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Sui polinomi ultrasferici a parametri complementari rispetto all'unità. (**)

1. – I polinomi ultrasferici, di parametro α,

$$P_n^{(\alpha)}(x) = \frac{(2\alpha, n)}{n!} \, {}_2F_1(-n, \; 2\alpha + n; \; \alpha + 1/2; \; (1-x)/2)$$

e le relative funzioni ultrasferiche

$$Q_n^{(\alpha)}(x) = \frac{\Gamma(2\alpha + n)}{2^{n+1}(\alpha, n+1)} x^{-2\alpha-n} \, {}_2F_1(\alpha + n/2, \quad \alpha + (n+1)/2; \quad \alpha + n + 1; \; 1/x^2)$$

sono vincolati dalla formula

$$Q_n^{(\alpha)}(x) = P_n^{(\alpha)}(x) \; Q_0^{(\alpha)}(x) - \Gamma(2\alpha) \; (x^2 - 1)^{-\alpha+1/2} \; R_{n-1}^{(\alpha)}(x) \,,$$

in cui i polinomi associati $R_n^{(\alpha)}(x)$, di grado uguale all'indice, soddisfano alla relazione ricorrente

$$(n+1) \; R_n^{(\alpha)}(x) - 2(\alpha + n)x \; R_{n-1}^{(\alpha)}(x) + (2\alpha + n - 1) \; R_{n-2}^{(\alpha)}(x) = 0 \,,$$

con $R_0^{(\alpha)}(x) = 1$ e $R_1^{(\alpha)}(x) = (\alpha + 1)x$.

I polinomi associati sono stati espressi come combinazione lineare di polinomi ultrasferici, da N. NIELSEN [5] a parametro $1 - \alpha$ e da G. N. WATSON [11] a parametro α.

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Ed io [9], estendendo un risultato di SCHÄFLI-HERMITE su gli associati dei polinomi di LEGENDRE, li ho espressi con la formula

$$R_n^{(\alpha)}(x) = \sum_0^n{}_r \frac{1}{r+1} P_r^{(1-\alpha)}(x)\ P_{n-r}^{(\alpha)}(x)\,,$$

in cui si presentano contemporaneamente i polinomi ultrasferici a parametri $1-\alpha$ e α, complementari rispetto all'unità. Questa espressione spinge naturalmente alla ricerca di altre formule in cui operino contemporaneamente i polinomi ultrasferici a parametri complementari rispetto all'unità, formule da porre a fianco di analoghe in cui si presentano polinomi ultrasferici a un solo parametro. E si procede così verso un nuovo tipo di generalizzazione di formule sui polinomi di LEGENDRE.

2. - In questa Nota prendiamo le mosse dalla formula nota $(m \leqslant n)$

$$(1)\qquad P_n^{(\alpha)}(x)\, P_m^{(\alpha)}(x) = \sum_0^m{}_k \frac{m+n+\alpha-2k}{m+n+\alpha-k} \frac{(m+n+2\alpha-2k,\ k)}{(m+n-2k+1,\ k)} \cdot$$

$$\cdot \frac{(\alpha,\ k)}{k\,!} \frac{(\alpha,\ m-k)}{(m-k)\,!} \frac{(\alpha,\ n-k)}{(n-k)\,!} \frac{(m+n-k)\,!}{(\alpha,\ m+n-k)} P_{m+n-2k}^{(\alpha)}(x)\,,$$

e stabiliamo dapprima la corrispondente a parametri complementari $(m \leqslant n)$

$$(2)\qquad P_n^{(\alpha)}(x)\ P_m^{(1-\alpha)}(x) = \sum_0^m{}_k \frac{m+n+\alpha-2k}{m+n+\alpha-k} \frac{(n+2\alpha-k,\ k)}{(n-k+1,\ k)} \cdot$$

$$\cdot \frac{(1-\alpha,\ k)}{k\,!} \frac{(1-\alpha,\ m-k)}{(m-k)\,!} \frac{(\alpha,\ n-k)}{(n-k)\,!} \frac{(m+n-k)\,!}{(\alpha,\ m+n-k)} P_{m+n-2k}^{(\alpha)}(x),$$

che insieme alla precedente si riduce per $\alpha = 1/2$ a quella di NEUMANN-ADAMS sui polinomi di LEGENDRE.

Poi stabiliamo le formule inverse $(m \leqslant n)$

$$(3)\qquad P_{n+m}^{(\alpha)}(x) = \frac{m\,!\ n\,!}{(m+n)\,!} \frac{1}{(\alpha,\ m)(\alpha,\ n)} \cdot$$

$$\cdot \sum_0^m{}_r \frac{m+n+2\alpha-2r}{m+n+2\alpha-r} \frac{(-\alpha,\ r)}{r\,!} (\alpha,\ m+n-r)(m+n+2\alpha-r,\ r) P_{n-r}^{(\alpha)}(x)\ P_{m-r}^{(\alpha)}(x)\,,$$

$$(4)\qquad P_{n+m}^{(\alpha)}(x) = \frac{m\,!\ n\,!}{(1-\alpha,\ m)(\alpha,\ n)} \cdot$$

$$\cdot \sum_0^m{}_r \frac{m+n-2r+1}{m+n-r+1} \frac{(\alpha-1,\ r)}{r\,!} \frac{(\alpha,\ m+n-r)}{(m+n-r)\,!} \frac{(n+2\alpha-r,\ r)}{(n-r+1,\ r)} P_{n-r}^{(\alpha)}(x)\ P_{m-r}^{(1-\alpha)}(x)\,,$$

che per $\alpha = {}^1/_2$ si riducono alla nota formula [1] $(m \leqslant n)$

$$P_{n+m}(x) = \frac{m!\,n!}{({}^1/_2,\ m)({}^1/_2,\ n)} \sum_0^m {}_r \frac{m+n-2r+1}{m+n-r+1} \frac{(-{}^1/_2,\ r)}{r!} \frac{({}^1/_2,\ m+n-r)}{(m+n-r)!} P_{n-r}(x)\, P_{m-r}(x).$$

