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Inversion Theorem for a Generalized Stieltjes Transform. (**)

1. — Introduction.

If we iterate the LAPLACE trasform, i.e., if we take

$$\chi(s) = \int_0^\infty e^{-st}\, \psi(t)\, dt,$$

where

(1) $$\psi(s) = \int_0^\infty e^{-st}\, \varphi(t)\, dt,$$

then

(2) $$\chi(s) = \int_0^\infty \frac{1}{s+t}\, \varphi(t)\, dt$$

and the equation (2) is referred to as the STIELTJES transform.

VARMA [8] has given a generalization of the LAPLACE transform in the form

(3) $$f(s) = \int_0^\infty (st)^{m-1/2}\, e^{-st/2}\ W_{k,m}(st)\ \psi(t)\, dt,$$

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where $W_{k,m}$ denotes a WITTAKER function. Further he has shown that if we take $f(s)$ to be the transform of $\psi(t)$ in the sense of (3) and $\psi(s)$ to be the ordinary LAPLACE transform of $\varphi(t)$ in the sense of (1), then

$$(4) \qquad f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty {}_2F_1 \left[\begin{matrix} 2m+1, \ 1 \\ m-k+3/2 \end{matrix} ; \ -\frac{t}{s} \right] \varphi(t) \, dt$$

provided that $\operatorname{Re}(2m+1) > 0$.

If $k + m = 1/2$, the hypergeometric function degenerates into a binomial expression and (4) reduces to (2). Also if $k - m = 1/2$ and $2m + 1 = \sigma$, (4) reduces to the form

$$(5) \qquad f_1(s) = \Gamma(\sigma) \, s^{\sigma-1} \int_0^\infty \frac{1}{(s+t)^\sigma} \varphi(t) \, dt$$

which is equivalent to the generalization of STIELTJES of order σ considered by POLLARD [6].

We shall call (4) as the VARMA's generalized STIELTJES transform of $\varphi(t)$. The object of this paper is to give an inversion theorem for (4).

2. — Inversion theorem.

Thereom A:

If $f(s)$ is the VARMA's generalized STIELTJES transform of $\varphi(t)$, i.e.,

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty {}_2F_1 \left[\begin{matrix} 2m+1, \ 1 \\ m-k+3/2 \end{matrix} ; \ -\frac{t}{s} \right] \varphi(t) \, dt,$$

then

$$(6) \qquad \frac{1}{2} [\varphi(t+) + \varphi(t-)] = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{c-i\tau}^{c+i\tau} \frac{\Gamma(m-k+l+1/2)}{\Gamma(2m+l) \, \Gamma(l) \, \Gamma(1-l)} t^{-l} \vartheta(l) \, dl$$

where

$$\vartheta(l) = \int_0^\infty s^{l-1} f(s) \, ds,$$

provided that

(i) $x^{c-1}\,\varphi(x) \in L\,(0,\ \infty)$,

(ii) $x^{l-1}\,f(x) \in L\,(0,\ \infty) \quad (l = c \pm i\tau, \quad -\infty < \tau < \infty)$,

(iii) $\varphi(x)$ is of bounded variation in the neighbourhood of the point $x = t\ (t > 0)$,

(iv) $\varphi(t) = O(t^{\varrho}), \qquad \mathrm{Re}\,(\varrho) > 0,\ (t \to 0)$

$= O\,(e^{-t^{\nu}}), \quad \mathrm{Re}\,(\nu) > 0,\ (t \to \infty)$,

(v) $1 > \mathrm{Re}\,(l) > 0,\ \mathrm{Re}\,(2m + l) > 0$ and $m - k + 3/2 \neq 0, -1, -2, \ldots$.

If $\varphi(t)$ is continuous at $x = t$, we have

$$\varphi(t) = \frac{1}{2\pi i} \lim_{\tau\to\infty} \int_{c-i\tau}^{c+i\tau} \frac{\Gamma\,(m - k + l + 1/2)}{\Gamma\,(2m + l)\ \Gamma\,(l)\ \Gamma\,(1 - l)}\, t^{-l}\,\vartheta(l)\ \mathrm{d}l.$$

Proof:

Multiplying both sides of (4) by s^{l-1} and integrating with respect to s from 0 to infinity, we have

$$\int_0^{\infty} s^{l-1} f(s)\ \mathrm{d}s = \frac{\Gamma\,(2m + 1)}{\Gamma\,(m - k + 3/2)} \int_0^{\infty} s^{l-2}\ \mathrm{d}s \int_0^{\infty} {}_2F_1\left[\begin{matrix} 2m + 1 \\ m - k + 3/2 \end{matrix};\ -\frac{t}{s}\right] \varphi(t)\ \mathrm{d}t =$$

$$= \frac{\Gamma\,(2m + 1)}{\Gamma\,(m - k + 3/2)} \int_0^{\infty} \varphi(t)\ \mathrm{d}t \int_0^{\infty} s^{l-2}\,{}_2F_1\left[\begin{matrix} 2m + 1,\ 1 \\ m - k + 3/2 \end{matrix};\ -\frac{t}{s}\right] \mathrm{d}s,$$

