Riv. Mat. Univ. Parma (2) 6 (1965), 95-102

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On Some Dynamical Problems Arising in the Theory of Lubrication.

Part IV (**)

20. - Introduction.

The investigation is continued here, using, for ease of reference, progressive numbering of sections and consistent notation (lists of symbols are in Sect. **11** of Part II and Sect. **16** of Part III).

The matter under study is again the whirls of large amplitude of a rigid journal rotating in a bearing lubricated with an incompressible viscous fluid: here the case is considered of a heavy journal in a short bearing, with cavitating lubricant film. It is the case where, under steady conditions, the radial and transverse components of the force due to the lubricant are expressed in terms of eccentricity ratio A through formulae due to OCVIRK:

$$F_e = -\frac{R\, b^3\, \omega\, \eta}{c^2}\,\frac{A^2}{(1-A^2)^2}\,, \qquad F_n = \frac{\pi\, R\, b^3\, \omega\, \eta}{4c^2}\,\frac{A}{(1-A^2)^{3/2}}\,. \tag{20.1}$$

The steady configuration of the journal centre is therefore specified by the formulae

$$W = \frac{\pi}{4}\,\frac{R\, b^2\, \omega\, \eta}{c^2}\,\frac{A\{\,1 - A^2 + (16/\pi^2)A^2\,\}^{1/2}}{(1-A^2)^2}\,,$$

(20.2)

$$\tan \bar{\beta} = \frac{\pi}{4}\,(1-A^2)^{1/2}/A\,.$$

(*) Indirizzo: Centro Studi Calcolatrici Elettroniche del C.N.R. presso l'Università di Pisa, Pisa, Italia. — Ricevuto il 15-IV-1964.

(**) Part I is in Riv. Mat. Univ. Parma (2) **1** (1960), 1-20; Part II, ibid. (2) **4** (1963), 1-21; Part III, ibid. (2) **5** (1964), 49-60.

Here the notation is standard: for instance W is the load acting on the journal and $\bar{\beta}$ is the angle between the direction of the load (the downward vertical) and the vector $\Omega\bar{O}$, i. e. the oriented segment joining the bearing centre Ω with the steady position $\bar{O}$ of the journal centre.

Under dynamic conditions the formulae which express F_e, F_n must be much more complex, because the region of cavitation may rock in dependence with the kinematic state of the journal. Explicit formulae have been given nevertheless in an earlier paper (1); we write them down here again, because they play a decisive rôle in our developments:

$$\left\{\begin{aligned} F_e &= \frac{\eta R b^3}{2c^3}\{(\omega - 2\dot{\beta})e\, g_1 - 2\,\dot{e}\, g_2\} \\ F_n &= \frac{\eta R b^3}{2c^3}\{(\omega - 2\dot{\beta})e\, g_3 - 2\,\dot{e}\, g_1\}, \end{aligned}\right. \tag{20.3}$$

where

$$\left\{\begin{aligned} g_1 &= -\frac{2a\cos^3\alpha_1}{(1-a^2\cos^2\alpha_1)^2} \\ g_2 &= \frac{a\sin\alpha_1\cdot[3+(2-5a^2)\cos^2\alpha_1]}{(1-a^2)^2\,(1-a^2\cos^2\alpha_1)^2} + \\ &\qquad + \frac{1+2a^2}{(1-a^2)^{5/2}}\left\{\frac{\pi}{2} + \arctan\frac{a\sin\alpha_1}{(1-a^2)^{1/2}}\right\} \\ g_3 &= \frac{a\sin\alpha_1\cdot(1-2\cos^2\alpha_1 + a^2\cos^2\alpha_1)}{(1-a^2)(1-a^2\cos^2\alpha_1)^2} + \\ &\qquad + (1-a^2)^{-3/2}\left\{\frac{\pi}{2} + \arctan\frac{a\sin\alpha_1}{(1-a^2)^{1/2}}\right\}, \end{aligned}\right. \tag{20.4}$$

and α_1 has the following significance; the region occupied by the oil film (i. e., the region where the solution of REYNOLDS equation for the pressure is non negative) is $(\alpha_1, \alpha_1 + \pi)$, if the angles are counted as usual in the sense of rotation from the radius pointing to where is the maximum of film thickness.

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(1) G. CAPRIZ, *Sulle vibrazioni di aste rotanti*. Ann. Sc. Norm. Sup. Pisa (3) **17** (1963), 31-42.

