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G. Capriz, L. Galletti, A. Laratta (*)

# On Some Dynamical Problems Arising in the Theory of Lubrication. - III. (**)

## 16. - Introduction.

A further [1] contribution to the theory of lubrication is given here. For ease of reference the numbering of sections is continued from Parts I and II and the notation is made consistent with that used earlier (see Sect. 11); symbols introduced is this part are:

$A$, eccentricity ratio under steady load;

$W$, load on journal;

$w = W/(mc\omega^2)$, non-dimensional load ratio;

$\beta$, angular coordinate of $O$ in a fixed cylindrical system of reference with origin in $\Omega$ (as before); the axis from which angles are counted is taken now to be oriented as the load vector $W$ (usually the downward vertical).

We study here the behaviour of a rigid heavy rotor housed in a fully lubricated bearing during parallel whirl of large amplitude. The bearing is assumed to be either very long (case 1) or very short (case 2) so that we can take, for the forces due to the lubricant and acting on the journal, the explicit expressions obtained by Sommerfeld [i. e. expressions (7.1)] and respectively by Ocvirk [i. e. expressions (12.13)]. The attitude locus of the centre of the journal $O$ under steady load is in both cases a straight line through the centre of the

(*) Indirizzo: C.S.C.E., Istituto di Fisica, Università, Pisa, Italia.

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bearing $\Omega$: if the load is $W$, the eccentricity ratio in the steady state $A$ is given by

$$\frac{c^2 W}{6\pi R^3 b\eta\omega} = \frac{A}{\{1+(A^2/2)\}\,(1-A^2)^{1/2}} \tag{16.1}$$

in case 1 and by

$$\frac{2c^2 W}{\pi R b^3 \eta\omega} = \frac{A}{(1-A^2)^{3/2}}$$

in case 2; and the attitude angle $\bar{\beta}$ is equal $\pi/2$, in both cases.

A linear analysis shows that this static position of the journal under load is unstable ([2]). Numerical integration of the non-linear equations has led to some contrasting results: HOLMES found that with increasing time the whirl becomes nearly circular while the amplitude increases indefinitely (though at slowing rate) towards the clearance $c$ and that the whirl frequency tends to half the running frequency ([3]). For case 2 REDDI and TRUMPLER seem to have found instead that complete instability is not an absolute characteristic of the system: under some conditions the journal path may settle down to an orbit well within the clearance circle; furthermore that in cases of complete instability, the journal hits the bearing within a finite time ([4]).

We have studied the problem analytically and have carried out also many approximate calculations. Our results confirm the findings of HOLMES and contradict those of REDDI and TRUMPLER: the equations which describe the movement of the journal centre $O$ do not appear to admit solutions represen-

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([2]) See, for istance, R. HOLMES, *Oil-whirl characteristics of a rigid rotor in a* 360° *journal bearing*, Proc. Inst. Mech. Engns, Advance copy P II/63.

([3]) This last property is a necessary corollary of the first as can be proved through the following qualitative argument.

The rate at which work is done by the oil forces in a nearly circular whirl is approximately

$$\frac{\pi R b^3 \eta}{2c}\,\frac{a(\omega - 2\dot{\beta})\dot{\beta}}{(1-a^2)^{3/2}}\,;$$

this quantity is positive provided that $\dot{\beta} < \omega/2$. The amplitude of whirl (and hence the kinetic energy of the journal) may increase only if $\dot{\beta} < \omega/2$, but on the other hand the kinetic energy is approximately $\frac{1}{2}\,m a^2 c^2 \dot{\beta}^2$, a quantity that cannot exceed $\frac{1}{2}\,m c^2 \dot{\beta}^2$

It follows that the energy supply to the journal (work done by oil forces) must tend to zero, i.e. that $\dot{\beta} \to \omega/2$.

([4]) M. M. REDDI and P. R. TRUMPLER, *Stability of the high-speed journal bearing under steady load,* J. Eng. Industry, Trans. A.S.M.E., **84** (1962), 351-358.

ting paths of $O$ ending abruptly at the periphery of the clearance circle; it seems also unlikely that closed paths exist for $O$. We show, however, that such paths exist for a dynamic system which our system approaches when $W$ and $B_2 \left( = \dfrac{\pi R b^3 \eta}{2\sqrt{2} c^3 m\omega} \right)$ tend to infinity.

