(t)| \leqslant 2\,\omega(t)\,, \qquad \omega_1(t)\leqslant 4\,\omega(t)\,.$$

The proof is easy.

Lemma 2 (6). *If* $\{p_n\}$ *is non-negative and non-increasing, then for* $0\leqslant a\leqslant b\leqslant\infty$, $0\leqslant t\leqslant\pi$, *and any* n, *we have*

$$\Big|\sum_{k=a}^{b} p_k\, e^{i(n-k)t}\Big| \leqslant \begin{cases} K\,P(1/t) \\ K\,p_a/t & \text{for } a\geqslant[t^{-1}]. \end{cases}$$

Lemma 3 (7). *If* $\{p_n\}$ *is non-negative and non-increasing, then*

$$(P_0+P_1+\dots+P_n)\,p_n/P_n \leqslant P(1/t)\,,$$

for $t\leqslant\dfrac{1}{n}$.

Lemma 4. *If* $P_n\to\infty$ *as* $n\to\infty$, *then*

$$\sum_{n=m+1}^{\infty} p_n/(P_n\,P_{n-1}) = 1/P_m\,.$$

The proof is easy, since $p_n = P_n - P_{n-1}$ and $P_n\to\infty$ as $n\to\infty$.

Lemma 5. *If* $\int_0^\delta P^2(1/t)\,dt = O\big(P^2(1/\delta)\cdot\delta\big)$, *then*

$$\int_0^\delta P(1/t)\,dt = O\big(P(1/\delta)\cdot\delta\big)\,.$$

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(6) McFadden [1], Lemma (5.11).
(7) McFadden [1], Lemma (5.12).

P r o o f. By SCHWARZ's inequality we have

$$\int_0^\delta P(1/t)\,\mathrm{d}t \leqslant \Big\{\int_0^\delta \mathrm{d}t\Big\}^{1/2}\Big\{\int_0^\delta P^2(1/t)\,\mathrm{d}t\Big\}^{1/2}$$

$$\leqslant K\,\delta^{1/2}\,P(1/\delta)\,\delta^{1/2}$$

$$\leqslant K\,\delta\,P(1/\delta)\,.$$

L e m m a 6. *If* $\sum_n \frac{1}{n}\,\omega(1/n) < \infty$, *then*

$$\sum_{n=1}^{\infty} \omega(\pi/2^n) < \infty\,.$$

P r o o f. We have

$$\omega(\pi/2^n) = \frac{1}{\log 2}\,\omega(\pi/2^n)\int_{2^{n-1}/\pi}^{2^n/\pi} \frac{\mathrm{d}t}{t} \leqslant \frac{1}{\log 2}\int_{2^{n-1}/\pi}^{2^n/\pi} (1/t)\,\omega(1/t)\,\mathrm{d}t\,.$$

Therefore

$$\sum_n \omega(\pi/2^n) \leqslant \frac{1}{\log 2}\sum_{n=2}^{\infty}\int_{2^{n-1}/\pi}^{2^n/\pi} (1/t)\,\omega(1/t)\,\mathrm{d}t$$

$$\leqslant K\int_{2/\pi}^{\infty} (1/t)\,\omega(1/t)\,\mathrm{d}t\,. \qquad (2.1.1)$$

Again we have already $\sum_n \frac{1}{n}\;\omega(1/n) < \infty$. Now

$$\frac{1}{n}\;\omega\Big(\frac{1}{n}\Big) > \int_n^{n+1} \omega\Big(\frac{1}{t}\Big)\,\frac{1}{t}\,\mathrm{d}t\,.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n}\;\omega\Big(\frac{1}{n}\Big) > \int_1^{\infty} \frac{1}{t}\;\omega\Big(\frac{1}{t}\Big)\,\mathrm{d}t\,.$$

Thus

$$\int_{1}^{\infty} \frac{1}{t}\, \omega(1/t)\, dt \leqslant K .$$

Hence by (2.1.1) the result follows.

