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A Class of Polynomials Analogous to Lommel Polynomials. (**)

1. — Introduction.

In 1932, MITRA [2] considered a class of polynomials associated with the parabolic cylinder function $D_n(x)$ given by

$$D_n(x) = (-1)^n \, e^{1/(4x^2)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-1/(2x^2)} .$$

They were based on the recurrence relation

$$D_{n+1}(x) - x\, D_n(x) + n\, D_{n-1}(x) = 0$$

in the same way as LOMMEL polynomials are related to the BESSEL function by means of [4]

$$J_{\nu+1}(z) = (2\nu/z)\, J_\nu(z) - J_{\nu-1}(z).$$

Since in 1949, MITRA-SHARMA [3] considered a generalization of the parabolic cylinder function, it seems worthwhile to generate the class of polynomials analogous to LOMMEL polynomials corresponding to their generalization and to

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bring out some of its properties. As a matter of interest, we will thus be able to construct a class of polynomials satisfying a k-th order recurrence relation instead of order two [1].

2. — The genesis of the polynomials $S_{\alpha,\beta}(y)$.

Mitra and Sharma [3] define $\overline{D}_{km}(y)$ and $\overline{D}_{km+1}(y)$ by

$$\overline{D}_{km}(y) = e^{y^k/(2k)}\, \mathrm{D}_{(k)}^{(km)}\, e^{-y^k/k}$$

and

$$\overline{D}_{km+1}(y) = - e^{y^k/(2k)}\, \mathrm{D}_{(k)}^{(km+k-1)}\, e^{-y^k/k},$$

where $\mathrm{D}_{(k)}^{(k)}$ stands for the operator $\dfrac{d}{dy}\dfrac{1}{y^{k-2}}\dfrac{d}{dy}$ and $\mathrm{D}_{(k)}^{(km)}$ means repetition of $\mathrm{D}_{(k)}^{(k)}$ m times. Also

$$\mathrm{D}_{(k)}^{(km+1)} \equiv \frac{d}{dy}\, \mathrm{D}_{(k)}^{(km)}$$

and

$$\mathrm{D}_{(k)}^{(km+k-1)} \equiv \frac{1}{y^{k-2}}\frac{d}{dy}\, \mathrm{D}_{(k)}^{(km)}.$$

The functions $\overline{D}_{km}(y)$ and $\overline{D}_{km+1}(y)$ satisfy the recurrence relations

$$\overline{D}_{km}(y) - y^{k-1}\, \overline{D}_{km-k+1}(y) + (km - k + 1)\, \overline{D}_{km-k}(y) = 0 \tag{2.1}$$

and

$$\overline{D}_{km+1}(y) - y\, \overline{D}_{km}(y) + km\, \overline{D}_{km-k+1}(y) = 0 . \tag{2.2}$$

From (2.1) and (2.2) we have, by the classical method ([4], [1]), the following four relations:

$$(A) = 0, \qquad (B) = 0, \qquad (C) = 0, \qquad (D) = 0,$$

where (A), (B), (C), (D) are respectively the determinants

$$\begin{vmatrix} \overline{D}_{km+pk+1}(y) & -y & km+pk & 0 & \dots & 0 & 0 \\ 0 & 1 & -y^{k-1} & km+pk-k+1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -y^{k-1} & km+k+1 \\ (km+k)\,\overline{D}_{km+1}(y) & 0 & 0 & 0 & \dots & 1 & y \\ -y^{k-1}\,\overline{D}_{km+1}(y) + (km+1)\,\overline{D}_{km}(y) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix},$$

$$\begin{vmatrix} \overline{D}_{km+pk+1}(y) & -y & km+pk & 0 & \dots & 0 & 0 \\ 0 & 1 & -y^{k-1} & km+pk-k+1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (km+1)\,\overline{D}_{km}(y) & 0 & 0 & 0 & \dots & 1 & -y^{k-1} \\ km\overline{D}_{km-k+1}(y) - y\overline{D}_{km}(y) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix},$$

$$\begin{vmatrix} \overline{D}_{km+pk}(y) & -y^{k-1} & km+pk-k+1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -y & km+pk-k & \dots & 0 & 0 \\ 0 & 0 & 1 & -y^{k-1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (km+k)\,\overline{D}_{km+1}(y) & 0 & 0 & 0 & \dots & 1 & -y \\ (km+1)\,\overline{D}_{km}(y) - y^{k-1}\,\overline{D}_{km+1}(y) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix},$$

$$\begin{vmatrix} \overline{D}_{km+pk}(y) & -y^{k+1} & km+pk-k+1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -y & km+pk-k & \dots & 0 & 0 \\ 0 & 0 & 1 & -y^{k-1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -y & km+k \\ (km+1)\,\overline{D}_{km}(y) & 0 & 0 & 0 & \dots & 1 & -y^{k-1} \\ km\overline{D}_{km-k+1}(y) - y\overline{D}_{km}(y) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