These analytical results are obtained in Sect. **17**. Explicit reference is made there to the case of very short bearings; perfectly parallel arguments apply, however, also to the alternative case of long bearings; a brief summary of developments valid for that case is in Sect. **18**.

In the following Sect. **19** we describe approximate calculations which have led to the tracing of paths of $O$ for a number of special values of the parameters. These paths behave exactly according to the forecasts of Sects. **17**, **18**.

Details of the numerical procedures used are also given.

## 17. - Analysis of the case of short bearings.

The equations of motion of the journal under load within a short bearing are [see eqns. (12.14)]:

$$(17.1)\qquad \begin{cases} m\ddot{a} = ma\dot{\beta}^2 - 2\mathfrak{F}_2\,\dot{a}(1+2a^2)(1-a^2)^{-5/2} + (W/c)\cos\beta \\ ma\ddot{\beta} = -2m\dot{a}\dot{\beta} + \mathfrak{F}_2\,a(\omega - 2\dot{\beta})(1-a^2)^{-3/2} - (W/c)\sin\beta\,. \end{cases}$$

It appears from the first eqn. (17.1) that, in the cases where the whirling speed $\dot{\beta}$ is bounded and $a$ tends to 1 with increasing time, either (i) $\dot{a}$ tends to zero, as $a$ tends to 1, of an order not less than 5/2 in $1-a$, or (ii) the development of $a$ is approximately regulated for $a \sim 1$ by the equation

$$(17.2)\qquad m\ddot{a} = -2\mathfrak{F}_2\,\dot{a}\,\frac{1+2a^2}{(1-a^2)^{5/2}}\,,$$

because then the other two terms on the r.h.s. become negligible. But this second istance is contradictory; all solutions $a(t)$ of equation (17.2) tend for $t \to \infty$ to a value less than 1, against our hypothesis (5).

(5) In the phase space $(a, v)$ $[v = \dot{a}]$ the trajectories of (17.2) are graphs of the functions $v = \bar{v} - \dfrac{2\mathfrak{F}_2}{m}\displaystyle\int_{\bar{a}}^{a} \frac{1+2a^2}{1-a^2}\,da$. All these trajectories diverge to the point $(1, -\infty)$. The point $P \equiv \{a(t), \dot{a}(t)\}$ describes with increasing $t$ one, $\mathfrak{T}$, of these trajectories in the sense in which $|v|$ decreases. For $t \to \infty$, $P$ tends over $\mathfrak{T}$ to the point of intersection of $\mathfrak{T}$ with the axis $v = 0$.

Hence only the first alternative remains: but if

$$\dot{a} \sim K(1-a^2)^\alpha, \qquad \alpha \geqslant 5/2\,,$$

then the time $T$ taken for a to reach 1 is

$$T \sim K \int_{\bar{a}}^{1} \frac{da}{(1-a^2)^\alpha} = +\infty, \qquad \alpha \geqslant 5/2\,.$$

We conclude that the statement: « $a$ may reach 1 in a finite time » seems to be inconsistent with eqns (17.1).

To this negative result a positive one can be added through a closer examinations of the properties of those solutions of (17.1), which represent whirls of large amplitude. To study those solutions the following approximations are introduced [here as in Sect. **13**]: (i) $a$ and $\dot{\beta}$ are approximated respectively by $a = 1 - \delta$, $\dot{\beta} = \omega\sigma(1-\varepsilon)$, where $\sigma$ is a constant; $\delta$ and $\varepsilon$ are quantities small of the first order; (ii) the radial acceleration is approximated by $a\,\dot{\beta}^2$.

The second hypothesis seems to be justified by the first; in a nearly circular whirl with nearly constant rotational speed the ratio $\ddot{a}/a\dot{\beta}^2$ must be negligible.