Lemma 7. *If* $\sum_{n=1}^{\infty} 1/P_{2^n}^2 \leqslant K$ *and* $\sum_n \frac{1}{n}\, \omega(1/n) < \infty$, *then*

$$\sum_{n=1}^{\infty} \{\omega^{1/2}(\pi/2^n)\}/P_{2^n} \leqslant K .$$

Proof. By SCHWARZ's inequality and the result of Lemma 6, we get

$$\sum_{n=1}^{\infty} \frac{1}{P_{2^n}}\, \omega^{1/2}(\pi/2^n) \leqslant \{ \sum_{n=1}^{\infty} 1/P_{2^n}^2 \}^{1/2} \{ \sum_{n=1}^{\infty} \omega(\pi/2^n) \}^{1/2} \leqslant K.$$

Lemma 8. *If* $\sum_n \frac{1}{n}\, \omega(1/n) < \infty$, *then*

(i) $$\lim_{x \to \infty} \{\omega(1/x) \log x\} = 0 ,$$

(ii) $$\sum_{1}^{\infty} \frac{1}{2^{n/2}} \frac{1}{P_{2^n}} \{ \int_{1/\pi}^{2^n+1/\pi} \omega^2(1/t)\, dt \}^{1/2} < \infty ,$$

whenever $\sum_{n=1}^{\infty} 1/P_{2^n}^2 < \infty$.

Proof. The result (i) is due to SHAH (8).
We have, for a fixed n_0 (sufficiently large),

$$\int_{1/\pi}^{2^n+1/\pi} \omega^2(1/t)\, dt = O(1) + \int_{n_0}^{2^n+1/\pi} \omega^2(1/t)\, dt$$

$$= O(1) + K \int_{n_0}^{2^n+1/\pi} \frac{dt}{(\log t)^2}$$

$$= O(1) + K \frac{2^n}{n^2} .$$

(8) SHAH [5], Lemma 1 (i).

Thus to establish (ii), it is enough to prove that

$$\sum_1^\infty \frac{1}{2^{n/2}} \frac{1}{P_{2^n}} < \infty , \qquad \sum_1^\infty \frac{1}{2^{n/2}} \frac{1}{P_{2^n}} \frac{2^{n/2}}{n} < \infty .$$

Now

$$\sum_1^\infty \frac{1}{2^{n/2}} \frac{1}{P_{2^n}} \leqslant \frac{1}{P_2} \sum_1^\infty \frac{1}{2^{n/2}} \leqslant K .$$

And, by SCHWARZ's inequality, we get

$$\sum_1^\infty \frac{1}{n} \frac{1}{P_{2^n}} \leqslant \left\{ \sum_1^\infty \frac{1}{n^2} \right\}^{1/2} \left\{ \sum_1^\infty \frac{1}{P_{2^n}^2} \right\}^{1/2} \leqslant K ,$$

by the hypothesis that $\sum_1^\infty \frac{1}{P_{2^n}^2} \leqslant K$.

Lemma 9. *If $\{p_n\}$ is positive, monotonic non-increasing, then for all positive integers n, we have*

$$P_{2^n}/P_{2^{n-1}} \leqslant K .$$

Proof. We have

$$\begin{aligned} P_{2^n}/P_{2^{n-1}} &= (P_{2^{n-1}} + p_{2^{n-1}+1} + \dots + p_{2^n})/P_{2^{n-1}} \\ &\leqslant 1 + 2^{n-1} p_{2^{n-1}}/P_{2^{n-1}} \qquad \text{(since } p_n \text{ is monotonic non-increasing)} \\ &\leqslant 1 + P_{2^{n-1}}/P_{2^{n-1}} \leqslant 2 , \end{aligned}$$

since $n p_n \leqslant P_n$.