In other words α_1 is determined by the conditions

$$\alpha_1 = \arccos \frac{e \cdot (\omega - 2\dot{\beta})}{[(\omega - 2\dot{\beta})^2 e^2 + 4\,\dot{e}^2]^{1/2}} = \arcsin \frac{2\,\dot{e}}{[(\omega - 2\dot{\beta})^2 e^2 + 4\,\dot{e}^2]^{1/2}} . \tag{20.5}$$

The interest of formulae (20.3) lies in that they allow a discussion of the stability of the steady configuration specified by eqns (20.2) of the journal under load, more complete than that carried out by HOLMES [2]. HOLMES' analysis is based on the linearization of the expressions for F_e, F_n in the vicinity of their steady-state values and on the successive discussion of the properties of the solutions of the dynamic equations with the use of the rules of ROUTH-HURWITZ.

Because the phenomenon of journal whirl depends, within the limits of HOLMES' scheme, on two non-dimensional parameters only [for instance A and the non-dimensional load ratio $w = W/(m\,c\,\omega^2)$], it is possibile to determine, in the plane of those variables, the region of linear stability S and thus express the main result of the investigation very simply by means of a graph.

We have carried out a number of numerical integrations of the complete non-linear equations and found that generally stability is assured also in the large if A, w are chosen as coordinates of a point within S, and vice versa. The extension thus achieved of the validity of HOLMES' result is by no means trivial: notice, for instance, that under some circumstances during a whirl of large amplitude, considerable rocking of the film occurs.

21. - Dynamic equations. Linear stability.

With the usual notation the equations of motion of the journal centre in polar coordinates of origin Ω are:

$$\begin{cases} m\,c\,\ddot{a} = m\,c\,a\,\dot{\beta}^2 + F_e + W\cos\beta \\ m\,c\,a\,\ddot{\beta} = -2\,m\,c\,\dot{a}\,\dot{\beta} + F_n - W\sin\beta\,. \end{cases} \tag{21.1}$$

These equations can be put in a fully non-dimensional form introducing a variable $\tau = \omega\,t$:

$$\begin{cases} a'' = a\,\beta'^2 + M\,[(1 - 2\,\beta')\,a\,g_1 - 2a'\,g_2] + w\cos\beta \\ a\,\beta'' = -2\,a'\,\beta' + M\,[(1 - 2\,\beta')\,a\,g_3 - 2\,a'\,g_1] - w\sin\beta\,. \end{cases} \tag{21.2}$$

(2) In his paper: R. HOLMES, *The vibration of a rigid shaft on short sleeve bearings*, J. Mech. Eng. Science 2 (1960), 337-341.

Here a prime denotes a derivative towards τ, w is the parameter $W/(m\,c\,\omega^2)$ already quoted and M stands for the ratio $\eta\,R\,b^3/(2\,m\,c^3\,\omega)$:

$$w = W/(m\,c\,\omega^2), \qquad M = \eta\,R\,b^3/(2m\,c^3\,\omega)\,.$$

M is strictly related to the parameter B_2 used in Sect. **12** [see the third formula (12.15): $B_2 = \pi\,M/\sqrt{2}$].

As already remarked in Sect. **20**, eqns (21.2) admit of a constant solution:

$$a = A, \qquad \beta = \bar{\beta}, \tag{21.3}$$

where A and $\bar{\beta}$ are related to w and M through eqns (20.2).

In a neighbourhood of the solution (21.3), $a(\tau)$, $\beta(\tau)$ satisfy approximately a linear system which is obtained from (21.2) by substituting for $a(\tau)$ and $\beta(\tau)$ the expressions

$$a(\tau) = A + a_1(\tau), \qquad \beta(\tau) = \bar{\beta} + \beta_1(\tau)$$

and disregarding squares and products of a_1, β_1:

$$a_1'' = -\,M\,\pi\,(1 + 2\,A^2)(1 - A^2)^{-5/2}\,a_1' - 4\,M\,A\,(1 + A^2)(1 - A^2)^{-3}\,a_1 + $$
$$+\,4\,M\,A^2(1 - A^2)^{-2}\,\beta_1' - w\,\beta_1\sin\bar{\beta}\,,$$

$$\beta_1'' = -\,M\,\pi\,(1 - A^2)^{-3/2}\,\beta_1' - (w/A)\,\beta_1\cos\bar{\beta} + 4\,M\,(1 - A^2)^{-2}\,a_1' + $$
$$+\,(M\,\pi/2A)(1 + 2\,A^2)(1 - A^2)^{-5/2}\,a_1\,.$$