Successivamente deduciamo dalle (2) e (3) la formula

$$(5) \qquad P_{n-1}^{(1-\alpha)}(x)\ P_{n+1}^{(\alpha)}(x) = \frac{(1-\alpha,\ n-1)}{(n-1)!} \frac{(\alpha,\ n+1)}{(n+1)!} \left[\frac{n!}{(\alpha,\ n)}\right]^2 \frac{1}{(\alpha,\ 2n)} \cdot$$

$$\cdot \sum_0^n {}_s (-1)^s \frac{2n+2\alpha-2s}{2n+2\alpha-s} \frac{(-\alpha,\ s)}{s!} (\alpha,\ 2n-s)(-2n-2\alpha+1,\ s) \cdot$$

$$\cdot\, {}_9F_8 \left[\begin{matrix} -2n-\alpha,\ -n-\frac{\alpha}{2}+1,\ -n+{}^1/_2,\ -n+1, \\ -n-\frac{\alpha}{2},\ -n-\alpha+{}^1/_2,\ -n-\alpha,\ -n,\ -2n, \end{matrix}\right.$$

$$\left.\begin{matrix} -n-\alpha+1,\ 1-\alpha,\ -n-2\alpha,\ -2n-2\alpha+s,\ -s;\ 1 \\ -n+\alpha+1,\ \alpha-s+1,\ -2n-\alpha+s+1; \end{matrix}\right] [P_{n-s}^{(\alpha)}(x)]^2 .$$

Quindi, la particolare eguaglianza notevole sui polinomi di LEGENDRE

$$(6) \qquad \frac{(n-1)!}{({}^1/_2,\ n-1)} P_{n-1}(x) \cdot \frac{(n+1)!}{({}^1/_2,\ n+1)} P_{n+1}(x) - \left[\frac{n!}{({}^1/_2,\ n)} P_n(x)\right]^2 =$$

$$= \frac{2(n-1)!\ (n-1)!}{({}^1/_2,\ n-1)({}^1/_2,\ n)} \sum_0^{n-1} {}_s \frac{[P_s(x)]^2}{(2s-1)(2s+3)},$$

da cui segue la disuguaglianza tipo TURÁN $(n > 1)$

$$(7) \qquad \left[\frac{n!}{({}^1/_2,\ n)} P_n(x)\right]^2 - \frac{(n-1)!}{({}^1/_2,\ n-1)} P_{n-1}(x) \cdot$$

$$\cdot \frac{(n+1)!}{({}^1/_2,\ n+1)} P_{n+1}(x) < \frac{2(n-1)!\ (n-1)!}{3({}^1/_2,\ n-1)({}^1/_2,\ n)} .$$

Infine specializziamo le (1), (2), (3), (4), (5) con processo limite che conduce a formule sui polinomi di HERMITE, di cui le (9), (11), (12) del n. **9** sono nuove.

Pertanto, con la ricerca di formule nuove sui polinomi ultrasferici a parametri complementari rispetto all'unità, veniamo a stabilire nuovi e notevoli risultati anche per i polinomi di LEGENDRE e di HERMITE.

Ulteriori sviluppi del prodotto di due polinomi ultrasferici, a parametri complementari rispetto all'unità, si potrebbero fare estendendo una recente Nota di L. CARLITZ [3].

3. – BLAGOJ S. POPOV ha dato [7] la formula di composizione dei polinomi ultrasferici $(m \leqslant n)$

$$\text{(F)} \qquad P_n^{(\alpha,\alpha)}(x)\; P_m^{(\beta,\beta)}(x) = (m + 2\beta + 1,\; m) \cdot$$

$$\cdot \sum_0^m {}_k \binom{m+n-2k}{n-k} \frac{(n+\alpha-k+1,\, k)(m+\beta-k+1,\, k)(m+n+\alpha-2k+1,\, k)\, 2^{2k}}{(2m+2\beta-2k+1,\, 2k)(2n+2\alpha-2k+1,\, 2m)\, k!} \cdot$$

$$\cdot\, (n+2\alpha+1,\; m-k)(\beta+{}^1\!/_2,\; k)\, \frac{2m+2n+2\alpha-4k+1}{2m+2n+2\alpha-2k+1} \cdot$$

$$\cdot\, {}_4F_3 \begin{bmatrix} \beta-\alpha,\; k-m,\; {}^1\!/_2-\alpha,\; -k;\; 1 \\ n-k+1,\; k-m-n-2\alpha,\; {}^1\!/_2+\beta; \end{bmatrix} P_{m+n-2k}^{(\alpha,\alpha)}(x)\; .$$

Sostituiamo in essa α con $\alpha - {}^1\!/_2$, β con $\beta - {}^1\!/_2$, applichiamo le formule

$$P_n^{(\alpha-{}^1\!/_2,\, \alpha-{}^1\!/_2)}(x) = \frac{(\alpha+{}^1\!/_2,\; n)}{(2\alpha,\; n)}\, P_n^{(\alpha)}(x)\;, \qquad P_m^{(\beta-{}^1\!/_2,\, \beta-{}^1\!/_2)}(x) = \frac{(\beta+{}^1\!/_2,\; m)}{(2\beta,\; m)}\, P_m^{(\beta)}(x)\;,$$

e procediamo a varie trasformazioni per dare alla (F) una forma più semplice ed espressiva.