2.2. - Proof of the Theorem. From (1.1.1) we have

$$\begin{aligned} t_n - t_{n-1} &= \sum_{k=0}^{n-1} \left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \right) a_{n-k} \\ &= \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) a_{n-k} . \end{aligned}$$

For the FOURIER series of $f(t)$, at $t = x$,

$$a_{n-k} = A_{n-k}(x) = \frac{1}{\pi}\int_0^\pi \Phi(t)\ \cos(n-k)t\,dt\,,$$

so that

$$t_n - t_{n-1} = \frac{1}{\pi}\int_0^\pi \Phi(t)\left\{\frac{1}{P_n P_{n-1}}\sum_{k=0}^{n-1}(P_n p_k - p_n P_k)\ \cos(n-k)t\right\}dt$$

$$= \frac{1}{\pi}\,\frac{1}{P_n P_{n-1}}\int_0^\pi \Phi(t)\left\{\left(\sum_{k=0}^{\infty} P_n p_k - \sum_{k=n}^{\infty} P_n p_k - \sum_{k=0}^{n-1} p_n P_k\right)\ \cos(n-k)t\right\}dt\,.$$

Thus

$$\pi\,|\,t_n - t_{n-1}\,| \leqslant \frac{1}{P_{n-1}}\left|\int_0^\pi \Phi(t)\left(\sum_{k=0}^{\infty} p_k\ \cos(n-k)t\right)dt\ \right|$$

$$+\frac{1}{P_{n-1}}\left|\int_0^{1/n} \Phi(t)\left(\sum_{k=n}^{\infty} p_k\ \cos(n-k)t\right)dt\ \right|$$

$$+\frac{p_n}{P_n P_{n-1}}\left|\int_0^{1/n} \Phi(t)\ \left(\sum_{k=0}^{n-1} P_k\ \cos(n-k)t\right)dt\ \right|$$

$$+\frac{1}{P_{n-1}}\left|\ \int_{1/n}^{\pi} \Phi(t)\left(\sum_{k=n}^{\infty} p_k \cos(n-k)t + \frac{p_n}{P_n}\sum_{k=0}^{n-1} P_k\ \cos(n-k)t\right)dt\ \right|$$

$$= T_1(n) + T_2(n) + T_3(n) + T_4(n)\,, \qquad \text{say.}$$

Now, in order to prove our theorem we show that

$$\sum_{n=1}^{\infty} T_i(n) \leqslant K \qquad (i = 1,\ 2,\ 3,\ 4).$$

Applying Lemma 2, we have

$$\sum_{n=1}^{\infty} T_2(n) \leqslant \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} |\Phi(t)| \, \Big| \sum_{k=n}^{\infty} p_k \cos(n-k)t \Big| \, dt$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} \omega(t) \, P(1/t) \, dt$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \, \omega(1/n) \int_0^{1/n} P(1/t) \, dt$$

$$\leqslant K \sum_{n=1}^{\infty} \omega(1/n) \, P_n/(n \, P_{n-1}) \qquad \text{(by Lemma 5)}$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{n} \, \omega(1/n) \leqslant K \,,$$

by virtue of the hypothesis (ii).

Also by Lemma 3, we get

$$\sum_{n=1}^{\infty} T_3(n) = \sum_{n=1}^{\infty} \frac{p_n}{P_n \, P_{n-1}} \left| \int_0^{1/n} \Phi(t) \sum_{k=0}^{n-1} P_k \cos(n-k)t \, dt \right|$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \, \frac{P_0 + P_1 + \ldots + P_{n-1}}{P_n} \, p_n \int_0^{1/n} |\Phi(t)| \, dt$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} \omega(t) \, P(1/t) \, dt \leqslant K,$$

proceeding as in the proof of $\sum_{n=1}^{\infty} T_2(n) \leqslant K$.

Applying ABEL's transformation we get (9)

$$\sum_{k=n}^{\infty} p_k \cos(n-k)t = \tfrac{1}{2} p_n + \sum_{k=n}^{\infty} \Delta p_k \frac{\sin(n-k+\frac{1}{2})t}{2\sin(t/2)}$$

and

$$\sum_{k=0}^{n-1} P_k \cos(n-k)t = \sum_{k=0}^{n-1} p_k \frac{\sin(n-k+\frac{1}{2})t}{2\sin(t/2)} - \frac{1}{2} P_{n-1}.$$

