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The Absolute Riesz Summability of Fourier Series. (**)

1.1. - Definitions.

Let $\sum a_n$ be a given infinite series, and let $\lambda_n = \lambda(n)$ be a positive monotonic function of n, tending to infinity with n. We write

$$A_\lambda(\omega) = A_\lambda^0(\omega) = \sum_{\lambda_n \leq \omega} a_n, \qquad A_\lambda^r(\omega) = \sum_{\lambda_n \leq \omega} (\omega - \lambda_n)^r a_n \qquad (r > 0).$$

The series $\sum a_n$ is said to be summable $(\mathrm{R}, \lambda_n, r)$, $r \geqslant 0$, to sum s, if $A_\lambda^r(\omega)/\omega^r \to s$, as $\omega \to \infty$, and is said to be absolutely summable $(\mathrm{R}, \lambda_n, r)$, $r \geqslant 0$, or summable $|\mathrm{R}, \lambda_n, r|$, $r \geqslant 0$, if $A_\lambda^r(\omega)/\omega^r \in \mathrm{BV}(A, \infty)$ (1), where A is a finite positive number (2).

The sequence $\{\lambda_n\}$ is called the « type » and the number r is called the « order ».

An equivalent definition is obtained, as follows, by a suitable extension of the definition of the type $\lambda(x)$ at points other than those given by $x = n$ ($n = 1, 2, \ldots$), and a corresponding change in the variable involved in the « RIESZ mean » $A_\lambda^r(\omega)/\omega^r$.

Let $\lambda = \lambda(\omega)$ be a differentiable, monotonic increasing function of ω in (A, ∞), where A is a positive constant, and let $\lambda(\omega)$ tend to infinity with ω.

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(1) By « $f(x) \in \mathrm{BV}(h, k)$ » we mean that $f(x)$ is a function of bounded variation over the interval (h, k).

(2) OBRECHKOFF [6], [7].

We write

$$c_r(\omega) = \sum_{n \leq \omega} \{ \lambda(\omega) - \lambda(n) \}^r a_n \qquad (r > 0) .$$

Then $\sum_1^\infty a_n$ is said to be summable $| \mathrm{R}, \lambda, r |$, $r > 0$, if the integral

$$\int_A^\infty | \mathrm{d}[c_r(\omega)/\{ \lambda(\omega) \}^r] |$$

converges.

Now, for $r > 0$, $m < \omega < m+1$,

$$\frac{\mathrm{d}}{\mathrm{d}\omega} [c_r(\omega) / \{ \lambda(\omega) \}^r] = \frac{r \lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \sum_{n \leq \omega} \{ \lambda(\omega) - \lambda(n) \}^{r-1} \lambda(n) a_n .$$

Hence, the summabiltiy $| \mathrm{R}, \lambda, r |$, $r > 0$, is equivalent to the convergence of the integral

$$\int_A^\infty | [r \lambda'(\omega)/\{ \lambda(\omega) \}^{r+1}] \sum_{n \leq \omega} \{ \lambda(\omega) - \lambda(n) \}^{r-1} \lambda(n) a_n | \mathrm{d}\omega .$$

Summability $| \mathrm{R}, \lambda, 0 |$ is equivalent to absolute convergence, whatever be the type $\lambda(\omega)$.

For convenience, *we shall adopt here the alternative definition given above.*

1.2. – Let $f(t)$ be a periodic function with period 2π and integrable (L), that is, in the sense of LEBESGUE, over $(-\pi, \pi)$. We can, without any loss of generality, write the FOURIER series of $f(t)$ as

$$\sum_{n=1}^\infty (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^\infty A_n(t),$$

assuming that the constant term is zero.

Throughout the present paper we use the following notations:

$$\varphi(t) = (1/2)\{ f(x+t) + f(x-t) \},$$

$$\Phi_\alpha(t) = \{ 1/\Gamma(\alpha) \} \int_0^t (t-u)^{\alpha-1} \varphi(u) \, \mathrm{d}u \qquad (\alpha > 0), \qquad \Phi_0(t) = \varphi(t),$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1) \, t^{-\alpha} \, \Phi_\alpha(t) \qquad (\alpha \geqslant 0),$$

$$E^{(\varrho)}(\omega, t) = \left(\frac{\partial}{\partial t}\right)^\varrho E(\omega, t), \qquad \{ F(t, n) \}_\varrho = \left(\frac{\partial}{\partial t}\right)^\varrho F(t, n),$$

$$E(\omega,\ t) = \sum_{n<\omega} \{ e(\omega) - e(n) \}^{\beta-1} e(n) \cos nt,$$

$$F(\omega,\ t,\ \varrho,\ s) = \sum_{n<\omega} \{ e(\omega) - e(n) \}^{\beta-1} e(n)\, n^s (\cos nt)_\varrho,$$

$$S_n^\varkappa(t,\ \varrho) = \text{the } n^{\text{th}} \text{ Cesàro sum of order } \varkappa \text{ of } \begin{cases} (1/2) + \sum_1^\infty \cos nt & (\varrho = 0) \\ \sum_1^\infty (\cos nt)_\varrho & (\varrho \geqslant 1), \end{cases}$$

$$S^\varkappa(x;\ t,\ \varrho) = \begin{cases} (1/2)\ x^\varkappa + \sum_{n<x} (x-n)^\varkappa \cos nt & (\varrho = 0) \\ \sum_{n<x} (x-n)^\varkappa (\cos nt)_\varrho & (\varrho \geqslant 1), \end{cases}$$

$$\Gamma(1 + [\alpha] - \alpha)\, g(\omega,\ u) = \int_u^\pi (t-u)^{[\alpha]-\alpha} E^{([\alpha])}(\omega,\ t)\, \mathrm{d}t$$

($[x]$ = the integral part of x, so that, if x is a positive integer, $[x] = x$),

$$F(x) = \frac{x\, e^{(1)}(x)}{e(x)},$$

$$G(\omega,\ u) = \int_0^u \frac{v^\alpha}{\{F(k/v)\}^\alpha} \frac{\mathrm{d}}{\mathrm{d}v} g(\omega,\ v)\, \mathrm{d}v,$$

$$H(\omega,\ u) = \int_u^\pi \frac{v^\alpha}{\{F(k/v)\}^\alpha} \frac{\mathrm{d}}{\mathrm{d}v} g(\omega,\ v)\, \mathrm{d}v.$$

2. - Introduction.

In 1950 Wang proved the following general result regarding the ordinary Riesz summability of the Fourier series of a periodic Lebesgue integrable function with period 2π, using the rapidly increasing type

$$\exp\,(\log \omega)^{1+(1/\alpha)} \qquad (\alpha = 1,\ 2,\ 3,\ \ldots).$$

Theorem A [1]. If $\alpha > 0$, and

$$\Phi_\alpha(t) = o(t^\alpha/\log(1/t)), \quad \text{as} \quad t \to +0,$$

then the Fourier series of $f(t)$, at $t = x$, is summable (R, $\exp\,(\log \omega)^{1+(1/\alpha)}$, $\alpha + 1$).

[1] Wang [12].

The analogue for absolute summability of this theorem of WANG was obtained in 1954 by PATI in the form of the following theorem, of which the particular case when $\alpha = 1$ had been proved in 1951 by MOHANTY (1).

Theorem B (2). If α is an integer $\geqslant 1$, and $\varphi_\alpha(t) \log (k/t) \in \mathrm{BV}(0, \pi)$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{R}, \exp (\log \omega)^{1+(1/\alpha)}, \alpha + 1|$.

Generalising this theorem of his, PATI subsequently proved the following sharper result.

Theorem C (3). If α is an integer $\geqslant 1$ and $\varphi_\alpha(t) \log (k/t) \in \mathrm{BV}(0, \pi)$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{R}, \exp (\log \omega)^{1+(1/\alpha)}, \alpha + \delta|$, for every $\delta > 0$.

The extension of Theorem C to cover the case of *non-integral* positive values of α was carried out by MATSUMOTO, without apparently having learnt of Theorem C. While the technique of PATI's theorem is a resultant of some of the original techniques of MOHANTY and those developed by PATI in his proof of Theorem B and some techniques used by WANG in proving a sharper result than the case $\alpha = 1$ of Theorem A, in which the order 2 is replaced by $1 + \delta$ for every $\delta > 0$, one finds MATSUMOTO using certain results and techniques of PATI along with new techniques and several detailed order-estimates of his own. MATSUMOTO's theorem reads as follows:

Theorem D (4). If $\varphi_\alpha(t) \{ \log (k/t) \}^{\alpha \cdot (\Delta - 1)} \in \mathrm{BV}(0, \pi)$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{R}, \exp (\log \omega)^{\Delta}, \beta|$, where $\beta > \alpha > 0$ and $\Delta \geqslant 1$.

It may be noted that, if $\Delta = 1$, the type considered by MATSUMOTO reduces to ω and Theorem D becomes equivalent to the following well known theorem of BOSANQUET for $|\mathrm{C}|$ summability of FOURIER series (5).

Theorem E (6). If $\varphi_\alpha(t) \in \mathrm{BV}(0, \pi)$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{C}, \beta|$, where $\beta > \alpha \geqslant 0$.

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(1) MOHANTY [5].

(2) PATI [8]; the constant k is taken, without loss of generality to be greater than $\pi e^{\alpha+2}$ for convenience of analysis.

(3) PATI [9]; the same remark applies to the constant here as in Theorem B.

(4) MATSUMOTO [4]; the constant k is taken, without loss of generality, to be greater than $\pi e^{\alpha(\Delta-1)+1}$, for convenience of analysis.

(5) This equivalence is due to the fact tath summability $|\mathrm{R}, n, k|$, $k \geqslant 0$, is equivalent to summability $|\mathrm{C}, k|$; see HYSLOP [2].

(6) BOSANQUET [1].

We discover, however, that the technique of proof given by MATSUMOTO for establishing Theorem D does not seem to work if $\Delta = 1$ (1).

The object of the present paper is to replace the type $\exp(\log \omega)^{\Delta}$, $\Delta > 1$, by a general type $e(\omega)$ and axiomatize the various conditions required to be imposed on $e(\omega)$ which suffice to ensure that the FOURIER series of $f(t)$, at $t = x$, is summable $|R, e(\omega), \beta|$, for $\beta > \alpha > 0$, whenever the hypothesis $\varphi_{\alpha}(t)\{F(k/t)\}^{\alpha} \in \mathrm{BV}(0, \pi)$ holds.

3. – Our theorem, from which MATSUMOTO's theorem follows in the special case in which the type is $\exp(\log \omega)^{\Delta}$, $\Delta > 1$, is the following:

Theorem.

Let $e(\omega)$ satisfy the following conditions:

(3.1) $e^{(1)}(\omega)/\omega^{\delta}$ *is monotonic increasing, and hence, by* (3.4),

(3.2) $e(\omega)/\omega^{\delta}$ *is monotonic increasing, for arbitrary* $\delta \geqslant 0$;

(3.3) $\omega\, e^{(1)}(\omega)/e(\omega)$ *is monotonic increasing;*

(3.4) $e(\omega)/e^{(1)}(\omega)$ *is monotonic increasing;*

(3.5) $\omega^{\varepsilon}\, e(\omega)/\{\omega\, e^{(1)}(\omega)\}$ *is monotonic increasing for arbitrarily small* $\varepsilon > 0$;

(3.6) $e^{(i)}(x) = \{e^{(1)}(x)/e(x)\}^{i}\, e(x) +$ *(a linear combination of sum of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors)* $(i = 2, 3, \ldots)$;

(3.7) $\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\lambda}/\omega^{\mu} = O(1)$ *and ultimately monotonic diminishing, where λ and μ are both positive, but arbitrary, the emphasis being on large λ and small μ, and, as a consequence of* (3.7),

(3.7)′ $e^{(1)}(\omega)/e(\omega) = O(1/\omega^{1-\varepsilon})$, *as* $\omega \to \infty$, *for arbitrarily small* $\varepsilon > 0$ (2);

(3.8) $F(x) \equiv x\, e^{(1)}(x)/e(x)$ *is such that, with*

$$\Lambda(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x}{F(k/x)}\right\}^{\alpha} \quad \textit{and} \quad \Omega(x) \equiv \Lambda(x)\, x^{[\alpha]-\alpha+1};$$

(1) The idea of the type $\exp(\log \omega)^{\Delta}$ struck to MATSUMOTO, as admitted by him, essentially on studying a paper by KANNO on the RIESZ summability of FOURIER series (KANNO [3]). And with this very idea, SINHA [10] had also given a theorem on the absolute RIESZ summability of FOURIER series which is identical with the special case of Theorem D when α is integral, and is closely parallel to PATI's earlier work (PATI [9]).

(2) (3.7)′ follows also from (3.5).

(i) $F\left(\frac{x}{k}F(x)\right)\Big/F(x) = O(1)$, *as* $x \to \infty$;

(ii) $x^{\lambda}\{F(x)\}^{-\alpha}\Big/e\left(\frac{x}{k}F(x)\right) = O(1)$, *as* $x \to \infty$, *for arbitrary positive* λ

(k *is a suitably chosen constant*);

(iii) $A(ts)/s^{\alpha-1}$ *is a monotonic non-diminishing function for every fixed* t, $0 < t < \pi$;

(iv) $[\Omega^{(\varrho-1)}(t)]_{t=0} = 0, \quad 1 \leqslant \varrho \leqslant [\alpha]$;

(v) $\Omega^{([\alpha])}(t)$ *is a linear combination of positive functions of* t, *say* $H_i(t)$ ($i = 1, 2, 3, \ldots$) *such that, for each* i,

(a) $$\begin{cases} H_i^{(1)}(t) & \text{is monotonic non-increasing,} \\ t\,H_i^{(1)}(t) & \text{is monotonic non-diminishing,} \end{cases}$$

(b) $$\sum_n \frac{1}{n^2}\left|H_i^{(1)}\left(\frac{1}{n}\right)\right| < \infty .$$

Then, if $\varphi_\alpha(t)\{F(k/t)\}^\alpha \in \text{BV}\,(0, \pi)$, $\alpha > 0$, *the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|\,\text{R}, e(\omega), \beta\,|$ *for every* $\beta > \alpha > 0$.

These conditions on $e(\omega)$ are stipulated to hold for all values of ω, sufficiently large, without any loss of generality, since $e(\omega)$ may be modified over any finite interval in any suitable manner and yet not affect the summability process.

We shall require a number of Lemmas for the proof of the Theorem. These are stated and proved in sections **4**, **5**, **6**, ..., **12**, **13**.

Note: Throughout, we use $e(\omega)$ in the sense defined in the Theorem by the hypotheses (3.1), (3.2), ..., (3.7), (3.8).

4. - Lemma 1 [1].

$$S_n^{\varkappa}(t, \varrho) = \{\Omega(n, t, \varkappa)\}_\varrho + \{W(n, t, \varkappa)\}_\varrho \qquad (\varrho = 0, 1, 2, \ldots),$$

where

$$\Omega(n, t, \varkappa) = \sin\left\{\left(n + \frac{\varkappa+1}{2}\right)t - \frac{\varkappa\pi}{2}\right\}\Big/\left(2\sin\frac{t}{2}\right)^{\varkappa+1},$$

$$\{W(n, t, \varkappa)\}_\varrho = \begin{cases} 0 & \text{for } \varkappa = 0 \\ O(n^{\varkappa-1}t^{-\varrho-2}) & \text{for } \varkappa = 1, 2, \ldots . \end{cases}$$

(1) PATI [8], Lemma 3.

5. - Lemma 2.

As $\omega \to \infty$, *for* $0 < t < \pi$,

(i) $e(\omega) - e(\omega - t^{-1}) = O[t^{-1} e^{(1)}(\omega)]$,

(ii) $e(\omega) - e(\omega - t^{-1}) \geqslant t^{-1} e^{(1)}(\omega - t^{-1})$.

Proof. We have

$$e(\omega) - e(\omega - t^{-1}) = \int_{\omega - t^{-1}}^{\omega} e^{(1)}(x)\, dx = O[e^{(1)}(\omega) \int_{\omega - t^{-1}}^{\omega} dx] = O[t^{-1} e^{(1)}(\omega)] .$$

This proves (i). The proof of (ii) is easy.

6. - Lemma 3.