Si ha dapprima

$$P_n^{(\alpha)}(x)\; P_m^{(\beta)}(x) = \frac{(2\alpha,\; n)(2\beta,\; m)(m+2\beta,\; m)}{(\alpha+{}^1\!/_2,\; n)(\beta+{}^1\!/_2,\; m)} \cdot$$

$$\cdot \sum_0^m {}_k \binom{m+n-2k}{n-k} \frac{(n+\alpha-k+{}^1\!/_2,\, k)(m+\beta-k+{}^1\!/_2,\, k)(m+n+\alpha-2k+{}^1\!/_2,\, k)\, 2^{2k}}{(2m+2\beta-2k,\; 2k)(2n+2\alpha-2k,\; 2m)\, k!} \cdot$$

$$\cdot\, \frac{(n+2\alpha,\; m-k)(\beta,\; k)(\alpha+{}^1\!/_2,\; m+n-2k)}{(2\alpha,\; m+n-2k)}\, \frac{m+n+\alpha-2k}{m+n+\alpha-k} \cdot$$

$$\cdot\, {}_4F_3 \begin{bmatrix} \beta-\alpha,\; k-m,\; 1-\alpha,\; -k;\; 1 \\ n-k+1,\; k-m-n-2\alpha+1,\; \beta; \end{bmatrix} P_{m+n-2k}^{(\alpha)}(x)\; .$$

D'altra parte:

$$(2\beta,\ m)(m+2\beta,\ m)=(2\beta,\ 2m),$$

$$\frac{(2\beta,\ m)(m+2\beta,\ m)}{(2m+2\beta-2k,\ 2k)}=\frac{(2\beta,\ 2m)}{(2m+2\beta-2k,\ 2k)}=(2\beta,\ 2m-2k)\ ,$$

$$\frac{(m+\beta-k+{}^1/_2,\ k)}{(\beta+{}^1/_2,\ m)}=\frac{1}{(\beta+{}^1/_2,\ m-k)}\ ,$$

$$(2\beta,\ 2m-2k)=2^{2m-2k}\,(\beta,\ m-k)(\beta+{}^1/_2,\ m-k)\ ,$$

$$\frac{(2\beta,\ m)(m+2\beta,\ m)(m+\beta-k+{}^1/_2,\ k)}{(2m+2\beta-2k,\ 2k)(\beta+{}^1/_2,\ m)}=2^{2m-2k}(\beta,\ m-k);$$

$$(\alpha+{}^1/_2,\ m+n-2k)(m+n+\alpha-2k+{}^1/_2,\ k)=(\alpha+{}^1/_2,\ m+n-k),$$

$$\frac{(n+\alpha-k+{}^1/_2,\ k)}{(\alpha+{}^1/_2,\ n)}=\frac{1}{(\alpha+{}^1/_2,\ n-k)}\ ,$$

$$(2\alpha,\ n)(n+2\alpha,\ m-k)=(2\alpha,\ m+n-k)\ ,$$

$$(2n+2\alpha-2k,\ 2m)=2^{2m}\,(n+\alpha-k,\ m)(n+\alpha-k+{}^1/_2,\ m),$$

$$(\alpha+{}^1/_2,\ m+n-2k)(m+n+\alpha-2k+{}^1/_2,\ k)\ \frac{(n+\alpha-k+{}^1/_2,\ k)}{(\alpha+{}^1/_2,\ n)}\ \cdot$$

$$\cdot\ \frac{(2\alpha,\ n)(n+2\alpha,\ m-k)}{(2n+2\alpha-2k,\ 2m)}=\frac{(\alpha+{}^1/_2,\ m+n-k)\ (2\alpha,\ m+n-k)}{(\alpha+{}^1/_2,\ n-k)2^{2m}(n+\alpha-k,\ m)(n+\alpha-k+{}^1/_2,\ m)}=$$

$$=\frac{(2\alpha,\ m+n-k)}{2^{2m}\,(n+\alpha-k,\ m)}\ .$$

Risalendo e sostituendo si ottiene la formula ($m\leqslant n$):

$$\text{(F}')\qquad P_n^{(\alpha)}(x)\,P_m^{(\beta)}(x)=\sum_0^m{}_k\ \frac{m+n+\alpha-2k}{m+n+\alpha-k}\ \frac{(m+n+2\alpha-2k,\ k)}{(m+n-2k+1,\ k)}\ \cdot$$

$$\cdot\ \frac{(\alpha,\ n-k)}{(n-k)\,!}\ \frac{(\beta,\ m-k)}{(m-k)\,!}\ \frac{(\beta,\ k)}{k\,!}\ \frac{(m+n-k)\,!}{(\alpha,\ m+n-k)}\ \cdot$$

$$\cdot\ {}_4F_3\left[\begin{matrix}\beta-\alpha,\ k-m,\ 1-\alpha,\ -k;\ 1\\ n-k+1,\ k-m-n-2\alpha+1,\ \beta;\end{matrix}\right]\ P_{m+n-2k}^{(\alpha)}(x)\ .$$

4. – Dalla (F′), facendo $\beta = \alpha$, quindi ${}_4F_3 = 1$, si ha la formula (1), pure dedotta da Popov nella forma che discende dalla (F).