Then from the first eqn. (17.1) follows for $\delta$

$$\delta'/\delta^{5/2} = -(2/B_2)(\sigma^2 + w\cos\beta)\,, \tag{17.3}$$

[as in Part II, a prime indicates a derivative towards

$$\tau = \omega t; \qquad \text{also } B_2 = \frac{\pi R b^3 \eta}{(2\sqrt{2}c^3 m\omega}\,, \qquad \text{whereas } w = \frac{W}{mc\omega^2}\Big]$$

and from the second eqn. (17.1) follows that: (i) $\sigma$ must be equal to $1/2$; (ii) $\varepsilon$ is related to $\delta$ through the equation:

$$\varepsilon = (4w/B_2)\delta^{3/2}\sin\beta\,. \tag{17.4}$$

Taking $\beta$ as independent variable in (17.3) one obtains the expression of $\delta$ in terms of $\beta$:

$$\delta = \frac{\bar{\delta}}{\{1 + (3\bar{\delta}^{3/2}/2B_2)[\beta - \bar{\beta} + 4w(\sin\beta - \sin\bar{\beta})]\}^{2/3}} \tag{17.5}$$

[$\bar{\delta}$ is here the initial value of $\delta$, i.e. the value of $\delta$ when $\beta = \bar{\beta}$].

A number of remarks can be deduced from eqns. (17.3), (17.4). First of all that, if $w$ is sufficiently small (less than 1/4), $\delta$ decreases steadily to zero as the time increases [see Fig. 1]. If $w$ is large the value of $\delta$ may oscillate, [increasing while $\beta$ is between $(4m + 1)\ \pi/2$ and $(4m + 3)\ \pi/2$ and decreasing

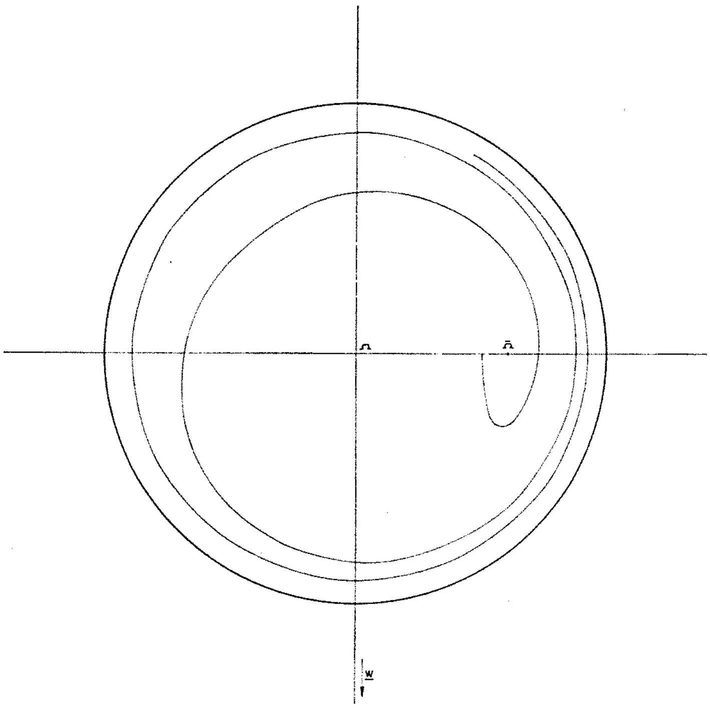


Fig. 1 - Case 1

while $\beta$ is between $(4n + 3)\ \pi/2$ and $(4n + 5)\ \pi/2)$] though the amplitude of the oscillation tends to zero as the time increases [see Fig. 2]; also when $w$ is large, our assumption on the smallness of $\delta$ becomes invalid, if care is not taken to choose $\bar{\delta}$ appropriately small.

Thus we have confirmed HOLMES' results at least for the cases where formula (17.5) gives a reasonable approximation to the solution. In particular it appears that in all cases where $\delta$ happens to assume a sufficiently small value (i. e. the amplitude of the whirl is sufficiently large at a certain time) $\delta$ tends asymptotically to zero as $t \to \infty$; at the same time the whirling speed tends to $\omega/2$.