When we expand the determinant (A) in terms of the elements of the first column, we have after a little simplification

$$(2.3) \qquad \overline{D}_{km+pk+1}(y) + (km+1)\,\overline{D}_{km}(y)\cdot S_{km,pk-k+1}(y) - \\ - \overline{D}_{km+1}(y)\cdot S_{km-k+1,pk}(y) = 0,$$

where

$$(-1)^{2p-1}\, S_{km,pk-k+1}(y) = (E), \qquad (-1)^{2p}\, S_{km-k+1,pk}(y) = (F)$$

and (E), (F) are respectively the determinants

$$\begin{vmatrix} -y & km+pk & 0 & \dots & 0 & 0 \\ 1 & -y^{k-1} & km+pk-k+1 & \dots & 0 & 0 \\ 0 & 1 & -y & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -y^{k-1} & km+k+1 \\ 0 & 0 & 0 & \dots & 1 & -y \end{vmatrix},$$

$$\begin{vmatrix} -y & km+pk & 0 & \dots & 0 & 0 & 0 \\ 1 & -y^{k-1} & km+pk-k+1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -y & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -y^{k-1} & km+k+1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -y & km+k \\ 0 & 0 & 0 & \dots & 0 & 1 & -y^{k-1} \end{vmatrix}.$$

Here the suffixes $pk-k+1$ and pk in $S_{km,pk-k+1}(y)$ and $S_{km-k+1,pk}(y)$ respectively denote the degree of the respective polynomials.

Similarly, from (B), we have

$$(2.4) \qquad \overline{D}_{km+pk+1}(y) + km\,\overline{D}_{km-k+1}(y)\cdot S_{km-k+1,pk}(y) - \\ - \overline{D}_{km}(y)\cdot S_{km-k,pk+1}(y) = 0\,.$$

From (C) and (D) we respectively have

$$(2.5)\qquad \overline{D}_{km+pk}(y) + (km+1)\,\overline{D}_{km}(y)\cdot S_{km,pk-k}(y) - \overline{D}_{km+1}(y)\cdot S_{km-k+1,pk-1}(y) = 0$$

and

$$(2.6)\qquad \overline{D}_{km+pk}(y) + (km+k)\,\overline{D}_{km+1}(y)\cdot S_{km+1,pk-k+1}(y) - \overline{D}_{km+k}(y)\cdot S_{km,pk-k}(y) = 0,$$

where

$$(-1)^{2p-2}\, S_{km,pk-k}(y) = (G), \qquad (-1)^{2p-1}\, S_{km-k+1,pk-1}(y) = (H)$$

and (G), (H) are respectively the determinants

$$\begin{vmatrix} -y^{k-1} & km+pk-k+1 & 0 & \dots & 0 & 0 \\ 1 & -y & km+pk-k & \dots & 0 & 0 \\ 0 & 1 & -y^{k-1} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -y^{k-1} & km+k+1 \\ 0 & 0 & 0 & \dots & 1 & -y \end{vmatrix},$$

$$\begin{vmatrix} -y^{k-1} & km+pk+1-k & 0 & \dots & 0 & 0 & 0 \\ 1 & -y & km+pk-k & \dots & 0 & 0 & 0 \\ 0 & 1 & -y^{k-1} & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -y^{k-1} & km+k+1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -y & km+k \\ 0 & 0 & 0 & \dots & 0 & 1 & -y^{k-1} \end{vmatrix},$$

3. — Recurrence relations and other properties.

From (2.3) and (2.4) we have the relation

$$(3.1)\qquad (km+1)\ S_{km,pk-k+1}(y) - y\ S_{km-k+1,pk}(y) + S_{km-k,pk+1}(y) = 0,$$

and (2.5) and (2.6) give

$$(3.2)\qquad (km+k)\ S_{km+1,pk-k-1}(y) - y^{k-1}\ S_{km,pk-k}(y) + S_{km-k+1,pk-1}(y) = 0\ .$$

Also from (E), (F), (G), and (H) we have by expanding the determinants the following relations:

$$(3.3)\qquad S_{km,pk-k+1}(y) + (km+pk)\ S_{km,pk-2k+1}(y) = y\ S_{km,pk-k}(y),$$

$$(3.4)\qquad S_{km,pk-k}(y) + (km+pk-k+1)\ S_{km,pk-2k}(y) = y^{k-1}\ S_{km,pk-2k+1}(y),$$

$$(3.5)\qquad S_{km-k+1,pk}(y) + (km+pk)\ S_{km-k+1,pk-k}(y) = y\ S_{km-k+1,pk-1}(y),$$

$$(3.6)\qquad S_{km-k+1,pk-1}(y) + (km+pk-k+1)\ S_{km-k+1,pk-k-1}(y) = y^{k-1}\ S_{km-k+1,pk-k}(y)\ .$$