Si faccia poi $\beta = 1 - \alpha$. Si ha

$$ {}_4F_3 = {}_3F_2 \begin{bmatrix} 1 - 2\alpha, \ k - m, \ -k; \ 1 \\ n - k + 1, \ k - m - n - 2\alpha + 1; \end{bmatrix}, $$

e questa ultima espressione è un *saalschützziano.*

Applicando la formula

$$ {}_3F_2 \begin{bmatrix} a, b, -r; 1 \\ c, 1 + a + b - c - r; \end{bmatrix} = \frac{(c - a, r)(c - b, r)}{(c, r)(c - a - b, r)} $$

segue

$$ {}_4F_3 = \frac{(n + 2\alpha - k, k)(m + n - 2k + 1, k)}{(n - k + 1, k)(m + n + 2\alpha - 2k, k)} . $$

Quindi, risalendo alla (F′), si deduce la nuova formula (2).

5. – La formula (1) ha per inversa la (3), data al n. **2.** Per verificarla esprimiamo con la (1) il prodotto $P_{m-r}^{(\alpha)}(x) \ P_{n-r}^{(\alpha)}(x)$ del suo secondo membro.

Si ha

$$ P_{n+m}^{(\alpha)}(x) = \frac{m! \, n!}{(m + n)!} \ \frac{1}{(\alpha, m) \ (\alpha, n)} \cdot $$

$$ \cdot \sum_0^m {}_r \left[\frac{m + n + 2\alpha - 2r}{m + n + 2\alpha - r} \ \frac{(-\alpha, r)}{r!} (\alpha, m + n - r)(m + n + 2\alpha - r, r) \cdot \right. $$

$$ \cdot \sum_0^{m-r} {}_s \frac{m + n + \alpha - 2r - 2s}{m + n + \alpha - 2r - s} \ \frac{(m + n + 2\alpha - 2r - 2s, s)}{(m + n - 2r - 2s + 1, s)} \ \frac{(\alpha, s)}{s!} \ \frac{(\alpha, m - r - s)}{(m - r - s)!} \cdot $$

$$ \left. \cdot \frac{(\alpha, n - r - s)}{(n - r - s)!} \ \frac{(m + n - 2r - s)!}{(\alpha, m + n - 2r - s)} P_{m+n-2r-2s}^{(\alpha)}(x) \right] , $$

da cui, con $r+s=k$,

$$P^{(\alpha)}_{n+m}(x)=\frac{m!\ n!}{(m+n)!}\frac{1}{(\alpha,m)(\alpha,n)}\sum_0^m{}_k\left[\frac{(m+n+\alpha-2k)(\alpha,m-k)(\alpha,n-k)}{(m-k)!\ (n-k)!}P^{(\alpha)}_{m+n-2k}(x).\right.$$

$$\cdot\sum_0^k{}_r\frac{m+n+2\alpha-2r}{m+n+2\alpha-r}\frac{(-\alpha,r)}{r!}\frac{(\alpha,k-r)}{(k-r)!}\frac{(m+n+2\alpha-2k,k-r)}{(m+n-2k+1,k-r)}\cdot$$

$$\left.\cdot\frac{(\alpha,m+n-r)(m+n+2\alpha-r,r)(m+n-k-r)!}{(\alpha,m+n-k-r)(m+n+\alpha-k-r)}\right].$$

Con $k=0$ si ottiene al secondo membro il termine $P^{(\alpha)}_{n+m}(x)$, per cui resta da verificare la relazione

$$\sum_1^m{}_k\frac{(m+n+\alpha-2k)(\alpha,m-k)(\alpha,n-k)}{(m-k)!\ (n-k)!}P^{(\alpha)}_{m+n-2k}(x)\sum_0^k{}_r\frac{m+n+2\alpha-2r}{m+n+2\alpha-r}\frac{(-\alpha,r)}{r!}\cdot$$

$$\cdot\frac{(\alpha,k-r)}{(k-r)!}\frac{(m+n+2\alpha-2k,k-r)}{(m+n-2k+1,k-r)}\frac{(m+n-k-r)!}{(\alpha,m+n-k-r)}\frac{(\alpha,m+n-r)(m+n+2\alpha-r,r)}{m+n+\alpha-k-r}=0.$$

Intanto:

$$m+n+2\alpha-2r=(m+n+2\alpha)\frac{\left(\frac{-m-n-2\alpha+2}{2},r\right)}{\left(\frac{-m-n-2\alpha}{2},r\right)},$$

$$(\alpha,k-r)=\frac{(\alpha,k)}{(-1)^r(-\alpha-k+1,r)},\qquad (k-r)!=\frac{k!}{(-1)^r(-k,r)},$$

$$(m+n+2\alpha-2k,k-r)=\frac{(m+n+2\alpha-2k,k)}{(-1)^r(-m-n-2\alpha+k+1,r)},$$

$$(m+n-2k+1,k-r)=\frac{(m+n-2k+1,k)}{(-1)^r(-m-n+k,r)},$$

$$(m+n-k-r)!=\frac{(m+n-k)!}{(-1)^r(-m-n+k,r)},$$

$$(\alpha,m+n-k-r)=\frac{(\alpha,m+n-k)}{(-1)^r(-\alpha-m-n+k+1,r)},$$

$$(\alpha,m+n-r)=\frac{(\alpha,m+n)}{(-1)^r(-\alpha-m-n+1,r)},$$

$$(m+n+2\alpha-r,r)=(-1)^r(-m-n-2\alpha+1,r).$$

E risalendo, la precedente relazione diventa:

$$\sum_{1}^{m}{}_k \frac{(\alpha,\ k)(\alpha,\ m-k)(\alpha,\ n-k)}{k!\ (m-k)!\ (n-k)!}(m+n+\alpha-2k)(m+n-2k)!\ (m+n+\alpha-$$

$$-k+1,\ k-1)(m+n+2\alpha-2k,\ k)\cdot {}_5F_4\left[\begin{matrix}-m-n-2\alpha,\ \dfrac{-m-n-2\alpha+2}{2},\\ \dfrac{-m-n-2\alpha}{2};\ -m-n-\alpha+1,\end{matrix}\right.$$

$$\left.\begin{matrix}-\alpha,\ -m-n-\alpha+k,\ -k;\ 1\\ -\alpha-k+1, -m-n-2\alpha+k+1;\end{matrix}\right] P^{(\alpha)}_{m+n-2k}(x) = 0\,.$$

D'altra parte consideriamo la formula [2]

$${}_5F_4\left[\begin{matrix}a,\ 1+\dfrac{1}{2}a,\ c,\ d,\ -k;\ 1\\ \dfrac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a+k;\end{matrix}\right] = \frac{(1+a,\ k)(1+a-c-d,\ k)}{(1+a-c,\ k)(1+a-d,\ k)}\,,$$

e facciamo in essa

$$a = -m-n-2\alpha, \qquad c = -\alpha, \qquad d = -m-n-\alpha+k\,.$$

Segue:

$$1+a/2 = (-m-n-2\alpha+2)/2\,,$$

$$1+a-c = -m-n-\alpha+1, \qquad 1+a-d = -\alpha-k+1,$$

$$1+a+k = -m-n+2\alpha+k+1, \quad 1+a-c-d = -k+1,$$

$${}_5F_4\left[\begin{matrix}-m-n-2\alpha,\dfrac{-m-n-2\alpha+2}{2},\ -\alpha,\ -m-n-\alpha+k,\ -k;\ 1\\ \dfrac{-m-n-2\alpha}{2},\ -m-n-\alpha+1,\ -\alpha-k+1,\ -m-n+2\alpha+k+1;\end{matrix}\right] =$$

$$= \frac{(-m-n-2\alpha+1,\ k)(-k+1,\ k)}{(-m-n-\alpha+1,\ k)(-\alpha-k+1,\ k)} \qquad \text{con } k>0\,.$$

E poichè $(-k+1,\ k)=0$, da cui ${}_5F_4=0$, risulta verificata la relazione di sopra e stabilita la formula inversa (3).

6. – L'inversa della formula (2) è la (4) già presentata. E si può stabilire come la (3). Sostituendo il prodotto $P^{(\alpha)}_{n-r}(x)\ P^{(1-\alpha)}_{m-r}(x)$ con lo sviluppo (2) ed eliminato il termine $P^{(\alpha)}_{n+m}(x)$, resta da verificare la relazione

$$\sum_0^k {}_r\, \frac{m+n-2r+1}{m+n-r+1}\, \frac{(\alpha-1,\ r)}{r!}\, \frac{(1-\alpha,\ k-r)}{(k-r)!}\, \frac{(\alpha,\ m+n-r)}{(m+n-r)!}\ \cdot$$

$$\cdot\ \frac{(n+2\alpha-r,\ r)}{(n-r+1,\ r)}\, \frac{(n+2\alpha-k,\ k-r)}{(n-k+1,\ k-r)}\, \frac{(m+n-k-r)!}{(\alpha,\ m+n-k-r+1)}=0\,.$$

Operando le trasformazioni del paragrafo precedente, essa si può scrivere

$$\frac{(n-k)!\,(m+n-k)!}{n!\,(m+n)!\,k!}\,(1-\alpha,\ k)(m+n+\alpha-k+1,\ k-1)(n+2\alpha-k,\ k)\ \cdot$$

$$\cdot\ {}_5F_4\left[\begin{matrix} -m-n-1,\ \dfrac{-m-n+1}{2},\ \alpha-1,\ -m-n-\alpha+k,\ -k\,;1 \\ \dfrac{-m-n-1}{2},\ -m-n-\alpha+1,\ \alpha-k,\ -m-n+k; \end{matrix}\right]=0\,.$$

Inoltre è

$${}_5F_4=\frac{(-m-n,\ k)(1-k,\ k)}{(-m-n-\alpha+1,\ k)(\alpha-k,\ k)}=0 \qquad (k>0)\,.$$

Quindi risulta verificata la precedente relazione e stabilita la formula inversa (4).

7. – Riprendiamo le (2) e (3). Nella prima sostituiamo n con $n+1$ e facciamo $m=n-1$; nella seconda sostituiamo m ed n con $n-k$.

Così, per la (2), il prodotto $P^{(1-\alpha)}_{n-1}(x)\ P^{(\alpha)}_{n+1}(x)$ viene espresso con una combinazione lineare di termini del tipo $P^{(\alpha)}_{2n-2k}(x)$. A sua volta questo, per la (3), si esprime con una combinazione lineare di termini del tipo $[P^{(\alpha)}_{n-k-r}(x)]^2$. E nell'insieme il prodotto $P^{(1-\alpha)}_{n-1}(x)\ P^{(\alpha)}_{n+1}(x)$ risulta espresso con una combinazione lineare di termini del tipo $[P^{(\alpha)}_{n-s}(x)]^2$.