If $\varrho \geqslant 0$, *we have when* $[\beta] = 0$, *and as* $\omega \to \infty$, $t \to 0$:

$$(6.1)\ F(\omega, t, \varrho, s) = \sum_{l=0}^{\varrho} O[t^{-\varrho+l-\beta} e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \omega^{s+l}] + \sum_{l=0}^{\varrho} O[t^{-\varrho+l-\beta-1} \{ e^{(1)}(\omega) \}^{\beta} \omega^{l+s}]$$
$$+ \sum_{l=0}^{\varrho} O[t^{-\varrho+l-\beta-1} e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \omega^{s+l-1}] + O[t^{-\beta} e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \omega^{s+\varrho}]\ (^1);$$

and, when $\beta > 1$ (²),

$$(6.2)\qquad F(\omega, t, \varrho, s) = [O(t^{-[\beta]-\varrho}) + O(t^{-\varrho-2})] \{ e(\omega) \}^{\beta-1}$$
$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{\varrho} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[t^{-[\beta]-\varrho+l-1} \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma} \omega^{s+l-[\beta]}]$$
$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{\varrho} \sum_{\sigma=0}^{[\beta]} O[t^{-[\beta]-\varrho+l-1} \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma} \omega^{s+l-[\beta]}]$$
$$+ \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[t^{-\varrho-2} \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1} \omega^{s-1}]$$
$$+ \sum_{\sigma=0}^{[\beta]} O[t^{-\varrho-2} \{ e(\omega) \}^{\beta-1} e^{(1)}(\omega) \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1} \omega^{s}]$$
$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{\varrho} \sum_{i=1}^{[\beta]} O[t^{-\beta-\varrho+l} \{ e^{(1)}(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1} \omega^{s+l-[\beta]}] +$$

(¹) This order term is also already there in the first order expression $\sum_{l=0}^{\varrho}$ for $l = \varrho$.

(²) For the sake of clarity the indices of summations are always written irrespective of the fact whether the particular index of summation occurs explicitly or not.

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{\varrho} O[t^{-[\beta]-\varrho+l-1}\{ e(\omega) \}^{\beta}\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]}\, \omega^{s+l-[\beta]}]$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{\varrho} O[t^{-[\beta]-\varrho+l-1}\, e(\omega)\{ t^{-1} e^{(1)}(\omega) \}^{\beta-1}\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]}\, \omega^{s+l-[\beta]}] .$$

Proof of case (i): $[\beta]=0$.

$$(6.3)\quad F(\omega, t, \varrho, s) = [\sum_{1}^{[\omega-t^{-1}]} + \sum_{[\omega-t^{-1}]+1}^{[\omega]}]\{ e(\omega) - e(n) \}^{\beta-1} e(n)\, n^s (\cos nt)_{\varrho} = \sum_{i=1}^{2} D_i(\omega, t, \varrho, s),$$

say.

$$(6.4)\quad -D_1(\omega,\ t,\ \varrho,\ s) = \int_1^{\omega-t^{-1}} S^0(x;\ t,\ \varrho)\, \frac{d}{dx}\, [\{ e(\omega) - e(x) \}^{\beta-1} e(x)\, x^s]\, dx$$

$$- [\sum_{n\le\omega-t^{-1}} (\cos nt)_{\varrho}]\{ e(\omega) - e(\omega-t^{-1}) \}^{\beta-1} e(\omega-t^{-1})\, (\omega-t^{-1})^s$$

$$= -(\beta-1) \int_1^{\omega-t^{-1}} S^0(x;\ t,\ \varrho)\{ e(\omega) - e(x) \}^{\beta-2} e^{(1)}(x)\, e(x)\, x^s\, dx$$

$$+ \int_1^{\omega-t^{-1}} S^0(x;\ t,\ \varrho)\{ e(\omega) - e(x) \}^{\beta-1} e^{(1)}(x)\, x^s\, dx$$

$$+ s \int_1^{\omega-t^{-1}} S^0(x;\ t,\ \varrho)\{ e(\omega) - e(x) \}^{\beta-1} e(x)\, x^{s-1}\, dx$$

$$- [\sum_{n\le\omega-t^{-1}} (\cos nt)_{\varrho}]\{ e(\omega) - e(\omega-t^{-1}) \}^{\beta-1} e(\omega-t^{-1})\, (\omega-t^{-1})^s$$

$$= \sum_{i=1}^{3} C\, D_{1i}(\omega,\ t,\ \varrho,\ s) + O[t^{-\beta}\, e(\omega)\{ e^{(1)}(\omega) \}^{\beta-1}\, \omega^{s+\varrho}],$$

by Lemma 2 (ii), say. C denotes a constant coefficient attached to each term, and is not necessarily the same at each occurence.

Now, by Lemma 1, $S^0(x;\ t,\ \varrho) = [\sin\{ ([x]+1/2)t \}/\{ 2 \sin (t/2) \}]_{\varrho}$. Hence:

$$(6.5)\quad D_{11}(\omega,\ t,\ \varrho,\ s) = \int_1^{\omega-t^{-1}} [\sin\{ ([x]+1/2)t \}/\{ 2 \sin (t/2) \}]_{\varrho}\{ e(\omega) - e(x) \}^{\beta-2} \cdot$$
$$\cdot\ e^{1}(x)\, e(x)\, x^s\, dx$$

$$= \sum_{l=0}^{\varrho} C\, \frac{\{\cos (t/2)\}^n}{\{2 \sin (t/2)\}^{m+1}} \int_1^{\omega-t^{-1}} ([x]+1/2)^l\ {\sin \atop \cos}\{ ([x]+1/2)t \}\{ e(\omega) - e(x) \}^{\beta-2} \cdot$$
$$\cdot\ e^{(1)}(x)\, e(x)\, x^s\, dx$$

$$(0 \leqslant m,\ n \leqslant \varrho - l)$$

$$= \sum_{l=0}^{\varrho} O\Big[t^{-m-1}\{e(\omega)-e(\omega-t^{-1})\}^{\beta-2} e^{(1)}(\omega-t^{-1})\, e(\omega-t^{-1})\,(\omega-t^{-1})^{s+l} \cdot \max\Big|\int_{\xi}^{\omega-t^{-1}} {}^{\sin}_{\cos}\{([x]+1/2)t\}\,dx\Big|\Big]$$

$(1 \leqslant \xi \leqslant \omega - t^{-1})$, by (3.1) and (3.2),

$$= \sum_{l=0}^{\varrho} O\Big[t^{-m-2}\{t^{-1} e^{(1)}(\omega-t^{-1})\}^{\beta-2}\{e^{(1)}(\omega-t^{-1})\}^{2}\{e(\omega-t^{-1})/e^{(1)}(\omega-t^{-1})\} \cdot (\omega-t^{-1})^{s+l}\Big],$$ by Lemma 2 (ii),

$$= \sum_{l=0}^{\varrho} O\Big[t^{-m-\beta}\{e^{(1)}(\omega-t^{-1})\}^{\beta}\{e(\omega-t^{-1})/e^{(1)}(\omega-t^{-1})\}(\omega-t^{-1})^{s+l}\Big]$$

$$= \sum_{l=0}^{\varrho} O\Big[t^{-m-\beta}\{e^{(1)}(\omega)\}^{\beta}\{e(\omega)/e^{(1)}(\omega)\}\omega^{s+l}\Big],$$ by (3.1) and (3.4),

$$= \sum_{l=0}^{\varrho} O\Big[t^{-\varrho+l-\beta}\, e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\,\omega^{s+l}\Big],$$ since $m \leqslant \varrho - l$;

(6.6) $$D_{12}(\omega, t, \varrho, s) = \int_{1}^{\omega-t^{-1}} [\sin\{([x]+1/2)t\}/\{2\sin(t/2)\}]_{\varrho}\{e(\omega)-e(x)\}^{\beta-1} \cdot e^{1}(x)\, x^{s}\, dx$$

$$= \sum_{l=0}^{\varrho} C\, \frac{\{\cos(t/2)\}^{n}}{\{2\sin(t/2)\}^{m+1}} \int_{1}^{\omega-t^{-1}} ([x]+1/2)^{l}\ {}^{\sin}_{\cos}\{([x]+1/2)t\}\{e(\omega)-e(x)\}^{\beta-1} \cdot e^{(1)}(x)\, x^{s}\, dx$$

$(0 \leqslant m;\ n \leqslant \varrho - l)$

$$= \sum_{l=0}^{\varrho} O\Big[t^{-m-1}(\omega-t^{-1})^{l+s} e^{(1)}(\omega-t^{-1})\{e(\omega)-e(\omega-t^{-1})\}^{\beta-1} \cdot \max\Big|\int_{\eta}^{\omega-t^{-1}} {}^{\sin}_{\cos}\{([x]+1/2)t\}\,dx\Big|\Big]$$

$(1 \leqslant \eta \leqslant \omega - t^{-1})$, by (3.1),

$$= \sum_{l=0}^{\varrho} O\Big[t^{-m-2}(\omega-t^{-1})^{l+s} e^{(1)}(\omega-t^{-1})\{t^{-1} e^{(1)}(\omega-t^{-1})\}^{\beta-1}\Big],$$ by Lemma 2 (ii),

$$= \sum_{l=0}^{\varrho} O\Big[t^{-\varrho+l-\beta-1}\{e^{(1)}\omega)\}^{\beta}\,\omega^{l+s}\Big],$$ by (3.1);

(6.7) $$D_{13}(\omega, t, \varrho, s) = \sum_{l=0}^{\varrho} C\, \frac{\{\cos(t/2)\}^{n}}{\{2\sin(t/2)\}^{m+1}} \int_{1}^{\omega-t^{-1}} ([x]+1/2)^{l}\ {}^{\sin}_{\cos}\{([x]+1/2)t\} \cdot \{e(\omega)-e(x)\}^{\beta-1} e(x)\, x^{s-1}\, dx$$

$$= \sum_{l=0}^{\varrho} O[t^{-m-1}\, e(\omega - t^{-1})\, (\omega - t^{-1})^{l+s-1} \{ e(\omega) - e(\omega - t^{-1}) \}^{\beta-1} \cdot$$

$$\cdot \max \Big| \int_{\eta'}^{\omega - t^{-1}} {}^{\sin}_{\cos} \{ ([x] + 1/2)t \}\, dx \Big|]$$

$$(1 \leqslant \eta' \leqslant \omega - t^{-1}), \quad \text{by (3.2)},$$

$$= \sum_{l=0}^{\varrho} O[t^{-m-2}\, e(\omega - t^{-1})\, (\omega - t^{-1})^{l+s-1} \{ t^{-1}\, e^{(1)}(\omega - t^{-1}) \}^{\beta-1}], \text{ by Lemma 2 (ii)},$$

$$= \sum_{l=0}^{\varrho} O\, [t^{-m-2} \{ e(\omega - t^{-1})/e^{(1)}(\omega - t^{-1}) \} (\omega - t^{-1})^{l+s-1}\, t^{-\beta+1} \{ e^{(1)}(\omega - t^{-1}) \}^{\beta}]$$

$$= \sum_{l=0}^{\varrho} O[t^{-\varrho+l-\beta-1} \{ e(\omega)/e^{(1)}(\omega) \}\, \omega^{s+l-1} \{ e^{(1)}(\omega) \}^{\beta}], \quad \text{by (3.4) and (3.1)},$$

$$= \sum_{l=0}^{\varrho} O[t^{-\varrho+l-\beta-1}\, e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \omega^{s+l-1}];$$

(6.8)
$$D_2(\omega,\ t,\ \varrho,\ s) = \sum_{[\omega - t^{-1}]+1}^{[\omega]} \{ e(\omega) - e(n) \}^{\beta-1}\, e(n)\, n^s\, (\cos nt)_{\varrho}$$

$$= O[\int_{[\omega - t^{-1}]+1}^{[\omega]+1} \{ e(\omega) - e(x) \}^{\beta-1}\, e(x)\, x^{s+\varrho}\, dx]$$

$$= O[\int_{[\omega - t^{-1}]+1}^{[\omega]+1} \{ e(\omega) - e(x) \}^{\beta-1}\, e^{(1)}(x) \{ e(x)/e^{(1)}(x) \}\, x^{s+\varrho}\, dx]$$

$$= O[\{ e(\omega)/e^{(1)}(\omega) \}\, \omega^{s+\varrho} [\{ e(\omega) - e(x) \}^{\beta}]_{[\omega - t^{-1}]+1}^{[\omega]+1}], \quad \text{by (3.5)},$$

$$= O[\{ e(\omega)/e^{(1)}(\omega) \}\, \omega^{s+\varrho} \{ e(\omega) - e(\omega - t^{-1}) \}^{\beta}]$$

$$= O[\{ e(\omega)/e^{(1)}(\omega) \}\, \omega^{s+\varrho} \{ t^{-1} e^{(1)}(\omega) \}^{\beta}], \quad \text{by Lemma 2 (i)},$$

$$= O[t^{-\beta} e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1}\, \omega^{s+\varrho}].$$

Therefore, the Lemma 3 for the case $[\beta] = 0$ follows from (6.3), (6.4), (6.5), (6.6), (6.7) and (6.8).

Proof of case (ii): $\beta > 1$.

(6.9)
$$F(\omega,\ t,\ \varrho,\ s) = -\int_1^{\omega} S^0(x;\ t,\ \varrho)\, \frac{d}{dx} [\{ e(\omega) - e(x) \}^{\beta-1}\, e(x)\, x^s]\, dx$$

$$=\sum_{\varkappa=1}^{[\beta]-1}\left[\{(-1)^{\varkappa}/\varkappa!\}S^{\varkappa}(x;\ t,\ \varrho)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa}[\{e(\omega)-e(x)\}^{\beta-1}\ e(x)\,x^{s}]\right]_{1}^{\omega} \qquad (^{1})$$

$$+\{(-1)^{[\beta]}/([\beta]-1)!\}\int_{1}^{\omega}S^{[\beta]-1}(x;\ t,\ \varrho)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\beta]}[\{e(\omega)-e(x)\}^{\beta-1}\ e(x)\,x^{s}]\,\mathrm{d}x$$

$$=\sum_{\varkappa=1}^{[\beta]-1}\{(-1)^{\varkappa}/\varkappa!\}[J_{1\varkappa}(x;\ t,\ \varrho)]_{x=1}^{x=\omega}+\{(-1)^{[\beta]}/([\beta]-1)!\}J_{2}(\omega;\ t,\ \varrho),\quad \text{say.}$$

Now, by Lemma 1, we have

$$S^{\varkappa}(x;t,\varrho)=\sum_{n=1}^{[x]-\varkappa-1}\Delta^{\varkappa+1}(x-n)^{\varkappa}S_{n}^{\varkappa}(t,\varrho)+\Delta^{\varkappa}[x-([x]-\varkappa)]^{\varkappa}S_{[x]-\varkappa}^{\varkappa}(t,\varrho)+\ldots+$$
$$+\Delta^{2}[x-([x]-2)]^{\varkappa}S_{[x]-2}^{2}(t,\ \varrho)+\Delta\,[x-([x]-1)]^{\varkappa}S_{[x]-1}^{1}(t,\ \varrho)+$$
$$+(x-[x])^{\varkappa}S_{[x]}^{0}\ (t,\ \varrho)$$
$$=\sum_{j=0}^{\varkappa}C\,[\sin\{([x]-j+j/2+1/2)t-\pi j/2\}/\{2\sin(t/2)\}^{j+1}]_{\varrho}+$$
$$+O\,[\sum_{j=1}^{\varkappa}([x]-j)^{j-1}t^{-\varrho-2}]\,,$$

whence

$$(6.10)\quad S^{\varkappa}(x;\ t,\ \varrho)=C\sum_{j=0}^{\varkappa}C\sum_{l=0}^{\varrho}C\,[([x]-j/2+1/2)^{l}\,{\sin \atop \cos}\{([x]-j/2+1/2)t-\pi j/2\}\cdot$$
$$\cdot\{\cos(t/2)\}^{n}/\{2\sin(t/2)\}^{j+1+m}]+O\,[x^{\varkappa-1}/t^{\varrho+2}]\qquad(0\leqslant m,\ n\leqslant\varrho-l).$$

Hence (²)

$$(6.11)\qquad S^{\varkappa}(x;\ t,\ \varrho)=O[x^{l}/t^{\varkappa+1+\varrho-l}]+O[x^{\varkappa-1}/t^{\varrho+2}]\qquad(0\leqslant l\leqslant\varrho).$$

Now

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa}[\{e(\omega)-e(x)\}^{\beta-1}e(x)\,x^{s}]=\sum_{i=0}^{\varkappa}C_{i,\varkappa}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i}\{e(\omega)-e(x)\}^{\beta-1}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa-i}\{e(x)\,x^{s}\},$$

where C's stand for constants, *not necessarily the same at each occurrence.* Also we indicate for clarity the variables on which C depends in any special instance by using suffixes like i, $\varkappa$, etc., as in $C_{i,\varkappa}$.

(¹) When $[\beta]=1$, there will be no integration by parts, and hence this set of terms will not occur in the analysis.

(²) Cf. MATSUMOTO ([4], p. 227).