We now give some polynomials for $p = 0,\ 1$:

$$S_{km,1}(y) = y, \qquad S_{km-k+1,k}(y) = y^k - (km+k),$$

$$S_{km,k}(y) = y^k - (km+k+1), \qquad S_{km-k+1,k-1}(y) = y^{k-1},$$

$$S_{km-k+1,0}(y) = S_{km,0} = 1\ .$$

Relations (3.1) and (3.2) also enable us to define the polynomials for negative values of p. Thus

$$S_{km+1,-1}(y) = 0, \qquad S_{km,-k+1}(y) = 0\ .$$

From (3.3), it then follows that $S_{km,-k}(y) = 1$.

From these we can write the following ratios of these polynomials in the form of a continued fraction:

$$(3.7)\qquad \frac{S_{km,\,pk-k+1}(y)}{S_{km,\,pk-k}(y)} =$$

$$= y - \frac{km+pk}{y^{k-1}} - \frac{km+pk-k+1}{y} - \cdots - \frac{km+2k}{y^{k-1}} - \frac{km+k+1}{y},$$

$$(3.8)\qquad \frac{S_{km-k+1,\,pk}(y)}{S_{km-k+1,\,pk-1}(y)} =$$

$$= y - \frac{km+pk}{y^{k-1}} - \frac{km+pk-k+1}{y} - \cdots - \frac{km+2k}{y^{k-1}} - \frac{km+k+1}{y^{k-1}}.$$

Similarly we can write down the continued fraction equivalent to

$$\frac{S_{km-k+1,\,pk-1}(y)}{S_{km-k+1,\,pk-k}(y)} \quad \text{and} \quad \frac{S_{km,\,pk}(y)}{S_{km,\,pk-k+1}(y)}.$$

Again, writing $m-s$, $p+s$, and then $m+s$, $p-s$ for m and p respectively in succession in (3.3), we have on eliminating y:

$$(3.9)\qquad \begin{vmatrix} S_{km+ks,\,pk-sk-k}(y) & S_{km+ks,\,pk-sk-k+1}(y) \\ S_{km-ks,\,pk+sk-k}(y) & S_{km-ks,\,pk+sk-k+1}(y) \end{vmatrix} =$$

$$= (km+pk) \begin{vmatrix} S_{km+ks,\,pk-sk-2k+1}(y) & S_{km+ks,\,pk-sk-k}(y) \\ S_{km-ks,\,pk+sk-2k+1}(y) & S_{km-ks,\,pk+sk-k}(y) \end{vmatrix}.$$

Similarly, on applying the same method to (3.4) we have

$$(3.10)\qquad \begin{vmatrix} S_{km+ks,\,pk-sk-2k+1}(y) & S_{km+ks,\,pk-sk-k}(y) \\ S_{km-ks,\,pk+sk-2k+1}(y) & S_{km-ks,\,pk+sk-k}(y) \end{vmatrix} =$$

$$= (km+pk-k+1) \begin{vmatrix} S_{km+ks,\,pk-sk-2k}(y) & S_{km+ks,\,pk-sk-2k+1}(y) \\ S_{km-ks,\,pk+sk-2k}(y) & S_{km-ks,\,pk+sk-2k+1}(y) \end{vmatrix}.$$

From these we derive at once that

$$(3.11)\quad \begin{cases} \Delta_p(m,\, s,\, k;\, y) \equiv \begin{vmatrix} S_{km+ks,pk-sk-k}\,(y) & S_{km+ks,pk-sk-k+1}\,(y) \\ S_{km-ks,pk+sk-k}\,(y) & S_{km-ks,pk+sk-k+1}\,(y) \end{vmatrix} = \\ = (km + pk)\,(km + pk - k + 1)\;\Delta_{p-1}\,(m,\, s,\, k;\, y) = \\ = (km + pk)\,(km + pk - k + 1)\,(km + pk - k)\,(km + pk - 2k + 1) \ldots \\ \qquad \ldots (km + ks)\,(km + ks - k + 1)\,. \end{cases}$$

Also, (3.1) and (3.2) are similar to (2.1) and (2.2).

By repeating the same argument we can obtain a number of other relations. But, since our purpose is only to bring out the existence of the class of polynomials, we shall not give them here.

References.

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[4] G. N. WATSON: *A treatise on the theory of Bessel functions*, 2nd edition, Universitt Press, Cambridge 1952 (cf. pp. 294-303).