Application of ROUTH's rule to this linear system yields the following condition for stability: $w > G(A)$, where

$$G(A) = [16A^2 + \pi^2(1 - A^2)]^{1/2}\,[\pi^2(1 + 2A^2) -$$
$$-\,16A^2]^{-1}\{2A^2(1 + A^2)(2 + A^2)(1 - A^2)^{-1}\,(3 + A^2)^{-1} +$$
$$+\,(\pi^2/16)(2 + A^2)(1 + 2A^2)(3 + A^2)^{-1} -$$
$$-\,A^2(3 + A^2)(1 + 2A^2)(2 + A^2)^{-1}(1 - A^2)^{-1}\}\,,$$

if we express M and $\bar{\beta}$, wherever they appear, in terms of w, A using the second eqn (20.2) and a consequence of the first eqn (20.2):

$$M = 2(1 - A^2)^2A^{-1}[16\,A^2 + \pi^2\,(1 - A^2)]^{-1/2}\,w\,.$$

As we mentioned in the Introduction, it is possible to illustrate the condition of stability, by tracing the curve

$$w = G(A) \tag{2.14}$$

in the plane (A, w) [see Fig. 1 here and Fig. 3 of the paper quoted in footnote (2)]. Note that $G(0) = \pi/24$ and $G(A) \to 0$ for $A \to \bar{A}$, where $\bar{A}$ is less than 1, approximately $\bar{A} = 0.7556$; note also that $G(A) = \pi/24 + O(A^2)$: the boundary of the region of stability is very well approximated by the straight line $A = \pi/24$ in the neighbourhood of $A = 0$. Naturally there is a similarity between the curve of equation (2.14) in the plane (A, w) and the curve $\nu = \nu(A)$ appearing in the paper mentioned in footnote (1).

22. - Numerical integration of non-linear equations.

General qualitative properties of the solutions of large amplitude of system (21.2) are difficult to discover by analytical means, due to the complexity of the functions g_i, where, furthermore, the parameter α_1 must be specified using formulae (20.5). Recourse can be made to a direct numerical integration: we have used a predictor-corrector method (see Sect. 19) and carried out quite a few numerical experiments, with the usual precautions (see Table 1).

The values of A and w chosen for the experiments are the coordinates of points within the circles shown in Fig. 1. Four typical trajectories are shown in Figs. 2-5; two graphs of α_1 versus τ are traced in Fig. 6.

Table 1 gives case number, the values of A and w characterizing the case, the corresponding value of $\bar{\beta}$, the initial values of a and β (a and β' were all taken to be zero initially), the final value of τ (to mark the length of the experiment) and finally two indices u or s, 0 or 1 which describe the observed behaviour for large τ. u is for unstable, to mean that a tends towards 1, β' towards 1/2; s is for stable, to mean that a tends towards A, β towards $\bar{\beta}$. The indices 0 and 1 give a clue as to the behaviour of α_1: the first index is appended to cases where α_1 tends to zero, the second where α_1 shows large oscillations (between $-\pi/2$ and $\pi/2$). Where an index is missing no decisions could be made within the limits of the experiment. By cross-reference to Fig. 1, where unstable cases are marked with a cross and stable cases with a dot within the circle, the permanence in the large can be noticed of the rule of stability found by HOLMES within the limits of a perturbation anlysis. This permanence, within rather close bounds, is surprising, in view of the circumstance that in all unstable cases when $w \sim \pi/24$, rocking of the lubricant film occurs and when $A \sim 0.7$ the cavity appeared to rotate together with the vector ΩO. Obviously under linearly unstable conditions a progressive increase of the amplitude of vibration is not arrested by the rocking of the film.

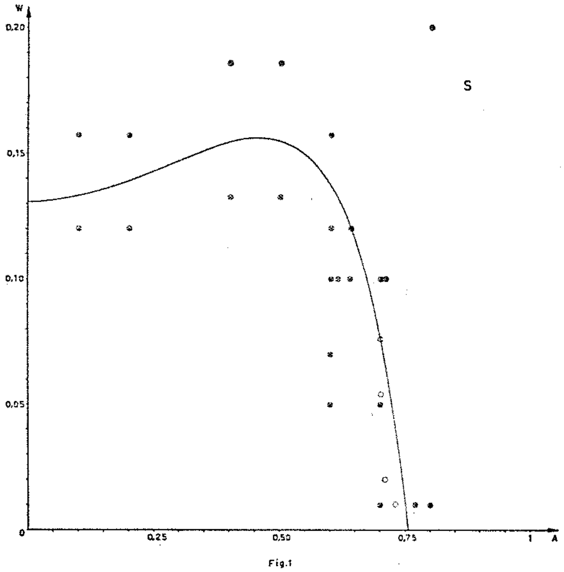

Fig. 1.