Now by FAA DI BRUNO's theorem (1) on the n-th derivative of a function of function, we see that $(\mathrm{d}/\mathrm{d}x)^i\{e(\omega)-e(x)\}^{\beta-1}$ is a sum of constant multiple of the terms of the type

$$\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^{\alpha_1}\{e^{(2)}(x)\}^{\alpha_2}\ldots\{e^{(i)}(x)\}^{\alpha_i},$$

where the α's are positive integers or zeros such that

$$\sum_{\nu=1}^{i}\alpha_\nu = r \leqslant i, \qquad \sum_{\nu=1}^{i}\nu\,\alpha_\nu = i\,.$$

Also

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa-i}\{e(x)\,x^s\} = \sum_{\sigma=0}^{\varkappa-i} C_{\sigma,\varkappa,i}\, e^{(\sigma)}(x)\, x^{s-\varkappa+\sigma+i}\,.$$

We write

$$\prod_{\nu=1}^{i}\{e^{(\nu)}(x)\}^{\alpha_\nu} = \{e^{(1)}(x)\}^{\alpha_1}\{e^{(2)}(x)\}^{\alpha_2}\ldots\{e^{(i)}(x)\}^{\alpha_i}\,.$$

Hence

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa}[\{e(\omega)-e(x)\}^{\beta-1}e(x)\,x^s] =$$

$$= \sum_{i=0}^{\varkappa} C_{i,\varkappa}\{e(\omega)-e(x)\}^{\beta-1-r}\prod_{\nu=1}^{i}\{e^{(\nu)}(x)\}^{\alpha_\nu}\sum_{\sigma=0}^{\varkappa-i}C_{\sigma,\varkappa,i}\,e^{(\sigma)}(x)\,x^{s-\varkappa+\sigma+i}\,.$$

Using the transformation $\sigma=\sigma'-i$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa}[\{e(\omega)-e(x)\}^{\beta-1}e(x)\,x^s] =$$

$$= \sum_{i=0}^{\varkappa} C_{i,\varkappa}\{e(\omega)-e(x)\}^{\beta-1-r}\prod_{\nu=1}^{i}\{e^{(\nu)}(x)\}^{\alpha_\nu}\sum_{\sigma'=i}^{\varkappa}C_{\sigma',\varkappa,i}\,e^{(\sigma'-i)}(x)\,x^{s-\varkappa+\sigma'},$$

---

(1) FAA DI BRUNO's Theorem. The n-th derivative of $\{F(x)\}^m$ is a sum of number of terms of the form

$$K\,\{F(x)\}^{m-r}\,\{F^{(1)}(x)\}^{\alpha_1}\,\{F^{(2)}(x)\}^{\alpha_2}\ldots\{F^{(n)}(x)\}^{\alpha_n},$$

where the K's are constants, $r\leqslant n$, and the α's are positive integers or zeros such that

$$\sum_{\nu=1}^{n}\alpha_\nu = r, \qquad \sum_{\nu=1}^{n}\nu\,\alpha_\nu = n,$$

further, if m is a positive integer, then $r\leqslant m$.

This is a particular case of a result due to FAA DI BRUNO on the n-th derivative of a function of function. See DE LA VALLÉE POUSSIN ([11], p. 89).

or, what is the same thing, dropping the primes,

$$\sum_{\sigma=0}^{\varkappa} x^{s-\varkappa+\sigma} \sum_{i=0}^{\sigma} C_{\sigma,\varkappa,i}\, e^{(\sigma-i)}(x) \{ e(\omega) - e(x) \}^{\beta-1-r} \prod_{\nu=1}^{i} \{ e^{(\nu)}(x) \}^{\alpha_\nu} .$$

Now the terms corresponding to $\sigma = 0$ and from what remains, $i = 0$, are given by

$$\{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C_{\sigma,\varkappa}\, e^{(\sigma)}(x)\, x^{s-\varkappa+\sigma}, \qquad r = 0, \text{ if } i = 0 .$$

Hence

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa} [\{ e(\omega) - e(x) \}^{\beta-1} e(x)\, x^s] = \{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C\, e^{(\sigma)}(x)\, x^{s-\varkappa+\sigma} +$$
$$+ \sum_{i=1}^{\varkappa} C \{ e(\omega) - e(x) \}^{\beta-1-r} \prod_{\nu=1}^{i} \{ e^{(\nu)}(x) \}^{\alpha_\nu} \sum_{\sigma=i}^{\varkappa} C\, e^{(\sigma-i)}(x)\, x^{s-\varkappa+\sigma} .$$

Let us also consider the set of terms corresponding to $\sigma = i$, then this set of terms is

$$\sum_{i=1}^{\varkappa} C \{ e(\omega) - e(x) \}^{\beta-1-r}\ e(x) \prod_{\nu=1}^{i} \{ e^{(\nu)}(x) \}^{\alpha_\nu}\ x^{s-\varkappa+i} .$$

Thus, writing the terms corresponding to $\sigma = 0$ and from what remains, $i = 0$; and $\sigma = i$, separately, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa} [\{ e(\omega) - e(x) \}^{\beta-1} e(x)\, x^s] = \{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C\, e^{(\sigma)}(x)\, x^{s-\varkappa+\sigma} +$$
$$+ \sum_{i=1}^{\varkappa} C \{ e(\omega) - e(x) \}^{\beta-1-r} e(x) \prod_{\nu=1}^{i} \{ e^{(\nu)}(x) \}^{\alpha_\nu} x^{s-\varkappa+i} +$$
$$+ \sum_{i=1}^{\varkappa} C \sum_{\sigma=i+1}^{\varkappa} C \{ e(\omega) - e(x) \}^{\beta-1-r} \prod_{\nu=1}^{i} \{ e^{(\nu)}(x) \}^{\alpha_\nu} e^{(\sigma-i)}(x)\, x^{s-\varkappa+\sigma}$$
$$= E_1 + E_2 + E_3, \qquad \text{say} .$$

Then, by hypothesis (3.6), we have

$$\prod_{\nu=1}^{i} \{ e^{(\nu)}(x) \}^{\alpha_\nu} =$$

$=\{e^{(1)}(x)\}^{\alpha_1}[\{e^{(1)}(x)/e(x)\}^2 e(x)]^{\alpha_2}\ldots[\{e^{(1)}(x)/e(x)\}^i e(x)]^{\alpha_i}+$ [a linear combination of multiples of this obtained by taking positive monotonic decreasing function of x (1) as multiplying factors] $=$

$=\{e^{(1)}(x)\}^i\{e(x)\}^r/\{e(x)\}^i+$ (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors) $=$

$=\{e^{(1)}(x)\}^i\{e(x)\}^{r-i}+$ (a linear combination of mutiples of this by taking positive monotonic diminishing functions of x as multiplying factors).

Now

$$E_3=\sum_{i=1}^{\varkappa} C\sum_{\sigma=i+1}^{\varkappa} C\{e(\omega)-e(x)\}^{\beta-1-r}\prod_{\nu=1}^{i}\{e^{(\nu)}(x)\}^{\alpha_\nu}e^{(\sigma-i)}(x)\,x^{s-\varkappa+\sigma}$$

$$=\sum_{i=1}^{\varkappa} C\sum_{\sigma=i+1}^{\varkappa} C\,[\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^i\{e(x)\}^{r-i}[\{e^{(1)}(x)/e(x)\}^{\sigma-i}e(x)]\,x^{s-\varkappa+\sigma}$$

$+$ (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors)], by (3.6),

$=\sum_{i=1}^{\varkappa} C\sum_{\sigma=i+1}^{\varkappa} C\,[\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^\sigma\{e(x)\}^{r-\sigma+1}x^{s-\varkappa+\sigma}+$ (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors)].

Next

$$E_2=\sum_{i=1}^{\varkappa} C\{e(\omega)-e(x)\}^{\beta-1-r}e(x)\prod_{\nu=1}^{i}\{e^{(\nu)}(x)\}^{\alpha_\nu}x^{s-\varkappa+i}$$

$=\sum_{i=1}^{\varkappa} C\,[\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^i\{e(x)\}^{r-i+1}x^{s-\varkappa+i}+$ (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors)].

Now, the last expressions of E_3 and E_2 can be combined and written in the form

$\sum_{i=1}^{\varkappa} C\sum_{\sigma=i}^{\varkappa} C\,[\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^\sigma\{e(x)\}^{r-\sigma+1}x^{s-\varkappa+\sigma}+$ (a linear combination of multiples of this obtained by taking positive monotonic diminishing functions of x as multiplying factors)].

---

(1) That is, as $x\to\infty$.

And, by (3.6),

$$E_1 = \{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C \, [\{ e^{(1)}(x)/e(x) \}^{\sigma} e(x) \, x^{s-\varkappa+\sigma} + \text{(a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of } x \text{ as multiplying factors)}] =$$

$$= \{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C \, [\{ e^{(1)}(x) \}^{\sigma} \{ e(x) \}^{1-\sigma} x^{s-\varkappa+\sigma} + \text{(a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of } x \text{ as multiplying factors)}].$$

Hence, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa} [\{ e(\omega) - e(x) \}^{\beta-1} e(x) \, x^s] =$$

$$= \sum_{i=1}^{\varkappa} C \sum_{\sigma=i}^{\varkappa} C \, [\{ e(\omega) - e(x) \}^{\beta-1-r} \{ e^{(1)}(x) \}^{\sigma} \{ e(x) \}^{r-\sigma+1} x^{s-\varkappa+\sigma} + \text{(a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of } x \text{ as multiplying factors)}] +$$

$$+ \{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C \, [\{ e^{(1)}(x) \}^{\sigma} \{ e(x) \}^{1-\sigma} x^{s-\varkappa+\sigma} + \text{(a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of } x \text{ as multiplying factors)}].$$

Now $\sum_{i=1}^{\varkappa} \sum_{\sigma=i}^{\varkappa} \ldots = \sum_{\sigma=1}^{\varkappa} \sum_{i=1}^{\sigma} \ldots$. Therefore, finally,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa} [\{ e(\omega) - e(x) \}^{\beta-1} e(x) \, x^s] = \tag{6.12}$$

$$= \sum_{\sigma=1}^{\varkappa} C \sum_{i=1}^{\sigma} C \, [\{ e(\omega) - e(x) \}^{\beta-1-r} \{ e^{(1)}(x) \}^{\sigma} \{ e(x) \}^{r-\sigma+1} x^{s-\varkappa+\sigma} + \text{(a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of } x \text{ as multiplying factors)}] +$$

$$+ \{ e(\omega) - e(x) \}^{\beta-1} \sum_{\sigma=0}^{\varkappa} C \, [\{ e^{(1)}(x) \}^{\sigma} \{ e(x) \}^{1-\sigma} x^{s-\varkappa+\sigma} + \text{(a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of } x \text{ as multiplying factors)}].$$

Now, for $\beta > 1$, and $\varkappa \leqslant [\beta] - 1$, since $r \leqslant i \leqslant \varkappa$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\varkappa} [\{ e(\omega) - e(x) \}^{\beta-1} e(x) \, x^s]_{x=\omega} = 0 \, .$$

Therefore, by (6.11),

$$(6.13)\qquad [J_{1\varkappa}(x;\ t,\ \varrho)]_1^\omega = [O(t^{-\varkappa-1-\varrho+l}) + O(t^{-\varrho-2})]\cdot O[\{e(\omega)\}^{\beta-1}] =$$

$$= [O(t^{-\varkappa-1-\varrho+l}) + O(t^{-\varrho-2})]\{e(\omega)\}^{\beta-1} = [O(t^{-\varkappa-1-\varrho}) + O(t^{-\varrho-2})]\{e(\omega)\}^{\beta-1},$$

since t is finite and positive and so is l,

$$= [O(t^{-[\beta]-\varrho}) + O(t^{-\varrho-2})]\{e(\omega)\}^{\beta-1}.$$

For the estimation of $J_2(\omega;\ t,\ \varrho)$, we use only the order estimate of $S^{[\beta]-1}(x;\ t,\ \varrho)$ and the expression for $(\mathrm{d}/\mathrm{d}x)^k[\{e(\omega)-e(x)\}^{\beta-1}e(x)\,x^s]$ for $\varkappa=[\beta]$. By (6.10) and (6.12),

$$(6.14)\qquad J_2(\omega;\ t,\ \varrho) = \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{\varrho}\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} C\int_1^\omega([x]-j/2+1/2)^l\cdot$$

$$\cdot\ {\sin \atop \cos}\{([x]-j/2+1/2)t-\pi j/2\}\cdot[\{\cos(t/2)\}^n/\{2\sin(t/2)\}^{j+1+m}]$$

$$[\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^\sigma\{e(x)\}^{r-\sigma+1}x^{s-[\beta]+\sigma} +$$

+ (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors)] $\mathrm{d}x$ +

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{\varrho}\sum_{\sigma=0}^{[\beta]} C\int_1^\omega([x]-j/2+1/2)^l\cdot$$

$$\cdot\ {\sin \atop \cos}\{([x]-j/2+1/2)t-\pi j/2\}\cdot[\{\cos(t/2)\}^n/\{2\sin(t/2)\}^{j+1+m}]\cdot$$

$$\cdot[\{e(\omega)-e(x)\}^{\beta-1}\{e^{(1)}(x)\}^\sigma\{e(x)\}^{1-\sigma}x^{s-[\beta]+\sigma} +$$

+ (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors)] $\mathrm{d}x$ $(0\leqslant m,\ n\leqslant\varrho-l)$

$$+ O[\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} C\int_1^\omega t^{-\varrho-2}x^{[\beta]-2}\{e(\omega)-e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^\sigma\{e(x)\}^{r-\sigma+1}x^{s-[\beta]+\sigma}\,\mathrm{d}x] +$$

$$+ O[\sum_{\sigma=0}^{[\beta]} C\int_1^\omega t^{-\varrho-2}x^{[\beta]-2}\{e(\omega)-e(x)\}^{\beta-1}\{e^{(1)}(x)\}^\sigma\{e(x)\}^{1-\sigma}x^{s-[\beta]+\sigma}\,\mathrm{d}x] =$$

$$= \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{\varrho}\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} J_{21}(\omega,t,\varrho,\sigma,r) + \text{etc.}\ (^1) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{\varrho}\sum_{\sigma=0}^{[\beta]} J'_{21}(\omega,t,\varrho,\sigma) + \text{etc.} +$$

$$+ O[\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} J_{22}(\omega,\ t,\ \varrho,\ \sigma,\ r)] + O[\sum_{\sigma=0}^{[\beta]} J'_{22}(\omega,\ t,\ \varrho,\ \sigma)].$$

---

(1) By « etc. » we mean terms of a similar character with positive monotonic diminishing functions of x as multiplying factors.

Case I: $r < [\beta]$.