It may be observed that in all cases where REDDI and TRUMPLER have concluded (on the strength of their approximate calculations) with the existence of a closed path for $O$, the value of $w$ was large. For large values of $w$ the field of validity of formula (17.5) is restricted. It is then necessary to discuss another asymptotic case for eqns (17.1): the case of very large $w$ and $B_2$ [from a mechanical point of view this can be described as the case where lubricant force and load predominate over the inertia force]. We want to show that then all paths

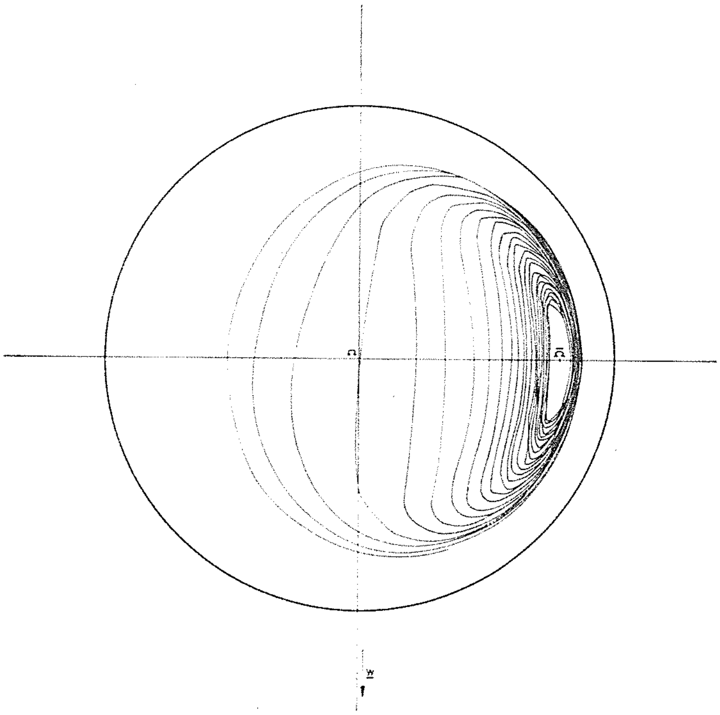


Fig. 2 - Case 10

of $O$ are closed paths; this result is only apparently in contrast with what we have already said about $\delta$: for almost circular paths the inertia force is predominantly a centrifugal force; if we cancel its contribution, small as it may be, in the dynamic equations, we force arbitrarily the actual conditons in favour of stability (6).

(6) Extensive numerical calculations for this case have been carried out by A.A. MILNE of the N.E.L., East Kilbride (Scotland).

Let us put first system (17.1) in a non-dimensional form:

$$a'' = a\beta'^2 - 2\sqrt{2}B_2 \frac{a'(1+2a^2)}{(1-a^2)^{5/2}} + w \cos\beta ,$$

$$a\beta'' = -2a'\beta' + \sqrt{2}B_2 \frac{a(1-2\beta')}{(1-a^2)^{3/2}} - w \operatorname{sen}\beta .$$

When $w$ and $B_2$ are large it can be reasonably presumed that the solutions of this system do not behave too differently from the solutions of the simpler system which is obtained by cancelling terms where these two large parameters do not appear:

$$a' \frac{1+2a^2}{(1-a^2)^{5/2}} = \frac{w}{2\sqrt{2}B_2} \cos\beta, \qquad \beta' = \frac{1}{2} - \frac{w}{2\sqrt{2}B_2} \frac{(1-a^2)^{3/2}}{a} \sin\beta .$$

The qualitative behaviour of the latter solutions can be easily decided through a study of the paths of the equations of the first order

$$\frac{da}{d\beta} = \frac{wa(1-a^2)^{5/2}\cos\beta}{(1+2a^2)[a\sqrt{2}B_2 - w(1-a^2)^{3/2}\sin\beta]} \tag{17.6}$$

in the phase-plane $(\beta, a)$ [see Fig. 3]; actually the correct manifold over which