Therefore

$$T_4(n) \leqslant \frac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} \frac{\Phi(t)}{2\sin(t/2)} \sum_{k=n}^{\infty} \Delta p_k \sin(n-k+\tfrac{1}{2})t \, dt \right|$$

$$+ \frac{p_n}{P_n P_{n-1}} \left| \int_{1/n}^{\pi} \frac{\Phi(t)}{2\sin(t/2)} \sum_{k=0}^{n-1} p_k \sin(n-k+\tfrac{1}{2})t \, dt \right|$$

$$+ \frac{1}{2} \frac{p_n}{P_{n-1}} \left(1 - \frac{P_{n-1}}{P_n}\right) \left| \int_{1/n}^{\pi} \Phi(t) \, dt \right|$$

$$= T_{41}(n) + T_{42}(n) + T_{43}(n), \qquad \text{say.}$$

By virtue of Lemma 2, since here we have $n \geqslant t^{-1}$,

$$T_{41}(n) \leqslant K \frac{\Delta p_n}{P_{n-1}} \int_{1/n}^{\pi} \frac{|\Phi(t)|}{2\sin(t/2)} t^{-1} \, dt \qquad \text{(by ABEL's lemma)}$$

$$\leqslant K \frac{\Delta p_n}{P_{n-1}} \int_{1/n}^{\pi} \omega(t) \, t^{-2} \, dt$$

$$= K \frac{\Delta p_n}{P_{n-1}} \int_{1/\pi}^{n} \omega(1/t) \, dt .$$

(9) MCFADDEN [1].

Therefore

$$\sum_{n=1}^{\infty} T_{41}(n) \leqslant K \sum_{n=1}^{\infty} \frac{\Delta p_n}{P_{n-1}} \left(K + \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right)\right)$$

$$\leqslant \frac{K}{P_0} \sum_{n=1}^{\infty} \Delta p_n + K \sum_{n=1}^{\infty} \frac{\Delta p_n}{P_{n-1}} \sum_{k=1}^{n} \omega(1/k)$$

$$\leqslant K + K \sum_{k=1}^{\infty} \omega(1/k) \sum_{n=k}^{\infty} \frac{\Delta p_n}{P_{n-1}}$$

$$\leqslant K + K \sum_{k=1}^{\infty} \omega(1/k) \frac{1}{P_{k-1}} \sum_{n=k}^{\infty} \Delta p_n$$

$$\leqslant K + K \sum_{k=1}^{\infty} \omega(1/k) \frac{p_k}{P_{k-1}}$$

$$\leqslant K + K \sum_{k=1}^{\infty} \omega(1/k) \frac{1}{k} \frac{P_{k-1}}{P_{k-1}}$$

(since p_k is monotonic non-increasing and therefore $k\, p_k \leqslant k\, p_{k-1} \leqslant P_{k-1}$)

$$\leqslant K + K \sum_{k=1}^{\infty} \frac{1}{k}\, \omega\left(\frac{1}{k}\right) \leqslant K,$$

by virtue of the hypothesis (ii).

Now we proceed to consider $\sum T_{42}(n)$.

$$T_{42}(n) \leqslant \frac{p_n}{P_n P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t)}{\sin(t/2)} \left| \sum_{k=0}^{n-1} p_k \sin(n - k + \tfrac{1}{2})t \right| \mathrm{d}t$$

$$\leqslant K \frac{p_n}{P_n P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t} P(1/t)\, \mathrm{d}t \qquad \text{(by Lemma 2)}$$

$$= K \frac{p_n}{P_n P_{n-1}} \int_{1/\pi}^{n} \omega\left(\frac{1}{t}\right) P(t) \frac{1}{t}\, \mathrm{d}t$$

$$\leqslant K \frac{p_n}{P_n P_{n-1}} \left\{K + \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) P_{k+1} \frac{1}{k}\right\}.$$

Therefore

$$\sum_{n=1}^{\infty} T_{42}(n) \leqslant K \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} + K \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) P_{k+1} \frac{1}{k}$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} + K \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) P_{k+1} \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}}$$

$$\leqslant K + K \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) \frac{P_{k+1}}{P_{k-1}} \qquad \text{(by Lemma 4)}$$

$$\leqslant K + K \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) \leqslant K,$$

by virtue of the hypothesis (ii).