All the terms of the type $J_{21}(\omega, t, \varrho, \sigma, r)$ involving positive monotonic decreasing factors in the integrands can be treated in the same manner, and since the maximum value of the monotonic decreasing factors can be taken outside and the second mean value theorem applied, the order estimates for them would be the same as that for $J_{21}(\omega, t, \varrho, \sigma, r)$ on grounds of technique followed as below. The remarks apply to $J'_{21}(\omega, t, \varrho, \sigma)$ also. We remember that $0 \leqslant m$, $n \leqslant \varrho - l$ and $0 \leqslant j \leqslant [\beta] - 1$.

$$J_{21}(\omega, t, \varrho, \sigma, r) = \int_1^{\omega} ([x] - j/2 + 1/2)^l \cdot \tag{6.15}$$

$$\cdot \frac{\sin}{\cos}\{([x] - j/2 + 1/2)t - \pi j/2\} \cdot [\{\cos(t/2)\}^n / \{2\sin(t/2)\}^{j+1+m}] \cdot$$

$$\cdot \{e(\omega) - e(x)\}^{\beta-1-r} \{e^{(1)}(x)\}^{\sigma} \{e(x)\}^{r-\sigma+1} x^{s-[\beta]+\sigma}\, dx$$

$$= O[t^{-[\beta]-\varrho+l} \,|\int_1^{\omega} \{e(\omega) - e(x)\}^{\beta-1-r} \{e^{(1)}(x)\}^{\sigma} \{e(x)\}^{r-\sigma+1} x^{s-[\beta]+\sigma} \cdot$$

$$([x] - j/2 + 1/2)^l \cdot \frac{\sin}{\cos}\{([x] - j/2 + 1/2)t - \pi j/2\}\, dx \,|] =$$

$$= O[t^{-[\beta]-\varrho+l} \{e(\omega)\}^{\beta-1-r} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma} \{e(\omega)\}^{r+1} \omega^{s+l-[\beta]} \cdot$$

$$\cdot \max |\int_{\xi}^{\xi'} \frac{\sin}{\cos}\{([x] - j/2 + 1/2)t - \pi j/2\}\, dx\,|] \quad (1 \leqslant \xi \leqslant \xi' \leqslant \omega),$$

by (3.3) and (3.2),

$$= O[t^{-[\beta]-\varrho+l-1} \{e(\omega)\}^{\beta-1-r} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma} \{e(\omega)\}^{r+1} \omega^{s+l-[\beta]}] =$$

$$= O[t^{-[\beta]-\varrho+l-1} \{e(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma} \omega^{s+l-[\beta]}].$$

Case II: $r = \sigma = [\beta]$.

$$J_{21}(\omega, t, \varrho, \sigma, r) = [\int_1^{\omega - t^{-1}} + \int_{\omega - t^{-1}}^{\omega}]([x] - j/2 + 1/2)^l \cdot \tag{6.16}$$

$$\cdot \frac{\sin}{\cos}\{([x] - j/2 + 1/2)t - \pi j/2\} \cdot$$

$$\cdot [\{\cos(t/2)\}^n / \{2\sin(t/2)\}^{j+1+m}] \{e(\omega) - e(x)\}^{\beta-1-[\beta]} \{e^{(1)}(x)\}^{[\beta]} e(x)\, x^s\, dx =$$

$$= J_{21;1}(\omega, t, \varrho, \sigma, r) + J_{21;2}(\omega, t, \varrho, \sigma, r), \qquad \text{say}.$$

$$J_{21;1}(\omega, t, \varrho, \sigma, r) = O[t^{-[\beta]-\varrho+l} \,|\int_1^{\omega - t^{-1}} \{e(\omega) - e(x)\}^{\beta-1-[\beta]} \{x\, e^{(1)}(x)/e(x)\}^{[\beta]} x^{s-[\beta]} \cdot \tag{6.17}$$

$$\cdot \{ e(x) \}^{[\beta]+1} ([x] - j/2 + 1/2)^l \frac{\sin}{\cos} \{ ([x] - j/2 + 1/2)t - \pi j/2 \} dx |] =$$

$$= O[t^{-[\beta]-\varrho+l} \{ e(\omega) - e(\omega - t^{-1}) \}^{\beta-1-[\beta]} \{ (\omega - t^{-1}) e^{(1)}(\omega - t^{-1})/e(\omega - t^{-1}) \}^{[\beta]} \cdot$$

$$\cdot \{ e(\omega - t^{-1}) \}^{[\beta]+1} (\omega - t^{-1})^{s+l-[\beta]} \cdot \max | \int_{\varrho}^{\omega - t^{-1}} \frac{\sin}{\cos} \{ ([x] - j/2 + 1/2)t - \pi j/2 \} dx |]$$

$(1 \leqslant \varrho \leqslant \omega - t^{-1})$, by (3.2) and (3.3),

$$= O[t^{-[\beta]-\varrho+l-1} \{ e^{(1)}(\omega - t^{-1}) t^{-1} \}^{\beta-1-[\beta]} \{ (\omega - t^{-1}) e^{(1)}(\omega - t^{-1})/e(\omega - t^{-1}) \}^{[\beta]} \cdot$$

$$\cdot \{ e(\omega - t^{-1}) \}^{[\beta]+1} (\omega - t^{-1})^{s+l-[\beta]}],$$ by Lemma 2 (ii),

$$= O(t^{-[\beta]-\varrho+l-1} \{ e^{(1)}(\omega - t^{-1}) \}^{\beta-[\beta]} t^{-\beta+1+[\beta]} \{ (\omega - t^{-1}) e^{(1)}(\omega - t^{-1})/e(\omega - t^{-1}) \}^{[\beta]} \cdot$$

$$\cdot \{ e(\omega - t^{-1})/e^{(1)}(\omega - t^{-1}) \} \{ e(\omega - t^{-1}) \}^{[\beta]} (\omega - t^{-1})^{s+l-[\beta]}] =$$

$$= O[t^{-\beta-\varrho+l} \{ e^{(1)}(\omega) \}^{\beta-[\beta]} \{ \omega e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \{ e(\omega)/e^{(1)}(\omega) \} \{ e(\omega) \}^{[\beta]} \omega^{s+l-[\beta]}],$$

by (3.3), (3.4) and (3.2),

$$= O[t^{-\beta-\varrho+l} \{ \omega e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1} \{ e^{(1)}(\omega) \}^{\beta} \omega^{s+l-[\beta]}];$$

(6.18) $$J_{21;2}(\omega, t, \varrho, \sigma, r) = O[t^{-[\beta]-\varrho+l} | \int_{\omega-t^{-1}}^{\omega} ([x] - j/2 + 1/2)^l \{ e(\omega) - e(x) \}^{\beta-1-[\beta]} e^{(1)}(x) \cdot$$

$$\cdot \{ x e^{(1)}(x)/e(x) \}^{[\beta]-1} \{ e(x) \}^{[\beta]} x^{s-[\beta]+1} \frac{\sin}{\cos} \{ [x] - j/2 + 1/2)t - \pi j/2 \} dx|] =$$

$$= O[t^{-[\beta]-\varrho+l} | \int_{\omega-t^{-1}}^{\omega} ([x] - j/2 + 1/2)^l \{ e(\omega) - e(x) \}^{\beta-1-[\beta]} e^{(1)}(x) \{ x e^{(1)}(x)/e(x) \}^{[\beta]-1} \cdot$$

$$\cdot \{ e(x) \}^{[\beta]} x^{s-[\beta]+1} dx|] =$$

$$= O[t^{-[\beta]-\varrho+l} \{ \omega e^{(1)}(\omega)/e(\omega) \}^{[\beta]-1} \{ e(\omega) \}^{[\beta]} \omega^{s-[\beta]+l+1} [\{ e(\omega) - e(x) \}^{\beta-[\beta]}]_{\omega-t^{-1}}^{\omega}],$$

by (3.3) and (3.2),

$$= O[t^{-[\beta]-\varrho+l} \{ e(\omega) \}^{[\beta]} \{ \omega e^{(1)}(\omega)/e(\omega) \}^{[\beta]-1} \{ e^{(1)}(\omega) t^{-1} \}^{\beta-[\beta]} \omega^{s-[\beta]+l+1}],$$

by Lemma 2 (i).

When $r < [\beta]$:

(6.15)′ $J'_{21}(\omega, t, \varrho, \sigma)$ + (a linear combination of multiples of this obtained by taking positive monotonic decreasing functions of x as multiplying factors) =

$$= O\Big[t^{-[\beta]-\varrho+l}\Big|\int_1^{\omega}\{e(\omega)-e(x)\}^{\beta-1}\{e^{(1)}(x)\}^{\sigma}\{e(x)\}^{1-\sigma}x^{s-[\beta]+\sigma}\cdot$$

$$\cdot([x]-j/2+1/2)^{l}\,{}^{\sin}_{\cos}\{([x]-j/2+1/2)t-\pi j/2\}\,dx\Big|\Big] =$$

$$= O\Big[t^{-[\beta]-\varrho+l}\{e(\omega)\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}e(\omega)\,\omega^{s+l-[\beta]}\cdot$$

$$\cdot\max\Big|\int_{\eta}^{\eta'}{}^{\sin}_{\cos}\{([x]-j/2+1/2)\,t-\pi j/2\}\,dx\Big|\Big]$$

$(1 \leqslant \eta \leqslant \eta' \leqslant \omega)$, by (3.3) and (3.2),

$$= O\big[t^{-[\beta]-\varrho+l-1}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{s+l-[\beta]}\big].$$

When $r = \sigma = [\beta]$:

(6.16)′ $$J'_{21}(\omega, t, \varrho, \sigma) = \Big\{\int_1^{\omega-t^{-1}} + \int_{\omega-t^{-1}}^{\omega}\Big\}([x]-j/2+1/2)^{l}\cdot$$

$$\cdot\,{}^{\sin}_{\cos}\{([x]-j/2+1/2)\,t-\pi j/2\}\cdot$$

$$\cdot[\{\cos(t/2)\}^{n}/\{2\sin(t/2)\}^{[\beta]+\varrho-l}][\{e(\omega)-e(x)\}^{\beta-1}\{e^{(1)}(x)\}^{\sigma}\{e(x)\}^{1-\sigma}x^{s}+$$

+ (a linear combination of multiples of this obtained by taking positive monotonic diminishing functions of x as multiplying factors)] dx =

$$= J'_{21;1}(\omega, t, \varrho, \sigma) + J'_{21;2}(\omega, t, \varrho, \sigma), \qquad \text{say}.$$

(6.17)′ $$J'_{21;1}(\omega, t, \varrho, \sigma) = O\Big[t^{-[\beta]-\varrho+l}\Big|\int_1^{\omega-t^{-1}}\{e(\omega)-e(x)\}^{\beta-1}\{x\, e^{(1)}(x)/e(x)\}^{[\beta]}e(x)\cdot$$

$$\cdot([x]-j/2+1/2)^{l}x^{s-[\beta]}\,{}^{\sin}_{\cos}\{([x]-j/2+1/2)t-\pi j/2\}\,dx\Big|\Big] =$$

$$= O\Big[t^{-[\beta]-\varrho+l}\{e(\omega)\}^{\beta-1}\{(\omega-t^{-1})\,e^{(1)}(\omega-t^{-1})/e(\omega-t^{-1})\}^{[\beta]}e(\omega-t^{-1})\,(\omega-t^{-1})^{s+l-[\beta]}\cdot$$

$$\cdot\max\Big|\int_{\psi}^{\psi'}{}^{\sin}_{\cos}\{([x]-j/2+1/2)t-\pi j/2\}\,dx\Big|\Big] \qquad (1 \leqslant \psi \leqslant \psi' \leqslant \omega - t^{-1}),$$

by (3.3) and (3.2),

$$= O[t^{-[\beta]-\varrho+l-1}\{e(\omega)\}^{\beta}\{\omega e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{s+l-[\beta]}] .$$

For $J_{21;2}(\omega, t, \varrho, \sigma)$ and similar terms with positive monotonic diminishing factors in the integrand, we note that

$$(6.18)' \qquad J_{21;2}(\omega, t, \varrho, \sigma) = O[t^{-[\beta]-\varrho+l} | \int_{\omega-t^{-1}}^{\omega} \{e(\omega) - e(x)\}^{\beta-1}\{x\, e^{(1)}(x)/e(x)\}^{[\beta]} e(x) \cdot$$

$$\cdot([x] - j/2 + 1/2)^{l} x^{s-[\beta]} \begin{matrix}\sin\\ \cos\end{matrix}\{([x] - j/2 + 1/2)t - \pi j/2\} dx|] =$$

$$= O[t^{-[\beta]-\varrho+l}\{e(\omega) - e(\omega - t^{-1})\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]} e(\omega)\, \omega^{s+l-[\beta]} \cdot$$

$$\cdot \max | \int_{\theta}^{\theta'} \begin{matrix}\sin\\ \cos\end{matrix}\{([x] - j/2 + 1/2)\, t - \pi j/2\} dx |]$$

$(\omega - t^{-1} \leqslant \theta \leqslant \theta' \leqslant \omega)$, by (3.2) and (3.3),

$$= O[t^{-[\beta]-\varrho+l-1} e(\omega)\{e^{(1)}(\omega)/t\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{s+l-[\beta]}] .$$

For $J_{22}(\omega, t, \varrho, \sigma, r)$ and similar terms with positive monotonic diminishing factors in the integrand, we observe that

$$(6.19) \qquad J_{22}(\omega, t, \varrho, \sigma, r) = O[t^{-\varrho-2}\int_{1}^{\omega} x^{[\beta]-2}\{e(\omega) - e(x)\}^{\beta-1-r}\{e^{(1)}(x)\}^{\sigma}\{e(x)\}^{r-\sigma+1} \cdot$$

$$\cdot\, x^{s-[\beta]+\sigma} dx] =$$

$$= O[t^{-\varrho-2}\int_{1}^{\omega}\{e(\omega) - e(x)\}^{\beta-1-r} e^{(1)}(x)\{x\, e^{(1)}(x)/e(x)\}^{\sigma-1}\{e(x)\}^{r} x^{s-1} dx] =$$

$$= O[t^{-\varrho-2}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\{e(\omega)\}^{r}\omega^{s-1} | \int_{1}^{\omega}\{e(\omega) - e(x)\}^{\beta-1-r} e^{(1)}(x)\, dx |] =$$

by (3.3) and (3.2),

$$= O[t^{-\varrho-2}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\{e(\omega)\}^{r}\omega^{s-1}[\{e(\omega) - e(x)\}^{\beta-r}]_{1}^{\omega}],$$

$$= O[t^{-\varrho-2}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\{e(\omega)\}^{r}\omega^{s-1}\{e(\omega)\}^{\beta-r}] , \text{ since } \beta - r > 0 ,$$

$$= O[t^{-\varrho-2}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\{e(\omega)\}^{\beta}\omega^{s-1}] .$$

For the part $J'_{22}(\omega, t, \varrho, \sigma)$, when $\sigma = 0$, and similar terms with positive monotonic decreasing factors in the integrand, we note that

$$(6.19)' \qquad J'_{22}(\omega, t, \varrho, \sigma) = O[t^{-\varrho-2}\int_1^\omega \{ e(\omega) - e(x) \}^{\beta-1} e(x)\, x^{s-2}\, dx] =$$

$$= O[t^{-\varrho-2}\{ e(\omega) \}^{\beta-1} e(\omega)\, \omega^{s-2}\int_\eta^{\eta'} dx] \qquad (1 \leqslant \eta \leqslant \eta' \leqslant \omega)$$

$$= O[t^{-\varrho-2}\{ e(\omega) \}^{\beta}\, \omega^{s-1}],$$

and for the part $J'_{22}(\omega, t, \varrho, \sigma)$ (when $\sigma \geqslant 1$) and terms of a similar kind with monotonic diminishing factors in the integrand, we observe that

$$(6.20) \qquad J'_{22}(\omega, t, \varrho, \sigma) = O[t^{-\varrho-2}\int_1^\omega \{ e(\omega) - e(x) \}^{\beta-1}\{ e^{(1)}(x) \}^{\sigma}\{ e(x) \}^{1-\sigma} x^{s+\sigma-2}\, dx]$$

$$= O[t^{-\varrho-2}\int_1^\omega \{ e(\omega) - e(x) \}^{\beta-1} e^{(1)}(x)\{ x\, e^{(1)}(x)/e(x) \}^{\sigma-1} x^{s-1}\, dx]$$

$$= O[t^{-\varrho-2}\{ e(\omega) \}^{\beta-1}\{ e^{(1)}(\omega)/\omega^{1-s} \}\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1}\int_\varrho^{\varrho'} dx] \qquad (1 \leqslant \varrho \leqslant \varrho' \leqslant \omega),$$

by (3.1) and (3.3),

$$= O[t^{-\varrho-2}\{ e(\omega) \}^{\beta-1} e^{(1)}(\omega)\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1} \omega^{s}].$$

Finally, collecting the estimates (6.13), (6.15), (6.15)′, (6.19) and (6.20) in case $r < [\beta]$, and (6.16), (6.16)′ in case $r = \sigma = [\beta]$, the lemma for the case $\beta > 1$ follows.

7. - Lemma 4.

If $\beta > 1$, *or, if* $\beta < 1$ *and* $s + \varrho + 1 > 0$, *then we have*

$$F(\omega, t, \varrho, s) = O[\{ e(\omega) \}^{\beta+1} \omega^{s+\varrho}/e^{(1)}(\omega)].$$

Proof.

$$F(\omega, t, \varrho, s) = O[\int_1^\omega \{ e(\omega) - e(x) \}^{\beta-1}\, e(x)\, x^{s+\varrho}\, dx] =$$

$$= O[\int_1^\omega \{ e(\omega) - e(x) \}^{\beta-1} e^{(1)}(x)\{ e(x)/e^{(1)}(x) \} x^{s+\varrho}\, dx], \qquad \text{by (3.5)},$$

$$= O[\{ e(\omega)/e^{(1)}(\omega) \}\, \omega^{s+\varrho}\, [\{ e(\omega) - e(x) \}^{\beta}]_1^{\omega}] = O[\{ e(\omega) \}^{\beta+1} \omega^{s+\varrho}/e^{(1)}(\omega)].$$

8. - Lemma 5.

We shall prove that, for $\beta > [\alpha] \geqslant 1$ *and* $[\alpha] - 1 \geqslant \varrho$,

$$\int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \mid E^{(\varrho)}(\omega, \pi) \mid d\omega < \infty .$$

Proof.