Precisamente si ha:

$$P_{n-1}^{(1-\alpha)}(x)\, P_{n+1}^{(\alpha)}(x) = \sum_0^{n-1}{}_k \left\{ \frac{2n+\alpha-2k}{2n+\alpha-k} \; \frac{(n+2\alpha-k+1,\ k)}{(n-k+2,\ k)} \; \frac{(1-\alpha,\ k)}{k!} \;\cdot\right.$$

$$\cdot \frac{(1-\alpha,\ n-k-1)}{(n-k-1)!} \; \frac{(\alpha,\ n-k+1)}{(n-k+1)!} \; \frac{(2n-k)!}{(\alpha,\ 2n-k)} \; \frac{(n-k)!\ (n-k)!}{(2n-k)!} \; \frac{1}{(\alpha,\ n-k)} \; \frac{1}{(\alpha,\ n-k)} \;\cdot$$

$$\left.\cdot \sum_0^{n-k}{}_r \frac{2n+2\alpha-2k-2r}{2n+2\alpha-2k-r} \; \frac{(-\alpha,\ r)}{r!} \; (\alpha,\ 2n-2k-r)(2n+2\alpha-2k-r,\ r)[P_{n-k-r}^{(\alpha)}(x)]^2 \right\} =$$

$$= \sum_0^{n}{}_s (2n+2\alpha-2s)[P_{n-s}^{(\alpha)}(x)]^2 \sum_0^{s}{}_k \frac{2n+\alpha-2k}{2n+\alpha-k} \; \frac{(1-\alpha,\ k)}{k!} \; \frac{(1-\alpha,\ n-k-1)}{(n-k-1)!} \;\cdot$$

$$\cdot \frac{(\alpha,\ n-k+1)}{(n-k+1)!} \; \frac{(n+2\alpha-k+1,\ k)}{(n-k+2,\ k)} \left[\frac{(n-k)!}{(\alpha,\ n-k)}\right]^2 \frac{(-\alpha,\ s-k)}{(s-k)!} \; \frac{(2n-k)!}{(\alpha,\ 2n-k)} \;\cdot$$

$$\cdot \frac{(2n+2\alpha-k-s,\ s-k)}{2n+2\alpha-k-s} \; \frac{(\alpha,\ 2n-k-s)}{(2n-2k)!} \,.$$

D'altra parte:

$$2n+\alpha-2k = (2n+\alpha)\, \frac{\left(-n-\frac{\alpha}{2}+1,\ k\right)}{\left(-n-\frac{\alpha}{2},\ k\right)} \,,$$

$$2n+\alpha-k = (2n+\alpha)\, \frac{(-2n-\alpha+1,\ k)}{(-2n-\alpha,\ k)} \,,$$

$$(1-\alpha,\ n-k-1) = \frac{(1-\alpha,\ n-1)}{(-1)^k(-n+\alpha+1,\ k)} \,, \quad (\alpha,\ n-k+1) = \frac{(\alpha,\ n+1)}{(-1)^k(-n-\alpha,\ k)} \,,$$

$$(n+2\alpha-k+1,\ k) = (-1)^k(-n-2\alpha,\ k), \quad (\alpha,\ n-k) = \frac{(\alpha,\ n)}{(-1)^k(-n-\alpha+1,\ k)} \,,$$

$$(-\alpha,\ s-k) = \frac{(-\alpha,\ s)}{(-1)^k(\alpha-s+1,\ k)} \,, \qquad (\alpha,\ 2n-k) = \frac{(\alpha,\ 2n)}{(-1)^k(-2n-\alpha+1,\ k)} \,,$$

$$(2n+2\alpha-k-s,\ s-k) = \frac{(2n+2\alpha-k-s,\ s)}{(-1)^k(-2n-2\alpha+k+1,\ k)} =$$

$$= \frac{(2n+2\alpha-k-s,\ s)(-2n-2\alpha+1,\ k)}{(-1)^k(-2n-2\alpha+k+1,\ k)(-2n-2\alpha+1,\ k)} =$$

$$= \frac{(-2n-2\alpha+1,\ k)(-1)^s(-2n-2\alpha+k+1,\ s)}{(-1)^k\ (-2n-2\alpha+1,\ 2k)} =$$

$$= \frac{(-1)^s(-2n-2\alpha+1,\ k+s)}{(-1)^k\ (-2n-2\alpha+1,\ 2k)} = \frac{(-1)^s(-2n-2\alpha+1,\ s)(-2n-2\alpha+s+1,\ k)}{(-1)^k\ 2^{2k}\ (-n-\alpha+\tfrac{1}{2},\ k)(-n-\alpha+1,\ k)},$$

$$2n+2\alpha-k-s = (2n+2\alpha-s)\ \frac{(-2n-2\alpha+s+1,\ k)}{(-2n-2\alpha+s,\ k)},$$

$$(\alpha,\ 2n-k-s) = \frac{(\alpha,\ 2n-s)}{(-1)^k(-2n-\alpha+s+1,\ k)}, \quad (n-k-1)\,! = \frac{(n-1)\,!}{(-1)^k(-n+1,\ k)},$$

$$(n-k+1)\,! = \frac{(n+1)\,!}{(-1)^k(-n-1,\ k)}, \qquad (n-k+2,\ k) = (-1)^k\ (-n-1,\ k),$$

$$(n-k)\,! = \frac{n\,!}{(-1)^k(-n,\ k)}, \qquad (s-k)\,! = \frac{s\,!}{(-1)^k(-s,\ k)},$$

$$(2n-k)\,! = \frac{(2n)\,!}{(-1)^k(-2n,\ k)}, \quad (2n-2k)\,! = \frac{(2n)\,!}{(-2n,\ 2k)} = \frac{(2n)\,!}{2^{2k}(-n,k)(-n+\tfrac{1}{2},k)}.$$