Next,

$$T_{43}(n) = \frac{1}{2} p_n \Delta\left(\frac{1}{P_{n-1}}\right) \left| \int_{1/n}^{\pi} \Phi(t)\, \mathrm{d}t \right|$$

$$\leqslant K p_n \Delta\left(\frac{1}{P_{n-1}}\right) \int_{1/\pi}^{n} \omega\left(\frac{1}{t}\right) \frac{1}{t^2}\, \mathrm{d}t$$

$$\leqslant K p_n \Delta\left(\frac{1}{P_{n-1}}\right) \left\{ K + \sum_{k=1}^{n} \frac{1}{k^2} \omega\left(\frac{1}{k}\right) \right\}.$$

Therefore

$$\sum_{n=1}^{\infty} T_{43}(n) \leqslant K \sum_{n=1}^{\infty} p_n \Delta\left(\frac{1}{P_{n-1}}\right) + K \sum_{n=1}^{\infty} p_n \Delta\left(\frac{1}{P_{n-1}}\right) \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \frac{1}{k^2}$$

$$\leqslant K p_1 \sum_{n=1}^{\infty} \Delta\left(\frac{1}{P_{n-1}}\right) + K p_1 \sum_{n=1}^{\infty} \Delta\left(\frac{1}{P_{n-1}}\right) \leqslant K,$$

by Lemma 4 and hypothesis (ii) of the theorem.

Now we notice that

$$T_1(n) = \frac{1}{P_{n-1}} \left| \int_0^\pi \Phi(t) \sum_{k=0}^\infty p_k \cos(n-k)t \, dt \right|$$

$$= \frac{1}{P_{n-1}} \left| \int_0^\pi \Phi(t) \left\{ \sum_{k=0}^\infty p_k \cos kt \cos nt + \sum_{k=0}^\infty p_k \sin kt \sin nt \right\} dt \right|$$

$$\leqslant \frac{1}{P_{n-1}} \left| \int_0^\pi \Phi(t)\, \alpha(t) \cos nt \, dt \right| + \frac{1}{P_{n-1}} \left| \int_0^\pi \Phi(t)\, \beta(t) \sin nt \, dt \right|$$

$$(2.2.1) \qquad = \frac{|\alpha_n| + |\beta_n|}{P_{n-1}} .$$

And, by Lemma 2,

$$\int_0^\pi \alpha^2(t) \, dt \leqslant \int_0^\pi P^2(1/t) \, dt \leqslant K ,$$

by virtue of the hypothesis (vi). Thus, since $\Phi(t)$ is bounded, α_n and β_n are the FOURIER coefficients of an even and odd function belonging to L^2. The FOURIER series of $\Phi(t+h)\,\alpha(t+h) - \Phi(t-h)\,\alpha(t-h)$ is

$$-\frac{4}{\pi} \sum_{n=1}^\infty \alpha_n \sin nh \sin nt .$$

Applying PARSEVAL's relation, we get (*)

$$\sum_{n=1}^\infty \alpha_n^2 \sin^2 nh \leqslant K \int_0^\pi \{\Phi(t+h)\,\alpha(t+h) - \Phi(t-h)\,\alpha(t-h)\}^2 \, dt$$

$$\leqslant K \int_0^\pi \alpha^2(t+h) \{\Phi(t+h) - \Phi(t-h)\}^2 \, dt$$

$$+ K \int_0^\pi \Phi^2(t-h) \{\alpha(t+h) - \alpha(t-h)\}^2 \, dt$$

$$= J_1(h) + J_2(h) , \qquad \text{say.}$$

(*) Cf. McFADDEN [1], p. 196.

If x_1, x_2 lie in $(0, 2\pi)$, then by definition $| f(x_1) - f(x_2) | \leqslant \omega(1/n)$ for $| x_1 - x_2 | \leqslant 1/n$. But, by virtue of hypothesis (i), we have, for sufficiently large n, $\omega(1/n) < \varepsilon$, an arbitrarily small number, and therefore $| f(x_1) - f(x_2) | < \varepsilon$. Thus $f(x)$ is continuous in $(0, 2\pi)$.