Now by Lemma 3, we have

$$E^{(\varrho)}(\omega, \pi) = F(\omega, \pi, \varrho, 0) = O[\{ e(\omega) \}^{\beta-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[\{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma} \omega^{l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} \sum_{\sigma=0}^{[\beta]} O[\{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma} \omega^{l-[\beta]}] +$$

$$+ \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[\{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1} \omega^{-1}] +$$

$$+ \sum_{\sigma=0}^{[\beta]} O[\{ e(\omega) \}^{\beta-1} e^{(1)}(\omega) \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} \sum_{i=1}^{[\beta]} O[\{ e^{(1)}(\omega) \}^{\beta} \{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \omega^{l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} O[\{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \omega^{l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} O[e(\omega) \{ e^{1}(\omega) \}^{\beta-1} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \omega^{l-[\beta]}] .$$

Substituting the estimate of $E^{(\varrho)}(\omega, \pi)$ from above in

$$\int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \mid E^{(\varrho)}(\omega, \pi) \mid d\omega ,$$

we have

$$\int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \mid E^{(\varrho)}(\omega, \pi) \mid d\omega = O(L_1) +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(L_2) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} \sum_{\sigma=0}^{[\beta]} O(L_3) + \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(L_4) +$$

$$+ \sum_{\sigma=0}^{[\beta]} O(L_5) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} \sum_{i=1}^{[\beta]} O(L_6) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} O(L_7) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]-1} O(L_8) ,$$

where the integrals L_r ($r = 1, 2, \ldots, 8$) are defined as in the body of proofs, given below, of their convergence, which is an immediate consequence, in each case, of the inequality displayed.

$$L_1 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1}\, d\omega \leqslant K\int_1^\infty \omega^{-2+\varepsilon}\, d\omega$$
$$(\varepsilon < 1)\ (^1),\ \text{by (3.7)}' \text{ and (3.2)}.$$

$$L_2 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\,\omega^{l-[\beta]}\, d\omega$$
$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\,\omega^{-[\beta]+l-1+\varepsilon}\, d\omega \qquad (\varepsilon < [\beta]-l)$$
$$\leqslant K\int_1^\infty \omega^{-[\beta]+l-1+\varepsilon}\, d\omega, \quad \text{by (3.7)}.$$

$$L_3 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\,\omega^{l-[\beta]}\, d\omega.$$

The case of L_3 is similar to that of L_2.

$$L_4 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\,\omega^{-1}\, d\omega$$
$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}/\omega^{\varepsilon}]\,\omega^{-2+\varepsilon}\, d\omega \leqslant K\int_1^\infty \omega^{-2+\sigma}\, d\omega$$
$$(\varepsilon < 1),\ \text{by (3.7)}.$$

$$L_5 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1} e^{(1)}\omega)\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\, d\omega$$
$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\omega^{-2+\varepsilon}\, d\omega \leqslant K\int_1^\infty \omega^{-2+\varepsilon}\, d\omega \quad (\varepsilon < 1),\ \text{by (3.7)}.$$

$$L_6 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e^{(1)}(\omega)\}^{\beta}\{e(\omega)/e^{(1)}(\omega)\}^{[\beta]+1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\,\omega^{l-[\beta]}\, d\omega$$
$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\omega^{-\beta+l+\varepsilon}\, d\omega \leqslant K\int_1^\infty \omega^{-\beta+l+\varepsilon}\, d\omega \quad (\varepsilon < \beta-l-1),\ \text{by (3.7)}.$$

$$L_7 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\,\omega^{l-[\beta]}\, d\omega$$
$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]+1}/\omega^{\varepsilon}]\,\omega^{-[\beta]+l-1+\varepsilon}\, d\omega \qquad (\varepsilon < [\beta]-l)$$

---

(1) K denotes an absolute constant not necessarily the same at each occurrence; ε is arbitrary but positive.

$$\leqslant K \int_1^\infty \omega^{-[\beta]+l-1+\varepsilon}\, d\omega, \qquad \text{by (3.7)}.$$

$$L_8 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}] e(\omega) \{e^{(1)}(\omega)\}^{\beta-1} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\, \omega^{l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta+[\beta]}/\omega^{\varepsilon}]\, \omega^{-[\beta]+l-\beta+\varepsilon}\, d\omega \qquad (\varepsilon < \beta + [\beta] - l - 1)$$

$$\leqslant K \int_1^\infty \omega^{-[\beta]+l-\beta+\varepsilon}\, d\omega, \qquad \text{by (3.7)}.$$

Hence the Lemma is proved.

9. - Lemma 6.

$g(\omega, u)$ is of the same order as the combined order-estimates of $F(\omega, u, [\alpha], \alpha - [\alpha] - 1)$ and $F(\omega, u, [\alpha]+1, \alpha - [\alpha] - 2)$ as obtained according to Lemma 3.

P r o o f. Now

$$\Gamma(1 + [\alpha] - \alpha)\, g(\omega, u) = \Big(\int_u^{u+1/n} + \int_{u+1/n}^{\pi}\Big) (t - u)^{[\alpha]-\alpha} E^{([\alpha])}(\omega, t)\, dt =$$

$$= \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n) \Big[(\cos n\theta)_{[\alpha]} \int_u^{u+1/n} (t - u)^{[\alpha]-\alpha}\, dt + n^{\alpha-[\alpha]} \int_{u+1/n}^{\varrho} (\cos nt)_{[\alpha]}\, dt\Big]$$

$$(u \leqslant \theta \leqslant u + 1/n) \quad \text{and} \quad (u + 1/n \leqslant \varrho \leqslant \pi)$$

$$= \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n) \left[n^{\alpha-[\alpha]-1} (\cos n\theta)_{[\alpha]} + n^{\alpha-[\alpha]} \left\{ \begin{matrix} n^{-1} (\cos n\nu)_{[\alpha]} \\ n^{-2} (\cos n\nu)_{[\alpha]+1} \end{matrix} \right. \right]$$

$$(u + n^{-1} \leqslant \nu \leqslant \pi).$$

Hence the result.

10. - Lemma 7.

For $[\beta] = 0$ and $\beta > 1$

$$\int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\, |\, g(\omega, \pi)\, |\, d\omega < \infty.$$

Proof.

For the case $[\beta]=0$, we have, by Lemma 6 and Lemma 3,

$$g(\omega,\pi)=O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-1}]+O[\{e^{(1)}(\omega)\}^{\beta}\omega^{\alpha-1}]+O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-2}]+$$

$$+\sum_{l=0}^{1}O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-2+l}]+\sum_{l=0}^{1}O[\{e^{(1)}(\omega)\}^{\beta}\omega^{\alpha-2+l}]+$$

$$+\sum_{l=0}^{1}O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-3+l}].$$

And for the case $\beta>1$, we have, by Lemma 6 and Lemma 3,

$$g(\omega,\ \pi)=O[\{e(\omega)\}^{\beta-1}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-1+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}\sum_{\sigma=0}^{[\beta]}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-1+l-[\beta]}]+$$

$$+\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-2}]+$$

$$+\sum_{\sigma=0}^{[\beta]}O[\{e(\omega)\}^{\beta-1}e^{(1)}(\omega)\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-1}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}\sum_{i=1}^{[\beta]}O[\{e^{(1)}(\omega)\}^{\beta}\{\omega e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\{e(\omega)/e^{(1)}(\omega)\}^{[\beta]+1}\omega^{\alpha-[\alpha]-1+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{\alpha-[\alpha]-1+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{\alpha-[\alpha]-1+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-2+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{\sigma=0}^{[\beta]}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-2+l-[\beta]}]+$$

$$+\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma}O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-3}]+$$

$$+\sum_{\sigma=0}^{\beta}O[\{e(\omega)\}^{\beta-1}e^{(1)}(\omega)\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-2}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{i=1}^{[\beta]} O[\{e^{(1)}(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\{e(\omega)/e^{(1)}(\omega)\}^{[\beta]+1}\,\omega^{\alpha-[\alpha]-2+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1} O[\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\,\omega^{\alpha-[\alpha]-2+l-[\beta]}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1} O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\,\omega^{\alpha-[\alpha]-2+l-[\beta]}]\,.$$

Substituting the order-estimate of $g(\omega, \pi)$ for the case $[\beta] = 0$ in the integral

$$\int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|\,g(\omega,\ \pi)\,|\,d\omega\,,$$

we have

$$\int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|\,g(\omega,\ \pi)\,|\,d\omega = O(M_1)+O(M_2)+O(M_3)+$$

$$+\sum_{l=0}^{1} O(M_4)+\sum_{l=0}^{1} O(M_5)+\sum_{l=0}^{1} O(M_6)\,,$$

where M_r $(r = 1, 2, \ldots, 6)$ are defined as in the body of proofs, given below of their convergence, which is an immediate consequence, in each case, of the inequality displayed.

$$M_1 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\,\omega^{\alpha-1}\,d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\,\omega^{-\beta+\alpha-1+\varepsilon}\,d\omega$$

$$\leqslant K\int_1^\infty \omega^{-\beta+\alpha-1+\varepsilon}\,d\omega \qquad (0<\varepsilon<\beta-\alpha),\ \text{by (3.7)}\,.$$

$$M_2 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e^{(1)}(\omega)\}^{\beta}\,\omega^{\alpha-1}\,d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta+1}/\omega^{\varepsilon}]\,\omega^{-\beta-2+\alpha+\varepsilon}\,d\omega$$

$$\leqslant K\int_1^\infty \omega^{-\beta+\alpha-2+\varepsilon}\,d\omega \qquad (0<\varepsilon<\beta-\alpha+1),\ \text{by (3.7)}\,.$$

$$M_3 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\,\omega^{\alpha-2}\,d\omega =$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^\beta/\omega^\varepsilon]\, \omega^{-\beta+\alpha-2+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-\beta+\alpha-2+\varepsilon}\, d\omega \qquad (0 < \varepsilon < \beta - \alpha + 1), \text{ by } (3.7).$$

$$M_4 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\, e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\, \omega^{\alpha+l-2}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^\beta/\omega^\varepsilon]\, \omega^{-\beta+\alpha+l-2+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-\beta+l+\alpha-2}\, d\omega \qquad (l = 0,\ 1;\ 0 < \varepsilon < \beta - \alpha - l + 1), \text{ by } (3.7).$$

$$M_5 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e^{(1)}(\omega)\}^\beta\, \omega^{\alpha+l-2}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta+1}/\omega^\varepsilon]\, \omega^{-\beta+l+\alpha-3+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-\beta+l+\alpha-3+\varepsilon}\, d\omega \qquad (l = 0, 1;\ 0 < \varepsilon < \beta - \alpha - l + 2), \text{ by } (3.7).$$

$$M_6 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\, e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\, \omega^{\alpha+l-3}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^\beta/\omega^\varepsilon]\, \omega^{-\beta-3+\alpha+l+\varepsilon}\, d\omega$$

$$\leqslant \int_1^\infty \omega^{-\beta-3+\alpha+l+\varepsilon}\, d\omega \qquad (l = 0,1;\ 0 < \varepsilon < \beta - \alpha - l + 2), \quad \text{by } (3.7).$$

Substituting the value of $g(\omega, \pi)$ for the case $\beta > 1$, in the integral

$$\int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\, |\, g(\omega, \pi)\, |\, d\omega,$$

we get

$$\int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\, |\, g(\omega, \pi)\, |\, d\omega = O(N_1) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(N_2) +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{\sigma=0}^{[\beta]} O(N_3) + \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(N_4) + \sum_{\sigma=0}^{[\beta]} O(N_5) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{i=1}^{[\beta]} O(N_6) +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} O(N_7) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} O(N_8) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(N_9) +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} \sum_{\sigma=0}^{[\beta]} O(N_{10}) + \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(N_{11}) + \sum_{\sigma=0}^{[\beta]} O(N_{12}) +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} \sum_{i=1}^{[\beta]} O(N_{13}) + \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} O(N_{14}) + \sum_{i=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} O(N_{15}),$$

where the integrals N_r $(r = 1, 2, \ldots, 15)$ are defined as in the body of proofs, given below, of their convergence, which is an immediate consequence, in each case, of the inequality displayed.

$$N_1=\int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1}\,d\omega \leqslant K\int_1^\infty \omega^{-2+\varepsilon}\,d\omega\ (0<\varepsilon<1),\ \text{by (3.7)}'\ \text{and (3.2)}.$$

In integrals $N_2, \dots, N_8$; $l=0, 1, \dots, [\alpha]$, and in integrals N_9 onwards $l =$ $= 0, 1, \dots, [\alpha]+1$; wherever they occur.

$$N_2 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\,\omega^{\alpha-[\alpha]-1+l-[\beta]}\,d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-2+\alpha-[\beta]+l+\varepsilon}\,d\omega \quad (0<\varepsilon<[\alpha]+1-\alpha+[\beta]-l)$$

$$\leqslant K\int_1^\infty \omega^{-[\alpha]-2+\alpha-[\beta]+l+\varepsilon}\,d\omega, \qquad \text{by (3.7)}.$$

$$N_3 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\,\omega^{\alpha-[\alpha]-1+l-[\beta]}\,d\omega = N_2,$$

$$N_4 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\,\omega^{\alpha-[\alpha]-2}\,d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-3+\alpha+\varepsilon}\,d\omega$$

$$\leqslant K\int_1^\infty \omega^{-[\alpha]-3+\alpha+\varepsilon}\,d\omega \qquad (0<\varepsilon<[\alpha]+2-\alpha),\ \text{by (3.7)}.$$

$$N_5 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1}e^{(1)}(\omega)\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\,\omega^{\alpha-[\alpha]-1}\,d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-3+\alpha+\varepsilon}\,d\omega$$

$$\leqslant K\int_1^\infty \omega^{-[\alpha]-3+\alpha+\varepsilon}\,d\omega \quad (0<\varepsilon<[\alpha]+2-\alpha),\ \text{by (3.7)}.$$

$$N_6 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e^{(1)}(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\{e(\omega)/$$

$$/e^{(1)}(\omega)\}^{[\beta]+1}\,\omega^{\alpha-[\alpha]-1+l-[\beta]}\,d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-1+\alpha-\beta+l+\varepsilon}\,d\omega \qquad (0<\varepsilon<[\alpha]-\alpha+\beta-l)$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-1+\alpha-\beta+l+\varepsilon}\, d\omega, \qquad \text{by (3.7)}.$$

$$N_7 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\,\omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]+1}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-2+\alpha-[\beta]+l+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-2+\alpha-[\beta]+l+\varepsilon}\, d\omega \quad (0<\varepsilon<[\alpha]+1-\alpha+[\beta]-l), \quad \text{by (3.7)}.$$

$$N_8 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\,\omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta+[\beta]}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-1+\alpha-[\beta]+l-\beta+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-1+\alpha-\beta-[\beta]+l+\varepsilon}\, d\omega \quad (0<\varepsilon<[\alpha]-\alpha+[\beta]-l+\beta), \quad \text{by (3.7)}.$$

Henceforth $l = 0, 1, \ldots, [\alpha]+1$.

$$N_9 = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\,\omega^{\alpha-[\alpha]-2+l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-3+\alpha-[\beta]+l+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-3+\alpha-[\beta]+l+\varepsilon}\, d\omega \quad (0<\varepsilon<[\alpha]+2-\alpha+[\beta]-l), \quad \text{by (3.7)}.$$

$$N_{10} = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\,\omega^{\alpha-[\alpha]-2+l-[\beta]}\, d\omega = N_9 .$$

$$N_{11} = \int_1^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\,\omega^{\alpha-[\alpha]-3}\, d\omega$$

$$= \int_1^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-4+\alpha+\varepsilon}\, d\omega \qquad (0<\varepsilon<[\alpha]+3-\alpha)$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-4+\alpha+\varepsilon}\, d\omega, \qquad \text{by (3.7)}.$$

$$N_{12} = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta-1} e^{(1)}(\omega) \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1} \omega^{\alpha-[\alpha]-2}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma+1}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-4+\alpha+\varepsilon}\, d\omega \qquad (0 < \varepsilon < [\alpha] + 3 - \alpha)$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-4+\alpha+\varepsilon}\, d\omega, \qquad \text{by } (3.7)\,.$$

$$N_{13} = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e^{(1)}(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot \{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1} \omega^{\alpha-[\alpha]-2+l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\beta}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-2+\alpha-\beta+l+\varepsilon}\, d\omega \quad (0 < \varepsilon < [\alpha] + 1 - \alpha + \beta - l)$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-2+\alpha-\beta+l+\varepsilon}\, d\omega, \qquad \text{by } (3.7)\,.$$

$$N_{14} = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \omega^{\alpha-[\alpha]-2+l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]+1}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-3+\alpha-[\beta]+l+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-3+\alpha-[\beta]+l+\varepsilon}\, d\omega \quad (0 < \varepsilon < [\alpha] + 2 - \alpha + [\beta] - l), \quad \text{by } (3.7)\,.$$

$$N_{15} = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}]\, e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \omega^{\alpha-[\alpha]-2+l-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\beta+[\beta]}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-2+\alpha-\beta-[\beta]+l+\varepsilon}\, d\omega$$

$$\leqslant K \int_1^\infty \omega^{-[\alpha]-2+\alpha-\beta-[\beta]+l+\varepsilon}\, d\omega \quad (0 < \varepsilon < [\alpha] + 1 - \alpha + \beta + [\beta] - l), \quad \text{by } (3.7)\,.$$

Hence the Lemma follows.