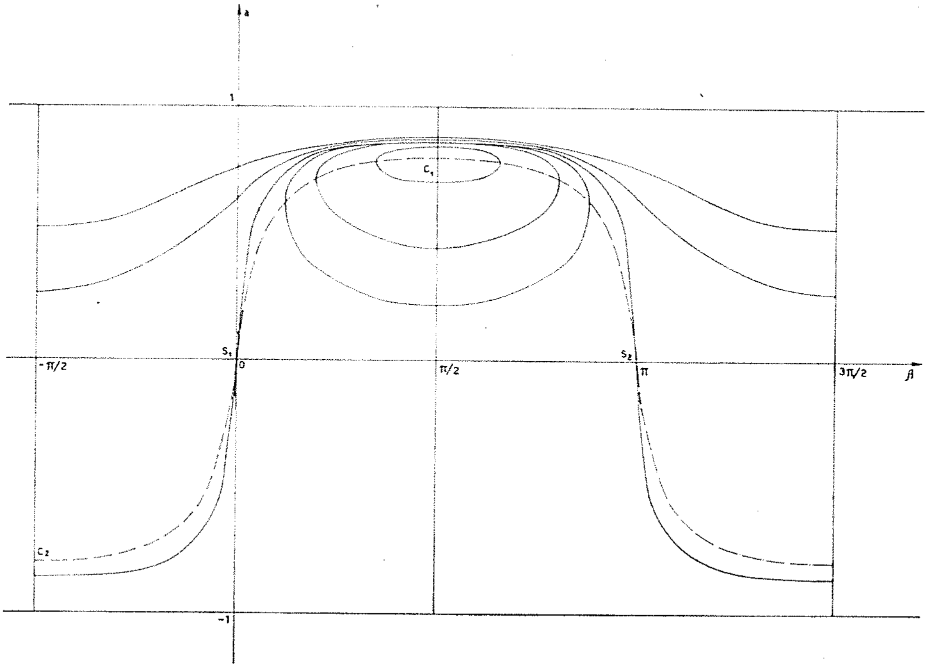



Fig. 3

the solutions of eqn (17.6) must be studied is the phase cylinder $\mathcal{C}$ obtained by folding the strip $\pi/2 \leqslant \beta < 3\pi/2$ and joining the straight lines $\beta = \pi/2$, $\beta = 3\pi/2$.

The singular points of (17.6) on the phase cylinder $\mathcal{C}$ are: (i) Two centres $C_1$, $C_2$ of coordinates

$$C_1 \equiv (\pi/2,\ A); \qquad C_2 \equiv (3\pi/2,\ -A)\,,$$

where $A$ is the real solution of the equation

$$\frac{A}{(1-A^2)^{3/2}} = \frac{w}{\sqrt{2}B_2}\,;$$

the paths in the immediate neighbourhood of $C_1$, $C_2$ are ellipses

$$(a-A)^2 + \frac{wA\,(1-A^2)^{5/2}}{2\sqrt{2}B_2(1+2A^2)}\left(\beta-\frac{\pi}{2}\right)^2 = \text{const.}$$

and, respectively,

$$(a+A)^2 + \frac{wA(1-A^2)^{5/2}}{2\sqrt{2}B_2(1+2A^2)}\left(\beta-\frac{3\pi}{2}\right)^2 = \text{const.}$$

(ii) Two saddle points $S_1$, $S_2$ of coordinates

$$S_1 \equiv (0,\ 0), \qquad S_2 \equiv (\pi,\ 0);$$

the paths in the immediate neighbourhood of $S_1$, $S_2$ are hyperbolae or, exceptionally, two straight lines; for istance, around $S_1$,

$$\beta_a - \frac{\sqrt{2}\,B_2}{2w}\,a^2 = \text{const.}\,,$$

and respectively

$$a = 0, \qquad a = \frac{2w}{\sqrt{2}\,B_2}\,\beta.$$

All paths of (17.6) on $\mathcal{C}$ are closed paths either of the first or second kind (the latter close around the cylinder); the only exception is the separatrix $a = 0$ through the saddle point.

To prove this, note that for all solutions of (17.6) the following relation of symmetry holds

$$a(\beta) = a(\pi - \beta)\,,$$

because if $a = f(\beta)$ is a solution of (17.6), so is also $a = f(\pi - \beta)$; furthermore $\mathrm{d}a/\mathrm{d}\beta = 0$ for $\beta = \pi/2$. Hence the field of paths to the right of $\beta = \pi/2$ is the mirror image of the field to the left of $\beta = \pi/2$.

The paths of the second kind are related to closed trajectories of $O$ encircling $\Omega$; the closed paths of the first kind instead leave $\Omega$ outside.