Now, since $f(t)$ is continuous and of bounded variation it follows that $\Phi(t)$ is also continuous and of bounded variation. Let V be the total variation of $\Phi(t)$ over $(0, 2\pi)$. We start from the inequality

$$\sum_{k=1}^{2N} \left\{ \alpha\left(t + \frac{k\pi}{N}\right) \right\}^2 \left\{ \Phi\left(t + k\frac{\pi}{N}\right) - \Phi\left(t + (k-1)\frac{\pi}{N}\right) \right\}^2$$

$$\leqslant K\, P^2(1/t) \sum_{k=1}^{2N} \left\{ \Phi\left(t + k\frac{\pi}{N}\right) - \Phi\left(t + (k-1)\frac{\pi}{N}\right) \right\}^2$$

(by virtue of Lemma 2),

$$\leqslant K\, P^2(1/t)\, \omega_1\left(\frac{\pi}{N}\right) \sum_{k=1}^{2N} \left| \Phi\left(t + k\frac{\pi}{N}\right) - \Phi\left(t + (k-1)\frac{\pi}{N}\right) \right|$$

$$\leqslant K\, P^2(1/t)\, \omega_1\left(\frac{\pi}{N}\right) V \qquad \text{(by Lemma 1)}$$

both sides of which we integrate over $(0, 2\pi)$, or what is the same thing, over $(-\pi, \pi)$. Now, on account of the periodicity (period 2π) and evenness of the integrands on the l.h.s. and the evenness of the integrands on the r.h.s., we get

$$2N \int_0^\pi \left\{ \alpha\left(t + \frac{\pi}{2N}\right) \right\}^2 \left\{ \Phi\left(t + \frac{\pi}{2N}\right) - \Phi\left(t - \frac{\pi}{2N}\right) \right\}^2 dt$$

$$\leqslant K\, \omega\left(\frac{\pi}{N}\right) \int_0^\pi P^2(1/t)\, dt \leqslant K\, \omega\left(\frac{\pi}{N}\right),$$

by virtue of the hypothesis (vi).

Hence, for $h = \pi/(2N)$,

$$J_1(h) = J_1(\pi/(2N)) \leqslant (K/N)\, \omega(\pi/N).$$

Next, we proceed to consider $J_2(h)$. We have

$$J_2(h) = K\int_0^{\pi} \Phi^2(t-h)\{\alpha(t+h)-\alpha(t-h)\}^2\, dt$$

$$\leqslant K\int_{-h}^{h} \Phi^2(t)\,\alpha^2(t+2h)\, dt + K\int_{-h}^{h} \Phi^2(t)\,\alpha^2(t)\, dt$$

$$+ K\int_{h}^{\pi} \Phi^2(t)\{\alpha(t+2h)-\alpha(t)\}^2\, dt$$

$$= J_{21}(h) + J_{22}(h) + J_{23}(h), \qquad \text{say.}$$

Considering these integrals separately, we have

$$J_{21}(h) \leqslant K\int_0^{h} \Phi^2(t)\,\alpha^2(t+2h)\, dt$$

$$\leqslant K\,\omega^2(h)\int_0^{h} \alpha^2(t+2h)\, dt,$$

$$\leqslant K\,\omega^2(h)\int_h^{3h} P^2(1/t)\, dt \qquad \text{(by Lemma 2)}$$

$$\leqslant K\,\omega^2(h)\,P^2(1/h)\,h,$$

by the hypothesis (vi).
Also,

$$J_{22}(h) = K\int_{-h}^{h} \Phi^2(t)\,\alpha^2(t)\, dt$$

$$= K\int_0^{h} \Phi^2(t)\,\alpha^2(t)\, dt$$

$$\leqslant K\,\omega^2(h)\int_0^{h} P^2(1/t)\, dt \qquad \text{(by Lemma 2),}$$

$$\leqslant K\,\omega^2(h)\,P^2(1/h)\,h,$$

by the hypothesis (vi).