11. - Lemma 8.

For $\beta > 1$, *we shall prove that*

$$\int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}]\, |\, E(\omega, \pi)\, |\, d\omega < \infty\,.$$

Proof.

Now $E(\omega, \pi) = F(\omega, \pi, 0, 0)$. By Lemma 3, for the case $\beta > 1$, we have

$$F(\omega, \pi, 0, 0) = O[\{e(\omega)\}^{\beta-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[\{e(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma} \omega^{-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{\sigma=0}^{[\beta]} O[\{e(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma} \omega^{-[\beta]}] +$$

$$+ \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[\{e(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1} \omega^{-1}] +$$

$$+ \sum_{\sigma=0}^{[\beta]} O[\{e(\omega)\}^{\beta-1} e^{(1)}(\omega) \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{i=1}^{[\beta]} O[\{e^{(1)}(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]} \{e(\omega)/e^{(1)}(\omega)\}^{[\beta]+1} \omega^{-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} O[\{e(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]} \omega^{-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} O[e(\omega) \{e^{(1)}(\omega)\}^{\beta-1} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]} \omega^{-[\beta]}] .$$

Substituting the estimate of $E(\omega, \pi)$ in the integral

$$\int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}] \, | E(\omega, \pi) | \, d\omega ,$$

we get

$$\int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}] \, | E(\omega, \pi) | \, d\omega = O(P_1) + \sum_{j=0}^{[\beta]-1} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(P_2) + \sum_{j=0}^{[\beta]-1} \sum_{\sigma=0}^{[\beta]} O(P_3) +$$

$$+ \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O(P_4) + \sum_{\sigma=0}^{[\beta]} O(P_5) + \sum_{j=0}^{[\beta]-1} \sum_{i=1}^{[\beta]} O(P_6) + \sum_{j=0}^{[\beta]-1} O(P_7) + \sum_{j=0}^{[\beta]-1} O(P_8),$$

where the integrals P_r $(r = 1, 2, \ldots, 8)$ are defined as in the body of proofs, given below, of their convergence, which is an immediate consequence, in each case, of the inequality displayed.

$$P_1 = \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta-1}] \{e(\omega)\}^{\beta-1} d\omega \leqslant K \int_1^{\infty} \omega^{-2+\varepsilon} d\omega \; (0 < \varepsilon < 1), \text{ by } (3.7)' \text{ and } (3.2).$$

$$P_2 = \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}] \{e(\omega)\}^{\beta} \{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma} \omega^{-[\beta]} \, d\omega =$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma+1}/\omega^{\varepsilon}]\, \omega^{-[\beta]-1+\varepsilon}\, d\omega \leqslant K \int_1^\infty \omega^{-[\beta]-1+\varepsilon}\, d\omega$$

$$(0 < \varepsilon < [\beta]), \text{ by } (3.7).$$

$$P_3 = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma}\, \omega^{-[\beta]}\, d\omega = P_2 .$$

$$P_4 = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1}\, \omega^{-1}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma}/\omega^{\varepsilon}]\, \omega^{-2+\varepsilon}\, d\omega \leqslant K \int_1^\infty \omega^{-2+\varepsilon}\, d\omega \quad (0 < \varepsilon < 1), \text{ by } (3.7).$$

$$P_5 = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta-1} e^{(1)}(\omega) \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma+1}/\omega^{\varepsilon}]\, \omega^{-2+\varepsilon}\, d\omega \leqslant K \int_1^\infty \omega^{-2+\varepsilon}\, d\omega \quad (0 < \varepsilon < 1), \text{ by } (3.7).$$

$$P_6 = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e^{(1)}(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]}$$

$$\{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1}\, \omega^{-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{ \omega e^{(1)}(\omega)/e(\omega) \}^{\beta}/\omega^{\varepsilon}]\, \omega^{-\beta+\varepsilon}\, d\omega \leqslant K \int_1^\infty \omega^{-\beta+\varepsilon}\, d\omega\, (0 < \varepsilon < \beta - 1), \text{ by } (3.7).$$

$$P_7 = \int_1^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]}\, \omega^{-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]+1}/\omega^{\varepsilon}]\, \omega^{-[\beta]-1+\varepsilon}\, d\omega \leqslant K \int_1^\infty \omega^{-[\beta]-1+\varepsilon}\, d\omega$$

$$(0 < \varepsilon < [\beta]), \text{ by } (3.7) .$$

$$P_8 = \int_1^\infty [e^{(1)}\omega)/\{ e(\omega) \}^{\beta+1}]\, e(\omega) \{ e^{(1)}\omega \}^{\beta-1} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]}\, \omega^{-[\beta]}\, d\omega$$

$$= \int_1^\infty [\{ \omega e^{(1)}(\omega)/e(\omega) \}^{\beta+[\beta]}/\omega^{\varepsilon}]\, \omega^{-\beta+\varepsilon}\, d\omega \leqslant K \int_1^\infty \omega^{-\beta+\varepsilon}\, d\omega$$

$$(0 < \varepsilon < \beta - 1), \text{ by } (3.7) .$$

Hence the Lemma is proved.

12. - Lemma 9.

The Fourier series of the special even function $\Omega^{([\alpha])}(t)$ *is absolutely convergent at* $t = 0$.

Proof.

Let

$$\Omega^{([\alpha])}(t) \sim \sum \alpha_n \cos nt\,.$$

Then

$$\alpha_n = \frac{2}{\pi}\int_0^\pi \Omega^{([\alpha])}(t) \cos nt \; dt =$$

$$= \frac{2}{\pi}\int_0^\pi \{\text{a linear combination of functions } H_i(t) \;\; (i = 1,\, 2,\, 3,\, \ldots)\} \cos nt \, dt,$$

by (3.8) (v). We have

$$\frac{2}{\pi}\int_0^\pi H_i(t) \cos nt \; dt = -\frac{2}{n\pi}\left(\int_0^{1/n} + \int_{1/n}^\pi\right) H_i^{(1)}(t) \sin nt \; dt = -\frac{2}{n\pi}(I_1 + I_2), \quad \text{say}\,.$$

Now, by hypothesis (3.8) (v) (a), we have

$$I_1 = \int_0^{1/n} \frac{\sin nt}{t}\, t\, H_i^{(1)}(t) \; dt = \frac{1}{n}\, H_i^{(1)}\left(\frac{1}{n}\right) \int_\varrho^{1/n} \frac{\sin nt}{t} \; dt = O\left[\frac{1}{n} \left| H_i^{(1)}\left(\frac{1}{n}\right)\right|\right] \quad \left(0 \leqslant \varrho \leqslant \frac{1}{n}\right).$$

And, by the same hypothesis, we have

$$I_2 = H_i^{(1)}\left(\frac{1}{n}\right) \int_{1/n}^{\eta} \sin nt \; dt = O\left[\frac{1}{n}\, H_i^{(1)}\left(\frac{1}{n}\right)\right] \qquad \left(\frac{1}{n} \leqslant \eta \leqslant \pi\right).$$

Thus

$$\alpha_n = \sum_i O\left[\frac{1}{n^2}\left| H_i^{(1)}\left(\frac{1}{n}\right)\right|\right].$$

Hence the Lemma follows by virtue of the hypothesis (3.8) (v) (b).

13. - Lemma 10 (1).

If τ be given by $F(k/u)/u$, then we shall prove that, for $\beta < 1$ and $[\alpha] + 1 \geqslant \beta > 1$,

$$[u^\alpha / \{F(k/u)\}^\alpha] \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\, |\, g(\omega,\, u)\, |\; d\omega = O(1)\,.$$

---

(1) This restriction on β for $\beta > 1$ under the overall assumption: $\beta > \alpha$ does not impair the generality of the *Theorem* in view of the first theorem of consistency for absolute RIESZ summability.

Proof.

When $\beta < 1$, and hence $[\beta] = 0$ and $[\alpha] = 0$, by Lemma 6 and Lemma 3, we have

$$g(\omega, u) = O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-1}/u^{\beta}] + O[\{e^{(1)}(\omega)\}^{\beta}\omega^{\alpha-1}/u^{\beta+1}] +$$

$$+ O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-2}/u^{\beta+1}] + \sum_{l=0}^{1} O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha+l-2}/u^{\beta-l+1}] +$$

$$+ \sum_{l=0}^{1} O[\{e^{(1)}(\omega)\}^{\beta}\omega^{\alpha+l-2}/u^{\beta-l+2}] + \sum_{l=0}^{1} O[e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha+l-3}/u^{\beta-l+2}]$$

and when $\beta > 1$, by Lemma 6 and Lemma 3, we have

$$g(\omega, u) = O[(u^{-[\beta]-[\alpha]} + u^{-[\alpha]-2})\{e(\omega)\}^{\beta-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[u^{-[\beta]-[\alpha]+l-1}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-1+l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{\sigma=0}^{[\beta]} O[u^{-[\beta]-[\alpha]+l-1}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-1+l-[\beta]}] +$$

$$+ \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[u^{-[\alpha]-2}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-2}] +$$

$$+ \sum_{\sigma=0}^{[\beta]} O[u^{-[\alpha]-2}\{e(\omega)\}^{\beta-1} e^{(1)}(\omega)\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{i=1}^{[\beta]} O[u^{-\beta-[\alpha]+l}\{e^{(1)}(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\{e(\omega)/e^{(1)}(\omega)\}^{[\beta]+1} \cdot$$

$$\cdot\, \omega^{\alpha-[\alpha]-1+l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} O[u^{-[\beta]-[\alpha]+l-1}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{\alpha-[\alpha]-1+l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} O[u^{-[\beta]-[\alpha]+l-1} e(\omega)\{e^{(1)}(\omega)/u\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{\alpha-[\alpha]-1+l-[\beta]}] +$$

$$+ O[(u^{-[\beta]-[\alpha]-1} + u^{-[\alpha]-3})\{e(\omega)\}^{\beta-1}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O[u^{-[\beta]-[\alpha]+l-2}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-2+l-[\beta]}] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]+1} \sum_{\sigma=0}^{[\beta]} O[u^{-[\beta]-[\alpha]+l-2}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma}\omega^{\alpha-[\alpha]-2+l-[\beta]}] +$$

$$+\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} O[u^{-[\alpha]-3}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-3}]+$$

$$+\sum_{\sigma=0}^{[\beta]} O[u^{-[\alpha]-3}\{e(\omega)\}^{\beta-1} e^{(1)}(\omega)\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\omega^{\alpha-[\alpha]-2}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{i=1}^{[\beta]} O[u^{-\beta-[\alpha]+l-1}\{e^{(1)}(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\{e(\omega)/e^{(1)}(\omega)\}^{[\beta]+1}\cdot$$
$$\cdot\,\omega^{\alpha-[\alpha]+l-[\beta]-2}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1} O[u^{-[\beta]-[\alpha]+l-2}\{e(\omega)\}^{\beta}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{\alpha-[\alpha]+l-[\beta]-2}]+$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1} O[u^{-[\beta]-[\alpha]+l-2} e(\omega)\{e^{(1)}(\omega)/u\}^{\beta-1}\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{[\beta]}\omega^{\alpha-[\alpha]+l-[\beta]-2}].$$

Substituting the estimate of $g(\omega,\ u)$ for the case $[\beta]=0$ in the integral

$$\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|\,g(\omega,\ u)\,|\,d\omega\,,$$

we have

$$[u^{\alpha}/\{F(k/u)\}^{\alpha}]\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|\,g(\omega,\ u)\,|\,d\omega=$$
$$=O(Q_1)+O(Q_2)+O(Q_3)+\sum_{l=0}^{1}O(Q_4)+\sum_{l=0}^{1}O(Q_5)+\sum_{l=0}^{1}O(Q_6)\,,$$

where the integrals Q_r $(r=1,\ 2,\ \dots,\ 6)$ are defined as in the body of proofs, given below, of their boundedness, which is an immediate consequence in each case, of the inequality displayed.

$$Q_1=\omega^{-\beta}\,\tau^{-\alpha}\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\omega^{\alpha-1}\,d\omega$$

$$=\tau^{-\alpha}\,u^{-\beta}\int_{\tau}^{\infty}[\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\,u^{-\beta+\alpha-1+\varepsilon}\,d\omega\qquad(0<\varepsilon<\beta-\alpha)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-\beta}\,[\{\tau\, e^{(1)}(\tau)/e(\tau)\}^{\beta}/\tau^{\varepsilon}]\,\tau^{-\beta+\alpha+\varepsilon},\qquad \text{by (3.7)}\,,$$

$$=K\,[e^{(1)}(\tau)/\{u\, e(\tau)\}]^{\beta}\leqslant K,\qquad \text{by (3.8) (i) (1)}.$$

---

(1) We observe that

$$e^{(1)}(\tau)/\{u\, e(\tau)\}=F(\tau)/(u\,\tau)=F\,\{F(k/u)/u\}\,/F(k/u)=O(1),$$

by (3.8) (i).

$$Q_2 = u^{-\beta-1}\,\tau^{-\alpha}\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e^{(1)}(\omega)\}^{\beta}\,\omega^{\alpha-1}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-\beta-1}\int_{\tau}^{\infty}[\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta+1}/\omega^{\varepsilon}]\,\omega^{-\beta+\alpha-2+\varepsilon}\,d\omega \quad (0<\varepsilon<\beta-\alpha+1)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-\beta-1}\,[\{\tau\, e^{(1)}(\tau)/e(\tau)\}^{\beta+1}/\tau^{\varepsilon}]\,\tau^{-\beta+\alpha-1+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\,[e^{(1)}(\tau)/\{u\, e(\tau)\}]^{\beta+1} \leqslant K\,.$$

$$Q_3 = u^{-\beta-1}\,\tau^{-\alpha}\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\,\omega^{\alpha-2}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-\beta-1}\int_{\tau}^{\infty}[\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\,\omega^{-\beta+\alpha-2+\varepsilon}d\omega \qquad (0<\varepsilon<\beta-\alpha+1)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-\beta-1}\,[\{\tau\, e^{(1)}(\tau)/e(\tau)\}^{\beta}/\tau^{\varepsilon}]\,\tau^{-\beta+\alpha-1+\varepsilon}\,, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{-1}\,[e^{(1)}(\tau)/\{u\, e(\tau)\}]^{\beta} \leqslant K, \qquad \text{by (3.8) (i) (}^{1}\text{)}.$$

In Q_4, Q_5, Q_6, $l = 0, 1$.

$$Q_4 = u^{-\beta+l-1}\,\tau^{-\alpha}\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\,\omega^{\alpha+l-2}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-\beta+l-1}\int_{\tau}^{\infty}[\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\,\omega^{-\beta+\alpha+l-2+\varepsilon}\,d\omega \quad (0<\varepsilon<\beta-\alpha-l+1)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-\beta+l-1}\,[\{\tau\, e^{(1)}(\tau)/e(\tau)\}^{\beta}/\tau^{\varepsilon}]\,\tau^{-\beta+\alpha+l-1+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{l-1}\,[e^{(1)}(\tau)/\{u\, e(\tau)\}]^{\beta} \leqslant K\,, \qquad \text{by (3.8) (i)}\,.$$

$$Q_5 = u^{-\beta+l-2}\,\tau^{-\alpha}\int_{\tau}^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e^{(1)}(\omega)\}^{\beta}\,\omega^{\alpha+l-2}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-\beta+l-2}\int_{\tau}^{\infty}[\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta+1}/\omega^{\varepsilon}]\,\omega^{-\beta+\alpha+l-3+\varepsilon}d\omega \quad (0<\varepsilon<\beta-\alpha-l+2)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-\beta+l-2}\,[\{\tau\, e^{(1)}(\tau)/e(\tau)\}^{\beta+1}/\tau^{\varepsilon}]\,\tau^{-\beta+\alpha+l-2+\varepsilon}\,, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{l-1}\,[e^{(1)}(\tau)/\{u\, e(\tau)\}]^{\beta+1} \leqslant K\,, \qquad \text{by (3.8) (i)}.$$

---

(1) We observe that $u\,\tau = F(k/u)$, which is monotonic increasing, as u diminishes, by (3.3).

$$Q_6 = u^{-\beta+l-2}\,\tau^{-\alpha}\int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]e(\omega)\{e^{(1)}(\omega)\}^{\beta-1}\,\omega^{\alpha+l-3}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-\beta+l-2}\int_\tau^\infty [\{\omega\, e^{(1)}(\omega)/e(\omega)\}^{\beta}/\omega^{\varepsilon}]\,\omega^{-\beta+\alpha+l-3+\varepsilon}\,d\omega \qquad (0<\varepsilon<\beta-\alpha-l+2)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-\beta+l-2}\,[\{\tau\, e^{(1)}(\tau)/e(\tau)\}^{\beta}/\tau^{\varepsilon}]\,\tau^{-\beta+\alpha+l-2+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{l-2}\,[e^{(1)}(\tau)/\{u\, e(\tau)\}]^{\beta} \leqslant K, \qquad \text{by (3.8) (i)}.$$

Hence the Lemma is proved for $\beta < 1$.