### 18. - The case of long bearings.

There is not much to change in what we have said in Sect. **17** when the case of long bearings is considered. For istance the dynamic equations (17.1) must be substituted by the system

$$\text{(18.1)} \qquad \begin{cases} m\ddot{a} = ma\dot{\beta}^2 - 2\mathfrak{F}_1\dot{a}(1-a^2)^{-3/2} + (W/c)\cos\beta \\ ma\ddot{\beta} = -2m\dot{a}\dot{\beta} + \mathfrak{F}_1 a(\omega - 2\dot{\beta})\left\{1 + (a^2/2)\right\}^{-1}(1-a^2)^{-1/2} - (W/c)\sin\beta\,, \end{cases}$$

but the equation for $\delta = 1 - a$, which takes the place of (17.3)

$$\delta'\delta^{-3/2} = -(1/B_1)\left\{(1/4) + w\cos\beta\right\},$$

where

$$B_1 = \frac{6\pi R^3 b\eta}{\sqrt{2}c^3 m\omega}$$

allows conclusions parallel to those of Sect. **17**.

Also the asymptotic case when load and lubricant forces prevail over the inertia forces (case of large $w$ and $B_1$) can be described starting from the equation

$$\text{(18.2)} \qquad \frac{\mathrm{d}a}{\mathrm{d}\beta} = \frac{2wa(1-a^2)^{3/2}\cos\beta}{2\sqrt{2}B_1 a - w(2+a^2)(1-a^2)\sin\beta}$$

[instead of (17.6)]. All paths of (18.2) on the phase cylinder are again closed.

### 19. - Results of numerical computations.

To help in bridging the gaps between the asymptotic cases considered in Sects. **17**, **18** quite a few numerical integrations of eqns (17.1) and (18.1) have been carried out.

The results of all computations bear out the analytical forecasts of complete instability and indefinite whirling with slowly increasing amplitude of the journal. It appears also that the rate of increase of the amplitude of oscillation is particularly slow as $w$ becomes larger.

For short bearings.

| Case | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | 0.6 | 0.6 | 0.6 | 0.8 | 0.8 | 0.33 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.625 |
| $w$ | 0.2 | 0.2 | 0.2 | 0.125 | 0.32 | 0.3472 | 0.78125 | 3.125 | 12.5 | 12.5 | 12.5 | 12.5 | 12.5 | 50 | 50 | 200 |
| $\alpha_0$ | 0.5 | 0.7 | 0.607 | 0.81 | 0.81 | 0.325 | 0.81 | 0.81 | 0.81 | 0.83 | 0.84 | 0.85 | 0.86 | 0.81 | 0.82 | 0.6 |
| $\beta_0$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ |

For long bearings.

| Case | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.33 | 0.33 | 0.33 |
| $w$ | 0.32 | 0.78125 | 3.125 | 12.5 | 12.5 | 12.5 | 12.5 | 50 | 50 | 50 |
| $\alpha_0$ | 0.81 | 0.82 | 0.81 | 0.81 | 0.83 | 0.84 | 0.85 | 0.325 | 0.8 | 0.85 |
| $\beta_0$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ | $\pi/2$ |

For the numerical integration of systems (17.1), (18.1) or, rather, of the equivalent systems of four equations of the first order (in $y^1 = a$, $y^2 = \mathrm{d}a/\mathrm{d}\tau$, $y^3 = \beta$, $y^4 = \mathrm{d}\beta/\mathrm{d}\tau$), HAMMINGS predictor-corrector method of fifth order was used (7):

predictor:

$$p^i_{k+1} = y^i_{k-3} + (4/3)\Delta\tau(2y'^i_k - y'^i_{k-1} + 2y'^i_{k-2});$$

modified predictor:

$$m^i_{k+1} = p^i_{k+1} - (112/121)(p^i_k - C^i_k);$$

corrector:

$$C^i_{k+1} = (1/8)[9y^i_k - y^i_{k-2} + 3\Delta\tau(m'^i_{k+1} + 2y'^i_k - y'^i_{k-1})];$$

modified corrector:

$$y^i_{k+1} = C^i_{k+1} + (9/121)(p^i_{k+1} - C^i_{k+1}) ;$$

for $i = 1, 2, 3, 4;\quad k = 3, 4, \ldots$ .