Next, by an application of the mean value theorem,

$$J_{23}(h) = K \int_h^{\pi} \Phi^2(t) \{\alpha(t+2h) - \alpha(t)\}^2 \, dt$$

$$\leqslant K h^2 \int_h^{\pi} \Phi^2(t) \{\alpha'(t+2\theta h)\}^2 \, dt \qquad (0 < \theta < 1)$$

$$\leqslant K h^2 \int_h^{\pi} \omega^2(t) \, \frac{1}{t^2} \, dt$$

$$\left(\text{by hypothesis (iii), since } 0 < 2\theta h < \frac{\pi}{2}\right),$$

$$\leqslant K h^2 \int_{1/\pi}^{1/h} \omega^2(1/t) \, dt.$$

Thus, taking, as before, $h = \pi/(2N)$, we obtain

$$\sum_{n=1}^{\infty} \alpha_n^2 \sin^2(n\pi/(2N)) \leqslant K \left\{ \frac{1}{N} \omega\left(\frac{\pi}{N}\right) + \frac{1}{N} \omega^2\left(\frac{\pi}{N}\right) P_N^2 + \frac{1}{N^2} \int_{1/\pi}^{2N/\pi} \omega^2\left(\frac{1}{t}\right) dt \right\}.$$

Putting $N = 2^{\nu}$, we have

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \alpha_n^2 \leqslant 2 \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \alpha_n^2 \sin^2(n\pi/2^{\nu+1}) \leqslant 2 \sum_{n=1}^{\infty} \alpha_n^2 \sin^2(n\pi/2^{\nu+1})$$

$$\leqslant K \left\{ \frac{1}{2^{\nu}} \omega\left(\frac{\pi}{2^{\nu}}\right) + \frac{1}{2^{\nu}} \omega^2\left(\frac{\pi}{2^{\nu}}\right) P_{2^{\nu}}^2 + \frac{1}{2^{2\nu}} \int_{1/\pi}^{2^{\nu+1}/\pi} \omega^2\left(\frac{1}{t}\right) dt \right\}.$$

Also, we have

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \frac{1}{P_{n-1}^2} \leqslant K \frac{1}{P_{2^{\nu-1}}^2} 2^{\nu-1}.$$

Therefore, by SCHWARZ's and MINKOWSKI's inequalities,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|\alpha_n|}{P_{n-1}} \leqslant \left\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} |\alpha_n|^2 \right\}^{1/2} \left\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{1}{P_{n-1}^2} \right\}^{1/2}$$

$$\leqslant K \left[\frac{1}{2^{\nu/2}} \, \omega^{1/2}\left(\frac{\pi}{2^\nu}\right) + \frac{1}{2^{\nu/2}} \, \omega\left(\frac{\pi}{2^\nu}\right) P_{2^\nu} + \frac{1}{2^\nu} \left\{ \int_{1/\pi}^{2^{\nu+1}/\pi} \omega^2\left(\frac{1}{t}\right) \mathrm{d}t \right\}^{1/2} \right] \frac{2^{\nu/2}}{P_{2^\nu - 1}}$$

$$\leqslant K \left[\frac{1}{P_{2^\nu}} \, \omega^{1/2}\left(\frac{\pi}{2^\nu}\right) + \omega\left(\frac{\pi}{2^\nu}\right) + \frac{1}{2^{\nu/2} P_{2^\nu}} \left\{ \int_{1/\pi}^{2^{\nu+1}/\pi} \omega^2\left(\frac{1}{t}\right) \mathrm{d}t \right\}^{1/2} \right],$$

by virtue of Lemma 9. Hence

$$\sum_2^\infty \frac{|\alpha_n|}{P_{n-1}} \leqslant K \left[\sum_{\nu=1}^\infty \frac{\omega^{1/2}(\pi/2^\nu)}{P_{2^\nu}} + \sum_{\nu=1}^\infty \omega\left(\frac{\pi}{2^\nu}\right) + \sum_{\nu=1}^\infty \frac{1}{2^{\nu/2} P_{2^\nu}} \left\{ \int_{1/\pi}^{2^{\nu+1}/\pi} \omega^2(1/t) \, \mathrm{d}t \right\}^{1/2} \right] \leqslant K,$$

by virtue of Lemmas 6, 7 and 8 (ii).

Similarly, we can demonstrate that

$$\sum_2^\infty \frac{|\beta_n|}{P_{n-1}} \leqslant K,$$

and therefore, from (2.2.1),

$$\sum_2^\infty T_1(n) \leqslant K.$$

This completes the proof of our theorem.

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