Substituting the estimate of $g(\omega, u)$ for the case $\beta > 1$ in the integral

$$\int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|\,g(\omega, u)\,|\,d\omega,$$

we have

$$[u^{\alpha}/\{F(k/u)\}^{\alpha}]\int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|\,g(\omega, u)\,|\,d\omega = O(R_1) +$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} O(R_2) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}\sum_{\sigma=1}^{[\beta]} O(R_3) + \sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} O(R_4) + \sum_{\sigma=0}^{[\beta]} O(R_5) +$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]}\sum_{i=1}^{[\beta]} O(R_6) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]} O(R_7) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]} O(R_8) + O(R_9) +$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} O(R_{10}) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{\sigma=0}^{[\beta]} O(R_{11}) + \sum_{\sigma=1}^{[\beta]}\sum_{i=1}^{\sigma} O(R_{12}) + \sum_{\sigma=0}^{[\beta]} O(R_{13}) +$$

$$+\sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1}\sum_{i=1}^{[\beta]} O(R_{14}) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1} O(R_{15}) + \sum_{j=0}^{[\beta]-1}\sum_{l=0}^{[\alpha]+1} O(R_{16}),$$

where the integrals R_r ($r = 1, 2, \ldots, 16$) are defined as in the body of proofs, given below, of their boundedness, which is an immediate consequence, in each case, of the inequality displayed.

$$R_1 = \tau^{-\alpha}\,(u^{-[\beta]-[\alpha]} + u^{-[\alpha]-2})\int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-[\beta]-[\alpha]}\int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{2}]\,d\omega + \tau^{-\alpha}\,u^{-[\alpha]-2}\int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{2}]\,d\omega$$

$$= R_1' + R_1''; \qquad \text{say}.$$

Now

$$R_1' = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]} \int_\tau^\infty [e^{(1)}(\omega)/\{\, e(\omega)\,\}^2]\, d\omega = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]} \{\, e(\tau)\,\}^{-1} \leqslant K,$$

by (3.8) (ii) ([1]),

$$R_1'' = \tau^{-\alpha}\, u^{-[\alpha]-2} \int_\tau^\infty (e^{(1)}(\omega)/\{\, e(\omega)\,\}^2)\, d\omega = \tau^{-\alpha}\, u^{-[\alpha]-2} \{\, e(\tau)\,\}^{-1} \leqslant K,$$

by (3.8) (ii).

In $R_2, \ldots, R_8$, $l = 0, 1, \ldots, [\alpha]$, wherever it occurs.

$$R_2 = u^{-[\beta]-[\alpha]+l-1}\, \tau^{-\alpha} \int_\tau^\infty [e^{(1)}(\omega)/\{\, e(\omega)\,\}^{\beta+1}]\{\, e(\omega)\,\}^{\beta}\{\, \omega\, e^{(1)}(\omega)/e(\omega)\,\}^{\sigma} \cdot \omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$= \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-1} \int_\tau^\infty [\{\, \omega\, e^{(1)}(\omega)/e(\omega)\,\}^{\sigma+1}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-2+\alpha-[\beta]+l+\varepsilon}\, d\omega$$

$$(0 < \varepsilon < [\alpha] + 1 - \alpha + [\beta] - l)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-1}\, [\{\, \tau\, e^{(1)}(\tau)/e(\tau)\,\}^{\sigma+1}/\tau^{\varepsilon}]\, \tau^{-[\alpha]-1+\alpha-[\beta]+l+\varepsilon}, \quad \text{by (3.7)},$$

$$= K\, [e^{(1)}(\tau)/\{\, u\, e(\tau)\,\}]^{\sigma+1}\, (u\tau)^{-[\beta]-[\alpha]+l+\sigma} \leqslant K, \qquad \text{as } Q_3 .$$

$$R_3 = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-1} \int_\tau^\infty [e^{(1)}(\omega)/\{\, e(\omega)\,\}^{\beta+1}]\{\, e(\omega)\,\}^{\beta}\{\, \omega\, e^{(1)}(\omega)/e(\omega)\,\}^{\sigma} \cdot \omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$\leqslant K, \qquad \text{as } R_2 .$$

$$R_4 = \tau^{-\alpha}\, u^{-[\alpha]-2} \int_\tau^\infty [e^{(1)}(\omega)/\{\, e(\omega)\,\}^{\beta+1}]\{\, e(\omega)\,\}^{\beta}\{\, \omega\, e^{(1)}(\omega)/e(\omega)\,\}^{\sigma-1}\, \omega^{\alpha-[\alpha]-2}\, d\omega$$

$$= \tau^{-\alpha}\, u^{-[\alpha]-2} \int_\tau^\infty [\{\, \omega\, e^{(1)}(\omega)/e(\omega)\,\}^{\sigma}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-3+\alpha+\varepsilon}\, d\omega \quad (0 < \varepsilon < [\alpha] + 2 - \alpha)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\alpha]-2}\, [\{\, \tau\, e^{(1)}(\tau)/e(\tau)\,\}^{\sigma}/\tau^{\varepsilon}]\, \tau^{-[\alpha]-2+\alpha+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]-2+\sigma}\, [e^{(1)}(\tau)/\{\, u\, e(\tau)\,\}]^{\sigma} .$$

([1]) We observe that

$$\tau^{-\alpha}\, u^{-[\beta]-[\alpha]}\, \{e(\tau)\}^{-1} = \{F(k/u)/u\}^{-\alpha}\, [e\, \{F(k/u)/u\}]^{-1}\, u^{-[\alpha]-[\beta]} =$$

$$= \{F(k/u)\}^{-\alpha}\, [e\, \{F(k/u)/u\}]^{-1}\, u^{-[\alpha]-[\beta]+\alpha} = O(1), \qquad \text{by (3.8) (ii).}$$

$$R_5 = \tau^{-\alpha}\, u^{-[\alpha]-2} \int_\tau^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}]\{ e(\omega) \}^{\beta-1} e^{(1)}(\omega) \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma-1} \cdot$$

$$\cdot\, \omega^{\alpha-[\alpha]-1}\, d\omega$$

$$= \tau^{-\alpha}\, u^{-[\alpha]-2} \int_\tau^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\sigma+1}/\omega^\varepsilon]\, \omega^{-[\alpha]-3+\alpha+\varepsilon}\, d\omega \quad (0<\varepsilon<[\alpha]+2-\alpha)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\alpha]-2}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{\sigma+1}/\tau^\varepsilon]\, \tau^{-[\alpha]-2+\alpha+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]-1+\sigma}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{\sigma+1}.$$

$$R_6 = \tau^{-\alpha}\, u^{-\beta-[\alpha]+l} \int_\tau^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}]\{ e^{(1)}(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot \{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1}\, \omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$= \tau^{-\alpha}\, u^{-\beta-[\alpha]+l} \int_\tau^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\beta}/\omega^\varepsilon]\, \omega^{-\beta-[\alpha]+\alpha+l-1+\varepsilon}\, d\omega$$

$$(0<\varepsilon<\beta+[\alpha]-\alpha-l)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-\beta-[\alpha]+l}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{\beta}/\tau^\varepsilon]\, \tau^{-\beta-[\alpha]+\alpha+l+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]+l}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{\beta}.$$

$$R_7 = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-1} \int_\tau^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}]\{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot\, \omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$= \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-1} \int_\tau^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]+1}/\omega^\varepsilon]\, \omega^{-[\alpha]-2+\alpha-[\beta]+l+\varepsilon}\, d\omega$$

$$(0<\varepsilon<[\alpha]+1-\alpha+[\beta]-l)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-1}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{[\beta]+1}/\tau^\varepsilon]\, \tau^{-[\alpha]-1+\alpha-[\beta]+l+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]+l}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{[\beta]+1}.$$

$$R_8 = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-\beta} \int_\tau^\infty [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}]\, e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot\, \omega^{\alpha-[\alpha]-1+l-[\beta]}\, d\omega$$

$$= \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-\beta} \int_\tau^\infty [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\beta+[\beta]}/\omega^\varepsilon]\, \omega^{-[\alpha]-1+\alpha-\beta-[\beta]+l+\varepsilon}\, d\omega$$

$$(0<\varepsilon<[\alpha]-\alpha+\beta+[\beta]-l)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-\beta}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{\beta+[\beta]}/\tau^\varepsilon]\, \tau^{-\beta-[\beta]-[\alpha]+\alpha+l+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]+l}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{\beta+[\beta]}.$$

$$R_9 = \tau^{-\alpha}\,(u^{-[\beta]-[\alpha]-1} + u^{-[\alpha]-3}) \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1}\,d\omega = R_9' + R_9'', \text{ say.}$$

Now,

$$R_9' = \tau^{-\alpha}\,u^{-[\beta]-[\alpha]-1} \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^2]\,d\omega = \tau^{-\alpha}u^{-[\beta]-[\alpha]-1}\{e(\tau)\}^{-1} \leqslant K, \text{ by (3.8) (ii)},$$

$$R_9'' = \tau^{-\alpha}\,u^{-[\alpha]-3} \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^2]\,d\omega = \tau^{-\alpha}\,u^{-[\alpha]-3}\,e(\tau) \leqslant K, \quad \text{by (3.8) (ii)}.$$

In R_{10}, ..., R_{16}, $l = 0, 1, ..., [\alpha] + 1$, wherever it occurs.

$$R_{10} = \tau^{-\alpha}\,u^{-[\beta]-[\alpha]+l-2} \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma} \cdot \cdot\, \omega^{\alpha-[\alpha]-2+l-[\beta]}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-[\beta]-[\alpha]+l-2} \int_\tau^\infty [\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-3+\alpha+l-[\beta]+\varepsilon}\,d\omega$$

$$(0 < \varepsilon < [\alpha] + 2 - \alpha - l + [\beta])$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-[\beta]-[\alpha]+l-2}\,[\{\tau\,e^{(1)}(\tau)/e(\tau)\}^{\sigma+1}/\tau^{\varepsilon}]\,\tau^{-[\alpha]-2+\alpha+l-[\beta]+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{-[\beta]-[\alpha]+l+\sigma-1}\,[e^{(1)}(\tau)/\{\tau\,e(\tau)\}]^{\sigma+1}.$$

$$R_{11} = \tau^{-\alpha}\,u^{-[\beta]-[\alpha]+l-2} \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma} \cdot \cdot\,\omega^{\alpha-[\alpha]-2+l-[\beta]}\,d\omega$$

$$\leqslant K, \qquad \text{as } R_{10}.$$

$$R_{12} = \tau^{-\alpha}\,u^{-[\alpha]-3} \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta}\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma-1}\,\omega^{\alpha-[\alpha]-3}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-[\alpha]-3} \int_\tau^\infty [\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-4+\alpha+\varepsilon}\,d\omega \quad (0 < \varepsilon < [\alpha] + 3 - \alpha)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-[\alpha]-3}\,[\{\tau\,e^{(1)}(\tau)/e(\tau)\}^{\sigma}/\tau^{\varepsilon}]\,\tau^{-[\alpha]-3+\alpha+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{-[\alpha]-3+\sigma}\,[e^{(1)}(\tau)/\{u\,e(\tau)\}]^{\sigma}.$$

$$R_{13} = \tau^{-\alpha}\,u^{-[\alpha]-3} \int_\tau^\infty [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\{e(\omega)\}^{\beta-1}\,e^{(1)}(\omega)\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma-1} \cdot \cdot\,\omega^{\alpha-[\alpha]-2}\,d\omega$$

$$= \tau^{-\alpha}\,u^{-[\alpha]-3} \int_\tau^\infty [\{\omega\,e^{(1)}(\omega)/e(\omega)\}^{\sigma+1}/\omega^{\varepsilon}]\,\omega^{-[\alpha]-4+\alpha+\varepsilon}\,d\omega \quad (0 < \varepsilon < [\alpha] + 3 - \alpha)$$

$$\leqslant K\,\tau^{-\alpha}\,u^{-[\alpha]-3}\,[\{\tau\,e^{(1)}(\tau)/e(\tau)\}^{\sigma+1}/\tau^{\varepsilon}]\,\tau^{-[\alpha]-3+\alpha+\varepsilon}, \qquad \text{by (3.7)},$$

$$= K\,(u\tau)^{-[\alpha]-2+\sigma}\,[e^{(1)}(\tau)/\{u\,e(\tau)\}]^{\sigma+1}.$$

$$R_{14} = \tau^{-\alpha}\, u^{-\beta-[\alpha]+l-1} \int_{\tau}^{\infty} [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e^{(1)}(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot \{ e(\omega)/e^{(1)}(\omega) \}^{[\beta]+1}\, \omega^{\alpha-[\alpha]-2+l-[\beta]}\, \mathrm{d}\omega$$

$$= \tau^{-\alpha}\, u^{-\beta-[\alpha]+l-1} \int_{\tau}^{\infty} [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\beta}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-2+\alpha+l-\beta+\varepsilon}\, \mathrm{d}\omega$$

$$(0 < \varepsilon < [\alpha] + 1 - \alpha - l + \beta)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-\beta-[\alpha]+l-1}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{\beta}/\tau^{\varepsilon}]\, \tau^{-[\alpha]-1+\alpha+l-\beta+\varepsilon}, \quad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]-1+l}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{\beta}.$$

$$R_{15} = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-2} \int_{\tau}^{\infty} [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \{ e(\omega) \}^{\beta} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot\, \omega^{\alpha-[\alpha]-2+l-[\beta]}\, \mathrm{d}\omega$$

$$= \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-2} \int_{\tau}^{\infty} [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]+1}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-3+\alpha+l+\varepsilon}\, \mathrm{d}\omega$$

$$(0 < \varepsilon < [\alpha] + 2 - \alpha - l)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-2}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{[\beta]+1}/\tau^{\varepsilon}]\, \tau^{-[\alpha]-2+\alpha+l+\varepsilon}, \quad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]-1+l}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{[\beta]+1}.$$

$$R_{16} = \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-\beta-1} \int_{\tau}^{\infty} [e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] e(\omega) \{ e^{(1)}(\omega) \}^{\beta-1} \{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{[\beta]} \cdot$$

$$\cdot\, \omega^{\alpha-[\alpha]-2+l-[\beta]}\, \mathrm{d}\omega$$

$$= \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-\beta-1} \int_{\tau}^{\infty} [\{ \omega\, e^{(1)}(\omega)/e(\omega) \}^{\beta+[\beta]}/\omega^{\varepsilon}]\, \omega^{-[\alpha]-2-\beta-[\beta]+l+\alpha+\varepsilon}\, \mathrm{d}\omega$$

$$(0 < \varepsilon < [\alpha] + 1 + \beta + [\beta] - l - \alpha)$$

$$\leqslant K\, \tau^{-\alpha}\, u^{-[\beta]-[\alpha]+l-\beta-1}\, [\{ \tau\, e^{(1)}(\tau)/e(\tau) \}^{\beta+[\beta]}/\tau^{\varepsilon}]\, \tau^{-[\alpha]-1-\beta-[\beta]+l+\alpha+\varepsilon}, \quad \text{by (3.7)},$$

$$= K\, (u\tau)^{-[\alpha]-1+l}\, [e^{(1)}(\tau)/\{ u\, e(\tau) \}]^{\beta+[\beta]}.$$

Hence the Lemma is completely proved.

14. - Proof of the Theorem (cf. n. **3**).

In view of Lemma 10 (cf. n. 13) and the first theorem for consistency for absolute RIESZ summability, we take, without any loss of generality: $\alpha < \beta \leqslant \leqslant [\alpha] + 1$, when $\beta > 1$.