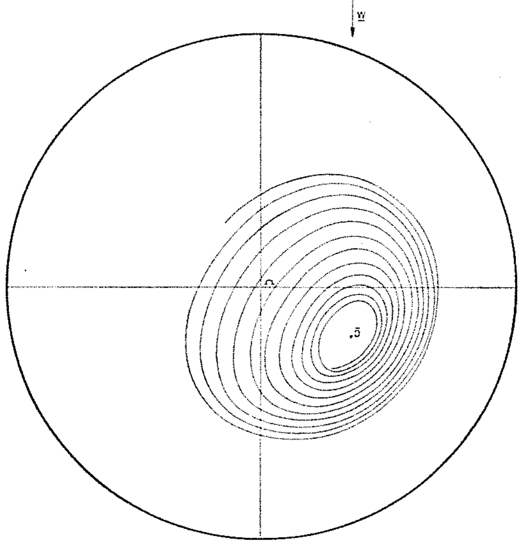

Fig. 2 - Case 6.

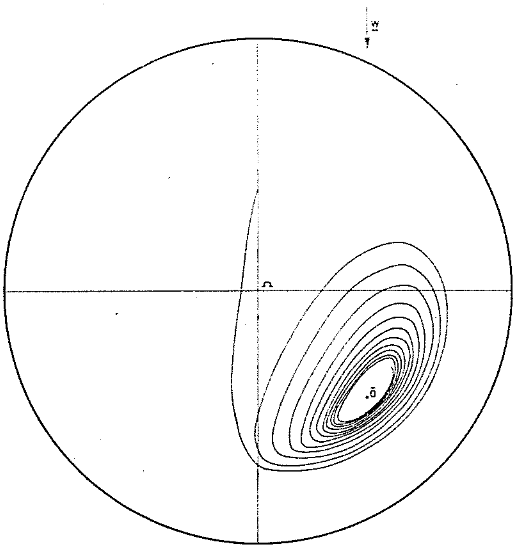

Fig. 3 - Case 9.

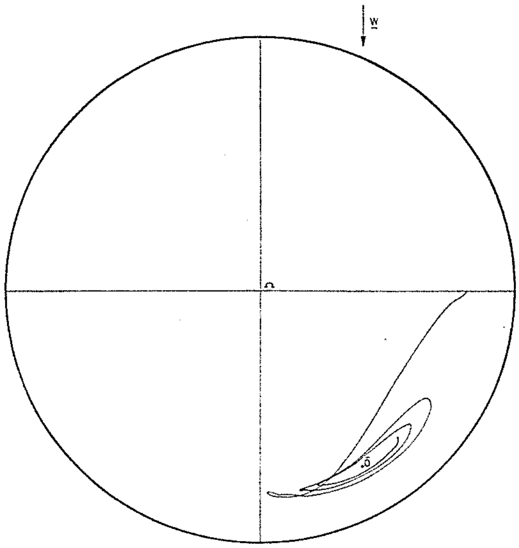

Fig. 4 - Case 22.

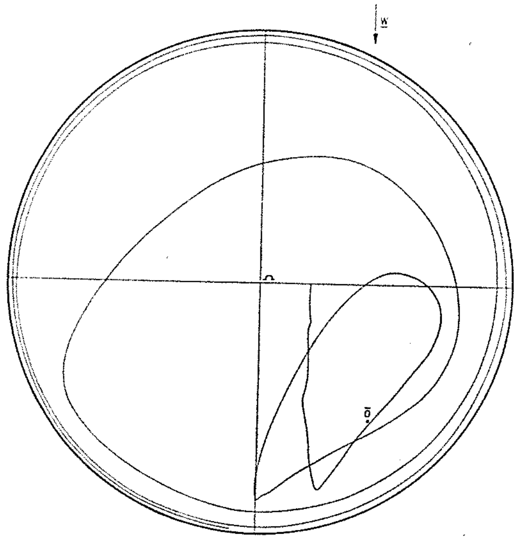

Fig. 5 - Case 25.