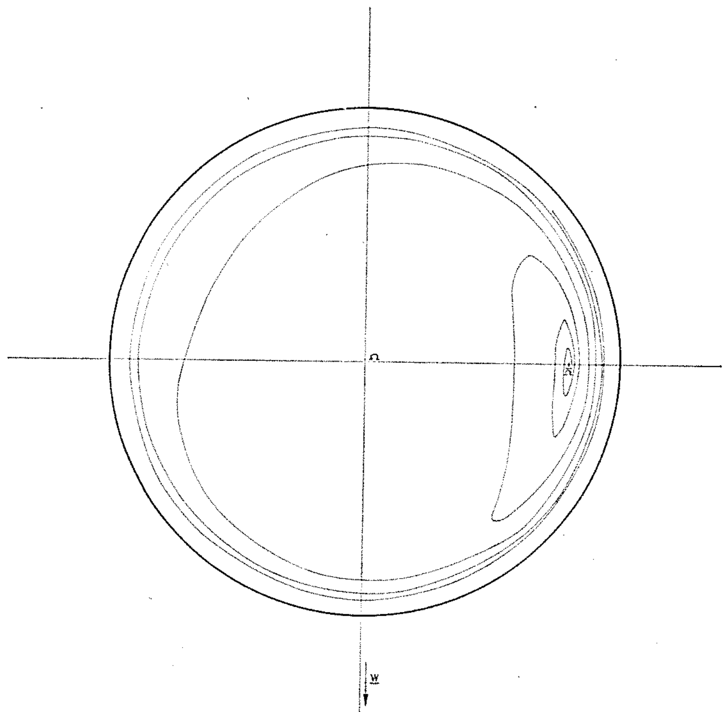
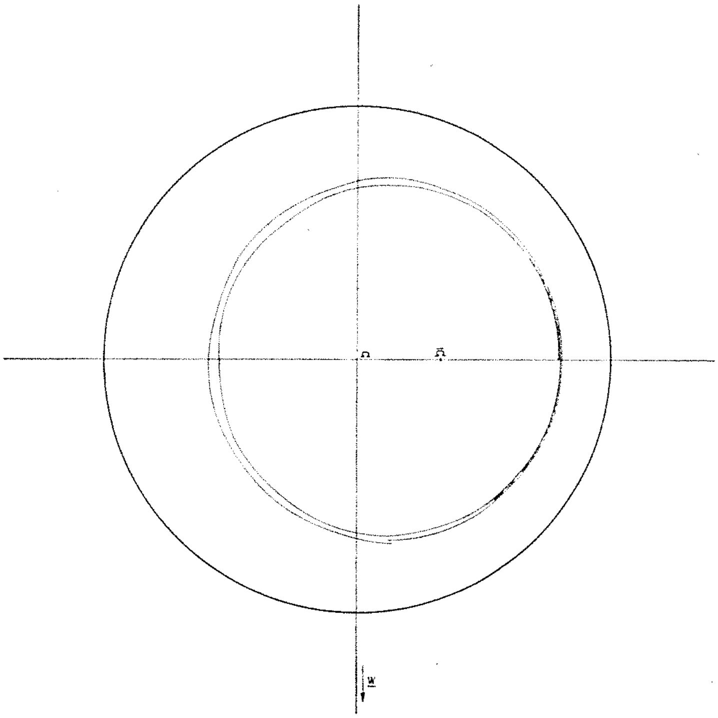


Fig. 4 - Case 7

(7) R. W. HAMMING, *Stable predictor-corrector methods for ordinary differential equations*, J. Assoc. Comput. Mach. 6 (1959), 37-47.

The starting values $y_1^i$, $y_2^i$, $y_3^i$, $(i = 1, 2, 3, 4)$ were computed by a RUNGE-KUTTA method. The control over errors was based on the differences $p_k^i - c_k^i$; it was noted that, over some intervals, the numerical integration could proceed only with very small steps, if satisfactory accuracy was to be achieved. As a consequence the machine programme was set up so that the step-size



Fig, 5 - Case 25

was chosen automatically: halved if $p_k^i - c_k^i$ was larger than $10^{-3}$, doubled if $p_k^i - c_k^i$ was smaller than $5.10^{-4}$. In the former case the restarting procedure made use again of the RUNGE-KUTTA method. Even so the determination of some trajectories took a relatively long time (around 30′ on the Computer of the University of Pisa C.E.P., with and addition time of approximately 20 $\mu$ sec). The tolerance values shown above are average. Some cases actually required smaller tolerance to avoid progressive worsening of the solution; also, to guard against complacency, many cases were repeated with different tolerances.

Examples of solutions are already shown in Figs. 1 and 2; further cases are in Fig. 4 and 5: note in particular the behaviour of the solution when $w$ is large (in Fig. 5 only the first and the last of nine orbits are shown).

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