We have

$$A_n(x) = (2/\pi) \int_0^\pi \varphi(t) \cos nt \, \mathrm{d}t \, .$$

Now we have to show that, under hypotheses of the theorem, the integral

$$I \equiv \int_1^\infty \left| \, [\beta \, e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \sum_{n<\omega} \{ e(\omega) - e(n) \}^{\beta-1} e(n) \, A_n(x) \, \right| \mathrm{d}\omega$$

is convergent. We have

$$I = \int_1^\infty [\beta \, e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \, \left| \sum_{n<\omega} \{ e(\omega) - e(n) \}^{\beta-1} e(n) \, A_n(x) \right| \mathrm{d}\omega$$

$$= (2/\pi) \int_1^\infty [\beta \, e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \, \left| \sum_{n<\omega} \{ e(\omega) - e(n) \}^{\beta-1} e(n) \int_0^\infty \varphi(t) \cos nt \, \mathrm{d}t \right| \mathrm{d}\omega \, ,$$

whence

$$I = (2/\pi) \int_1^\infty [\beta \, e^{(1)}(\omega)/\{ e(\omega) \}^{\beta+1}] \, \left| \int_0^\pi \varphi(t) \, E(\omega, \, t) \, \mathrm{d}t \right| \mathrm{d}\omega \, . \tag{14.1}$$

Now

$$\int_0^\pi \varphi(t) \, E(\omega, t) \, \mathrm{d}t = \Big[\sum_{\varrho=1}^{[\alpha]} (-1)^{\varrho-1} \, \Phi_\varrho(t) \, E^{(\varrho-1)}(\omega, t)\Big]_0^\pi + (-1)^{[\alpha]} \int_0^\pi \Phi_{[\alpha]}(t) \, E^{([\alpha])}(\omega, t) \, \mathrm{d}t.$$

Also

$$\int_0^\pi \Phi_{[\alpha]}(t) \, E^{([\alpha])}(\omega, \, t) \, \mathrm{d}t = \{ 1/\Gamma(1 + [\alpha] - \alpha) \} \int_0^\pi E^{([\alpha])}(\omega, \, t) \left\{ \int_0^t (t - u)^{[\alpha]-\alpha} \, \mathrm{d}\Phi_\alpha(u) \right\} \mathrm{d}t$$

$$= \{ 1/\Gamma(1+[\alpha] - \alpha) \} \int_0^\pi \mathrm{d}\Phi_\alpha(u) \int_u^\pi (t - u)^{[\alpha]-\alpha} \, E^{([\alpha])}(\omega, \, t) \, \mathrm{d}t = \int_0^\pi g(\omega, \, u) \, \mathrm{d}\Phi_\alpha(u) \, ,$$

whence

$$\int_0^\pi \Phi_{[\alpha]}(t) \, E^{([\alpha])}(\omega, \, t) \, \mathrm{d}t = [\Phi_\alpha(u) \, g(\omega, \, u)]_0^\pi - \int_0^\pi \Phi_\alpha(u) \frac{\mathrm{d}}{\mathrm{d}u} g(\omega, \, u) \, \mathrm{d}u \, . \tag{14.2}$$

Further

$$\Gamma(1 + \alpha) \int_0^\pi \Phi_\alpha(u) \frac{\mathrm{d}}{\mathrm{d}u} g(\omega, \, u) \, \mathrm{d}u = \int_0^\pi \varphi_\alpha(u) \, \{ F(k/u) \}^\alpha \, \frac{u^\alpha}{\{F(k/u)\}^\alpha} \, \frac{\mathrm{d}}{\mathrm{d}u} g(\omega, \, u) \, \mathrm{d}u$$

whence

$$(14.3)\qquad \Gamma(1+\alpha)\int_0^{\pi} \Phi_{\alpha}(u)\ \frac{\mathrm{d}}{\mathrm{d}u}\, g(\omega,\ u)\,\mathrm{d}u =$$

$$= [\varphi_{\alpha}(u)\{F(k/u)\}^{\alpha}\, G(\omega,\ u)]_0^{\pi} - \int_0^{\pi} \mathrm{d}[\varphi_{\alpha}(u)\{F(k/u)\}^{\alpha}]\, G(\omega,\ u)\,.$$

From (14.1), (14.2) and (14.3) it will suffice for the proof of our theorem to show that:

$$(14.4)\quad \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|E^{(\varrho)}(\omega,\pi)|\,\mathrm{d}\omega < \infty \quad (0 \leqslant \varrho \leqslant [\alpha]-1,\ \text{when } [\alpha] \geqslant 1),$$

$$(14.5)\qquad \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|g(\omega,\ \pi)|\,\mathrm{d}\omega < \infty\,,$$

$$(14.6)\qquad \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|G(\omega,\ \pi)|\,\mathrm{d}\omega < \infty\,,$$

and finally

$$(14.7)\qquad \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|G(\omega,\ u)|\,\mathrm{d}\omega = O(1) \qquad (0 < u < \pi)\,,$$

since by hypothesis $\varphi_{\alpha}(t)\{F(k/t)\}^{\alpha} \in \mathrm{BV}\,(0,\ \pi)$.

P r o o f of (14.4). When $\beta > [\alpha] \geqslant 1$ and $[\alpha]-1 \geqslant \varrho$, we have to show that

$$\int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|E^{(\varrho)}(\omega,\ \pi)|\,\mathrm{d}\omega < \infty\,.$$

The proof immediately follows by an appeal to Lemma 5.

P r o o f of (14.5). We have to show for $\beta > 1$ or $\beta < 1$,

$$\int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|g(\omega,\ \pi)|\,\mathrm{d}\omega < \infty\,.$$

The above has been proved in Lemma 7.

P r o o f of (14.6). We have to show that

$$(14.6)_1\qquad \int_1^{\infty} [e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\,|G(\omega,\ \pi)|\,\mathrm{d}\omega < \infty\,.$$

Now, with the notation $F(x) \equiv x\, e^{(1)}(x)/e(x)$, $\Lambda(v) \equiv (\mathrm{d}/\mathrm{d}v)[\{ v/F(k/v)\}^{\alpha}]$, and $\Omega(t) = t^{[\alpha]-\alpha+1}\Lambda(t)$, we have

$$(14.6)_2 \qquad G(\omega,\ \pi) = \int_0^{\pi} [v/F(k/v)]^{\alpha} \frac{\mathrm{d}}{\mathrm{d}v}\, g(\omega,\ v)\ \mathrm{d}v =$$

$$= [\{ v/F(k/v)\}^{\alpha} g(\omega,\ v)]_0^{\pi} - \int_0^{\pi} g(\omega,\ v)\,\Lambda(v)\ \mathrm{d}v = K\, g(\omega,\ \pi) - \int_0^{\pi} g(\omega,\ v)\,\Lambda(v)\ \mathrm{d}v,$$

$$(14.6)_3 \qquad \int_0^{\pi} g(\omega,v)\,\Lambda(v)\ \mathrm{d}v = \frac{1}{\Gamma(1+[\alpha]-\alpha)} \int_0^{\pi} \Lambda(v)\Big\{\int_v^{\pi}(t-v)^{[\alpha]-\alpha} E^{([\alpha])}(\omega,\ t)\ \mathrm{d}t\Big\}\,\mathrm{d}v =$$

$$= K\int_0^{\pi} E^{([\alpha])}(\omega,\ t)\ t^{[\alpha]-\alpha+1}\Big[\int_0^{1} s^{\alpha-1}(1-s)^{[\alpha]-\alpha}\{\Lambda(ts)/s^{\alpha-1}\}\,\mathrm{d}s\Big]\,\mathrm{d}t =$$

$$= K\int_0^{\pi} E^{([\alpha])}(\omega,\ t)\ t^{[\alpha]-\alpha+1}\Lambda(t)\Big[\int_{\varrho}^{1} s^{\alpha-1}(1-s)^{[\alpha]-\alpha}\,\mathrm{d}s\Big]\,\mathrm{d}t \quad (0 \leqslant \varrho \leqslant 1),\ \text{by (3.8) (iii)},$$

$$= K\int_0^{\pi} E^{([\alpha])}(\omega,\ t)\ t^{[\alpha]-\alpha+1}\Lambda(t)\ \mathrm{d}t = K\int_0^{\pi} E^{([\alpha])}(\omega,\ t)\,\Omega(t)\ \mathrm{d}t$$

$$= K\,[\sum_{\varrho=1}^{[\alpha]}(-1)^{\varrho-1} E^{([\alpha]-\varrho)}(\omega,\ t)\,\Omega^{(\varrho-1)}(t)]_0^{\pi} + K\,(-1)^{[\alpha]}\int_0^{\pi} E(\omega,\ t)\ \Omega^{([\alpha])}(t)\ \mathrm{d}t$$

$$= \sum_{\varrho=1}^{[\alpha]} K E^{([\alpha]-\varrho)}(\omega,\ \pi) + K\,(-1)^{[\alpha]}\sum_{n<\omega}\{e(\omega)-e(n)\}^{\beta-1} e(n)\int_0^{\pi}\Omega^{([\alpha])}(t)\cos nt\ \mathrm{d}t,$$

by (3.8) (iv).

From $(14.6)_1$, $(14.6)_2$ and $(14.6)_3$ it is clearly seen that for the proof of (14.6), we have only to show the following:

$$(14.6)_4 \qquad \int_1^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\ |\,g(\omega,\ \pi)\,|\ \mathrm{d}\omega < \infty,$$

$$(14.6)_5 \qquad \int_1^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\ |\,E^{(\varrho)}(\omega,\ \pi)\,|\ \mathrm{d}\omega < \infty \qquad \text{for } 0 \leqslant \varrho \leqslant [\alpha]-1,$$

and

$$(14.6)_6 \qquad \int_1^{\infty}[e^{(1)}(\omega)/\{e(\omega)\}^{\beta+1}]\ \Big|\sum_{n<\omega}\{e(\omega)-e(n)\}^{\beta-1} e(n)\int_0^{\pi}\Omega^{([\alpha])}(t)\cos nt\ \mathrm{d}t\,\Big|\ \mathrm{d}\omega < \infty.$$

Proof of $(14.6)_4$. This follows from Lemma 7.

Proof of $(14.6)_5$. This follows from Lemma 8 and Lemma 5.

Proof of $(14.6)_6$. By theorem of consistency for absolute RIESZ summability, the series $\sum_1^{\infty} u_n$, where u_n are the FOURIER co-efficients of the special even function $\Omega^{([\alpha])}(t)$, is summable $| \mathrm{R}, e(\omega), \beta |$, since as shown in Lemma 9, it is absolutely convergent.

Proof of (14.7). We put $\tau = F(k/u)/u$. Now

$$(14.7)_1 \quad \int_1^{\infty} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, | G(\omega, u) | \, d\omega = \int_1^{\tau} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, |G(\omega, u)| \, d\omega + $$

$$+ \int_{\tau}^{\infty} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, | G(\omega, \pi) - H(\omega, u) | \, d\omega \leqslant \int_{\tau}^{\infty} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, | G(\omega, \pi) | \, d\omega +$$

$$+ \int_1^{\tau} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, | G(\omega, u) | \, d\omega + \int_{\tau}^{\infty} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, | H(\omega, u) | \, d\omega .$$

By Lemma 6 and Lemma 4, we have

$$g(\omega, u) = O[\{ e(\omega) \}^{\beta+1} \omega^{\alpha-1}/e^{(1)}(\omega)] .$$

Now

$$G(\omega, u) = \int_0^u \frac{v^{\alpha}}{\{F(k/v)\}^{\alpha}} \frac{d}{dv} g(\omega, v) \, dv$$

$$= \left[\frac{v^{\alpha}}{\{F(k/v)\}^{\alpha}} g(\omega, v) \right]_0^u - \int_0^u \frac{d}{dv} \frac{v^{\alpha}}{\{F(k/v)\}^{\alpha}} \cdot g(\omega, v) \, dv$$

$$= O\left[\frac{u^{\alpha}}{\{F(k/u)\}^{\alpha}} \frac{\{e(\omega)\}^{\beta+1} \omega^{\alpha-1}}{e^{(1)}(\omega)} \right] ,$$

the first term in the integration by parts, being zero, since $v/F(k/v)$ vanishes at $v = 0$.

Therefore,

$$\int_1^{\tau} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \, | G(\omega, u) | \, d\omega = O\left[\int_1^{\tau} \frac{e^{(1)}(\omega)}{\{e(\omega)\}^{\beta+1}} \frac{u^{\alpha}}{\{F(k/u)\}^{\alpha}} \frac{\{e(\omega)\}^{\beta+1} \omega^{\alpha-1}}{e^{(1)}(\omega)} \, d\omega \right]$$

$$= O[\tau^{-\alpha} \int_1^{\tau} \omega^{\alpha-1} \, d\omega] = O(1) \qquad (0 < u < \pi) .$$

And

$$(14.7)_2 \qquad H(\omega, u) = \int_u^{\pi} \frac{v^{\alpha}}{\{F(k/v)\}^{\alpha}} \frac{d}{dv} g(\omega, v) \, dv = K \, g(\omega, \pi) - \frac{u^{\alpha}}{\{F(k/u)\}^{\alpha}} g(\omega, u) +$$

$+ K \int_u^{\pi} \Big[\frac{v^{\alpha-1}}{\{F(k/v)\}^{\alpha}}$ + (multiples of this obtained by taking positive monotonic diminishing functions of v as multiplying factors)$\Big] \; g(\omega, v), \, dv$.

Substituting the order-estimate of $g(\omega, v)$, we have, for the case $[\beta] = 0$,

$$\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha} g(\omega, v)\, dv = O\left[\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha} \frac{\{e^{(1)}(\omega)\}^{\beta-1} e(\omega)\, \omega^{\alpha-1}}{v^\beta}\, dv\right] +$$

$$+ O\left[\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha} \frac{\{e^{(1)}(\omega)\}^\beta\, \omega^{\alpha-1}}{v^{\beta+1}}\, dv\right] + O\left[\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha} \frac{e(\omega)\, \{e^{(1)}(\omega)\}^{\beta-1}\, \omega^{\alpha-2}}{v^{\beta+1}}\, dv\right] +$$

+ (other terms arising out of the order-estimate of $g(\omega, v)$ for the case $[\beta] = 0$ in Lemma 3).

In considering the order-estimate of each of the integrals above, we observe that, by hypothesis (3.7),

$(k/v)^\varepsilon / \{ F\,(k/v)\}^\alpha$ (ε is a conveniently chosen positive number in each case) is monotonic increasing as v diminishes, and therefore, taking the maximum value of this factor (which is obtained at $v = u$) outside the integral, each integral is less than

$$[(k/u)^\varepsilon / \{ F(k/u)\}^\alpha] \int_u^\pi v^q\, dv\,,$$

with appropriate values of q and with the corresponding factors involving ω. Integrating the last integral in each case, since the order of each such integral is determined at the lower limit u, now, we have

$$(14.7)_3 \quad \int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha}\, g(\omega, v)\, dv = \tau^{-\alpha} \cdot \{\text{terms in the order-estimate of } g(\omega, u) \text{ in Lemma 3}\}.$$

And for the case $\beta > 1$,

$$\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha}\, g(\omega, v)\, dv = O\left[\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha} \left(\frac{1}{v^{[\beta]+[\alpha]}} + \frac{1}{v^{[\alpha]+2}}\right) \{ e(\omega)\}^{\beta-1}\, dv\right] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{\sigma=1}^{[\beta]} \sum_{i=1}^{\sigma} O\left[\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/u)\}^\alpha} \frac{1}{v^{[\beta]+[\alpha]-l+1}} \{ e(\omega)\}^\beta \left\{\frac{\omega\, e^{(1)}(\omega)}{e(\omega)}\right\}^\sigma \omega^{\alpha-[\alpha]-1+l-[\beta]}\, dv\right] +$$

$$+ \sum_{j=0}^{[\beta]-1} \sum_{l=0}^{[\alpha]} \sum_{\sigma=0}^{[\beta]} O\left[\int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha} \frac{1}{v^{[\beta]+[\alpha]-l+1}} \{ e(\omega)\}^\beta \left\{\frac{\omega\, e^{(1)}(\omega)}{e\,(\omega)}\right\}^\sigma \omega^{\alpha-[\alpha]-1+l-[\beta]}\, dv\right] +$$

+ { other terms arising out of the order-estimate of $g(\omega, v)$ for the case $\beta > 1$ }.

Now following exactly the same lines of arguments as has been followed above in the case $\beta < 1$, we have

$$(14.7)_4 \quad \int_u^\pi \frac{v^{\alpha-1}}{\{F(k/v)\}^\alpha}\, g(\omega, v)\, dv = \tau^{-\alpha} \cdot \{\text{terms in the order-estimate of } g(\omega, u) \text{ in Lemma 3}\}.$$

From $(14.7)_2$, $(14.7)_3$ and $(14.7)_4$ it follows that

$$H(\omega, u) = \{\text{terms in the order-estimate of } g(\omega, \pi) \text{ in Lemma 3}\} + \tau^{-\alpha} \cdot \{\text{terms in the order-estimate of } g(\omega, u) \text{ in Lemma 3}\}.$$

Hence to prove (14.7), we need only appeal to (14.6), Lemma 7 and Lemma 10. Thus the *Theorem* is completely proved.

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