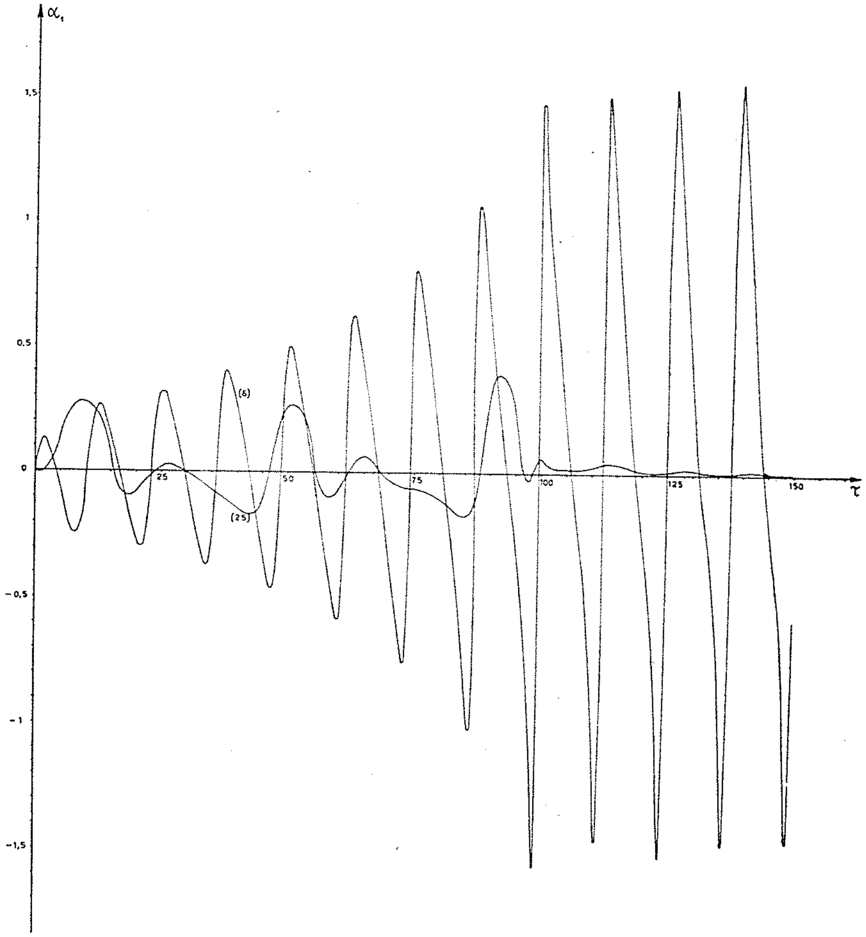

Fig. 6 - Behaviour of α for cases 6 and 25.

Table 1.

Case	1	2	3	4	5	6	7	8	9	10
A	0.1000	0.1000	0.2000	0.2000	0.4000	0.4000	0.5000	0.5000	0.6000	0.6000
w	0.1570	0.1200	0.1570	0.1200	0.1858	0.1325	0.1858	0.1325	0.1570	0.1200
$\bar{\beta}$	1.443	1.443	1.316	1.316	1.064	1.064	0.936	0.936	0.808	0.808
a_0	0.700	0.200	0.200	0.150	0.420	0.420	0.420	0.420	0.400	0.57
β_0	2.800	1.700	4.000	1.400	5.000	0.700	5.000	0.700	3.140	1.050
τ_f	150	150	150	150	150	150	150	150	150	150
	s	$u1$	$s0$	$u1$	$s0$	$u1$	$s0$	$u1$	$s0$	$u1$
Case	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
A	0.6395	0.6130	0.6395	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7300
w	0.1200	0.1000	0.1000	0.07569	0.05396	0.1000	0.0500	0.07569	0.05396	0.010[illegible]
$\bar{\beta}$	0.757	0.786	0.757	0.675	0.675	0.675	0.675	0.675	0.675	[illegible]
a_0	0.200	0.600	0.600	0.800	0.720	0.600	0.650	0.690	0.690	0.725
β_0	1.400	1.000	7.200	4.700	1.000	0.800	0.700	0.600	0.600	0.800
τ_f	150	150	150	150	150	150	150	150	150	300
	$s0$	$u1$	$u1$	u	0	$s0$	u	—	u	0
Case	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
A	0.7100	0.8000	0.6000	0.8000	0.7000	0.7300	0.7700	0.6000	0.6000	0.7100
w	0.0200	0.2000	0.1000	0.0100	0.0100	0.0100	0.0100	0.0500	0.0700	0.100
$\bar{\beta}$	0.661	0.532	0.808	0.532	0.675	0.634	0.573	0.808	0.808	0.661
a_0	0.300	0.810	0.650	0.200	0.200	0.725	0.700	0.620	0.500	0.500
β_0	1.400	1.570	1.050	1.570	1.570	0.600	3.140	1.050	1.000	1.300
τ_f	150	40	150	150	150	150	150	40	94	150
	0	s	u	$s0$	$u0$	$u0$	$u0$	u	u	$s0$

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