Risalendo, si trova che l'ultima somma rispetto all'indice k si può presentare nella forma ipergeometrica:

$$\frac{(1-\alpha,\ n-1)}{(n-1)\,!}\ \frac{(\alpha,\ n+1)}{(n+1)\,!}\left[\frac{n\,!}{(\alpha,\ n)}\right]^2 \frac{1}{(\alpha,\ 2n)}\cdot(-1)^s\ \frac{(-\alpha,\ s)}{s\,!}\ \frac{(-2n-2\alpha+1,\ s)}{2n+2\alpha-s}\ (\alpha, 2n-s).$$

$$\cdot\ {}_9F_8 \left[\begin{matrix} -2n-\alpha,\ -n-\frac{\alpha}{2}+1,\ -n+\tfrac{1}{2},\ -n+1,\ -n-\alpha+1,\ 1-\alpha, \\ -n-\frac{\alpha}{2},\ -n-\alpha+\tfrac{1}{2},\ -n-\alpha,\ -n,\ -2n,\ -n+\alpha+1, \end{matrix}\right.$$

$$\left.\begin{matrix} -n-2\alpha,\ -2n-2\alpha+s,\ -s;\ 1 \\ \alpha-s+1,\quad -2n-\alpha+s+1\ ; \end{matrix}\right].$$

Quindi risulta provata la (5).

8. – Consideriamo ora la formula ipergeometrica [2]:

$$
{}_9F_8\left[\begin{matrix} h, 1+\frac{h}{2}, \frac{a+1}{2}, 1+\frac{a}{2}, h+b-a, h+c-a, h+d-a, 1+a-w, -m; 1 \\ \frac{h}{2}, h+\frac{1-a}{2}, h-\frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, h-a+w, 1+h+m; \end{matrix}\right] =
$$

$$
= \frac{(1+h,\ m)(2h-2a,\ m)}{(2h-a,\ m)(h-a,\ m)}\ {}_6F_5\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ -m;\ 1 \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ w; \end{matrix}\right],
$$

con $w = 1 + 2a - 2h - m$, $h = 1 + 2a - b - c - d$. E facciamo in essa

$$
a = -2n,\quad b = -n+1,\quad c = 1,\quad d = -n-\alpha,\quad h = -2n-\alpha,\quad m = s\,.
$$

Così la ${}_9F_8$ coincide con la nostra della formula (5). Perchè, però, si possa ridurre alla ${}_6F_5$ occorre, per la condizione $h = 1 + 2a - b - c - d$, che sia $\alpha = {}^1/_2$.

Supponiamo soddisfatta questa condizione (relativa ai polinomi di LEGENDRE) e operiamo la riduzione.

Esaminiamo dapprima il rapporto $(s > 0)$

$$
R(s,\ t) = \frac{(2h-2a,\ m)}{(w,\ t)} = \frac{(-2\alpha,\ s)}{(2\alpha-s+1,\ t)} = (-1)^t(-2\alpha,\ s-t)\,,
$$

con t nullo o intero tale che sia $0 \leqslant t \leqslant s$.

Per $\alpha = {}^1/_2$ si ha

$$
R(s,\ t) = (-1)^t(-1,\ s-t)\,,
$$

quindi:

$$
R(s,\ s) = (-1)^s,\qquad R(s,\ s-1) = (-1)^s,\qquad R(s,\ t < s-1) = 0\,.
$$

Allora per la nostra ${}_9F_8$ con $\alpha = {}^1/_2$ si ha $(s > 0)$

$$
{}_9F_8 = \frac{(-2n+\frac{1}{2},\ s)\,(-1)^s}{(-2n-1,\ s)\,(-{}^1/_2,\ s)}\left[\frac{(-n+1,\ s)^2\,(-n-\frac{1}{2},\ s)(-s,\ s)}{(-n,\ s)^2\,(-n+{}^3/_2,\ s)} + \right.
$$

$$+\left.\frac{(-n+1,\ s-1)^2\,(-n-\tfrac{1}{2},\ s-1)(-s,\ s-1)}{(-n,\ s-1)^2\,(-n+{}^3/_2,\ s-1)}\right]=$$

$$=\frac{(-2n+\tfrac{1}{2},\ s)\ s\,!}{(-2n-1,\ s)(-\tfrac{1}{2},\ s)}\ \frac{n-s+1}{n^2}\ \frac{(2n+1)(2n-1)}{(2n-2s+1)(2n-2s+3)}\left[\frac{2n-2s+3}{2n-2s-1}\,\cdot\right.$$

$$\left.\cdot\,\frac{(n-s)^2}{(n-s+1)^2}-1\right]=\frac{(2n+1)(2n-1)\ s\,!\,(-2n+{}^1/_2,\ s)}{n^2(-2n-1,\ s)(-{}^1/_2,\ s)(2n-2s-1)(2n-2s+1)(2n-2s+3)}\,.$$

Risalendo viene:

$$P_{n-1}(x)\ P_{n+1}(x)=\frac{(\tfrac{1}{2},\ n-1)(\tfrac{1}{2},\ n+1)}{(n-1)\,!\,(n+1)\,!}\left[\frac{n\,!}{(\tfrac{1}{2},\ n)}\right]^2[P_n(x)]^2+$$

$$+\frac{(\tfrac{1}{2},n-1)(\tfrac{1}{2},\ n+1)}{(n-1)\,!\ (n+1)\,!}\left[\frac{n\,!}{(\tfrac{1}{2},\ n)}\right]^2\frac{1}{(\tfrac{1}{2},\ 2n)}\sum_{1}^{n}{}_s\,(-1)^s\,\frac{2n-2s+1}{2n-s+1}\,\frac{(-\tfrac{1}{2},\ s)}{s\,!}\,\cdot$$

$$\cdot\,({}^1/_2,\ 2n-s)(-2n,\ s)\,\frac{(2n+1)(2n-1)\ s\,!\ (-2n+\tfrac{1}{2},\ s)[P_{n-s}(x)]^2}{n^2(-2n-1,\ s)(-\tfrac{1}{2},\ s)(2n-2s-1)(2n-2s+1)(2n-2s+3)}\,,$$

da cui la formula

$$\frac{(n-1)\,!}{(\tfrac{1}{2},\ n-1)}\,P_{n-1}(x)\,\cdot\,\frac{(n+1)\,!}{(\tfrac{1}{2},\ n+1)}\,P_{n+1}(x)-\left[\frac{n\,!}{(\tfrac{1}{2},\ n)}\,P_n(x)\right]^2=$$

$$=\frac{2(n-1)\,!\ (n-1)\,!}{(\tfrac{1}{2},\ n-1)(\tfrac{1}{2},\ n)}\sum_{1}^{n}{}_s\,\frac{[P_{n-s}(x)]^2}{(2n-2s-1)(2n-2s+3)}\,,$$

equivalente alla (6).

Dalla (6) segue facilmente la disuguaglianza (7) tipo Turán [4].

9. – Le formule (1), (2), (3), (4), (5) si possono specializzare per polinomi di Hermite:

$$H_{2n}(x)=\frac{(-1)^n\,(2n)\,!}{2^n\cdot n\,!}\ {}_1F_1(-n;\ {}^1/_2;\ x^2/_2)\,,$$

$$H_{2n+1}(x)=\frac{(-1)^n(2n+1)\,!}{2^n\cdot n\,!}\ x\,{}_1F_1(-n;\ {}^3/_2;\ x^2/_2),$$

applicando le formule limite [8]

$$\lim_{\alpha\to\infty}\left\{\alpha^{-n/2}\,P_n^{(\alpha)}(x/\sqrt{\alpha})\right\}=\frac{2^{n/2}}{n\,!}\,H_n(x\sqrt{2})\,,$$

$$\lim_{\alpha\to\infty}\left\{\alpha^{-n/2}\,P_n^{(1-\alpha)}(x/\sqrt{\alpha})\right\}=\frac{2^{n/2}}{n\,!}\,i^nH_n(ix\sqrt{2})\qquad (i^2=-1)$$

e calcolando dei limiti particolari.

Si faccia pure la posizione

$$\mathcal{H}_n(x)=i^n\,H_n(-\,ix)=(-\,i)^n\,H_n(ix)\,.$$

Si ottengono infine le formule ($m\leqslant n$):

$$H_m(x)\,H_n(x)=\sum_0^m{}_k\binom{m}{k}\binom{n}{k}k\,!\,H_{m+n-2k}(x)\,,\tag{8}$$

$$\mathcal{H}_m(x)\,H_n(x)=\sum_0^m{}_k\binom{m}{k}\frac{(m+n-k)\,!}{(m+n-2k)\,!}\,H_{m+n-2k}(x)\tag{9}$$

che si può ancora presentare nella forma

$$H_m(x)\,\mathcal{H}_n(x)=\sum_0^m{}_k\,(-\,1)^k\binom{m}{k}\frac{(m+n-k)\,!}{(m+n-2k)\,!}\,\mathcal{H}_{m+n-2k}(x),\tag{9'}$$

$$H_{m+n}(x)=\sum_0^m{}_k(-\,1)^k\binom{m}{k}\binom{n}{k}k\,!\,H_{m-k}(x)\,H_{n-k}(x),\tag{10}$$

$$H_{m+n}(x)=\sum_0^m{}_k(-\,1)^k\binom{m}{k}\binom{m+n}{k}k\,!\,\frac{m+n-2k+1}{m+n-k+1}\,\mathcal{H}_{m-k}(x)\;\;H_{n-k}(x)\,,\tag{11}$$

che si può ancora scrivere

$$\mathcal{H}_{m+n}(x)=\sum_0^m{}_k\binom{m}{k}\binom{m+n}{k}k\,!\,\frac{m+n-2k+1}{m+n-k+1}\,H_{m-k}(x)\;\;\mathcal{H}_{n-k}(x),\tag{11'}$$

$$\mathcal{H}_{n-1}(x)\,H_{n+1}(x)=\sum_0^n{}_k(-1)^k\binom{n}{k}^2k\,!\;{}_3F_2(-\kappa,-n+\tfrac{1}{2},-n+1;-n,-2n;\,4)H^2_{n-s}(x).\tag{12}$$

Le (8) e (10) sono ben note formule di N. Nielsen [6]. Le (9), (11), (12) sono nuove e si ricollegano ad analoghe già da me assegnate [10].

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Summary.

Some new formulas for ultraspherical polynomials having complementary parameters respect to the unity, for Legendre *and* Hermite *polynomials, are established.*

* * *