provided that we can justify the change in the order of integration.
But we have [2, p. 79, (4)]

$$\int_0^{\infty} s^{l-2}\ {}_2F_1\left[\begin{matrix} 2m + 1,\ 1 \\ m - k + 3/2 \end{matrix};\ -\frac{t}{s}\right] \mathrm{d}s = t^{l-1}\,\frac{\Gamma(2m + l)\ \Gamma(l)\ \Gamma(m - k + 3/2)\ \Gamma(1 - l)}{\Gamma\,(2m + 1)\ \Gamma\,(m - k + l + 1/2)},$$

provided that $1 > \mathrm{Re}\,(l) > 0$, $\mathrm{Re}\,(2m + l) > 0$ and $m - k + 3/2 \neq 0,\ -1,\ -2, \ldots$.

Hence

$$\int_0^\infty s^{l-1} f(s)\, ds = \frac{\Gamma(2m+l)\ \Gamma(l)\ \Gamma(1-l)}{\Gamma(m-k+l+1/2)} \int_0^\infty t^{l-1} \varphi(t)\, dt,$$

where $\operatorname{Re}(2m+l) > 0$, $1 > \operatorname{Re}(l) > 0$ and $m-k+3/2 \neq 0, -1, -2, \dots$.

If we apply the MELLIN's inversion formula [7] to the integral

$$\int_0^\infty t^{l-1} \varphi(t)\, dt = \frac{\Gamma(m-k+l+1/2)}{\Gamma(2m+l)\ \Gamma(l)\ \Gamma(1-l)}\, \vartheta(l),$$

we have

$$\frac{1}{2}\left[\varphi(t+) + \varphi(t-)\right] = \frac{1}{2\pi i} \lim_{\tau\to\infty} \int_{c-i\tau}^{c+i\tau} \frac{\Gamma(m-k+l+1/2)}{\Gamma(2m+l)\ \Gamma(l)\ \Gamma(1-l)}\, t^{-l}\, \vartheta(l)\, dl,$$

where

$$\vartheta(l) = \int_0^\infty s^{l-1} f(s)\, ds$$

provided that

(i) $c^{c-1} \varphi\, x) \in L(0, \infty)$,

(ii) $x^{l-1} f(x) \in L(0, \infty)$ $(l = c \pm i\tau, \ -\infty < \tau < \infty)$,

(iii) $\varphi(x)$ is of bounded variation in the neighbourhood of the point $x = t$ $(t > 0)$.

To justify the change in the order of integration, let

$$A(s) = s^{l-2} \int_0^\varepsilon {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+3/2\end{matrix};\ -\frac{t}{s}\right] \varphi(t)\, dt,$$

and

$$B(t) = \varphi(t) \int_0^\infty {}_2F_1 \left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix} ;\ -\frac{t}{s} \right] \mathrm{d}s\,.$$

We see that $A(s)$ is uniformly convergent in $s \geqslant 0$, provided that

$$\mathrm{Re}\,(\varrho+1) > 0, \qquad \mathrm{Re}\,(l-1+\varrho) \geqslant 0,$$

since

$${}_2F_1 \left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix} ;\ -v \right] = O(1) \qquad (v \to 0).$$

Also $B(t)$ is uniformly convergent in $t \geqslant 0$, provided that

$$\mathrm{Re}\,(\varrho) \geqslant 0,\ \ 1 > \mathrm{Re}\,(l) > 0, \quad \mathrm{Re}\,(2m+l) > 0, \quad m-k+3/2 \neq 0,\ -1,\ -2,\ \ldots,$$

since

$${}_2F_1 \left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix} ;\ -v \right] = O(v^{-2m-1}) \qquad (\mathrm{Re}\,(m) < 0,\ v \to \infty)$$

$$= O\,(v^{-1}) \qquad (\mathrm{Re}\,(m) > 0,\ v \to \infty)\,.$$

Further we consider the integral

$$\int_T^\infty |\,\varphi(t)\,|\,\mathrm{d}t \int_{T'}^\infty \left| s^{l-2}\,{}_2F_1 \left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix} ;\ -t/s \right] \right| \mathrm{d}s\,,$$

where T and T' are large. This integral does not exceed a constant multiple of

$$\int_T^\infty |\,e^{-t^\nu}\,|\,\mathrm{d}t \int_{T'}^\infty |\,s^{l-2}\,|\,\mathrm{d}s$$

which tends to zero if $\mathrm{Re}\,(\nu) > 0$ and $\mathrm{Re}\,(l) < 1$.

3. — Particular cases.

(1) If $k + m = 1/2$, we have:

Theorem B: If

$$\chi(s) = \int_0^\infty \frac{1}{s+t}\,\varphi(t)\,dt,$$

then

$$\frac{1}{2}\left[\varphi(t+) + \varphi(t-)\right] = \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{c-i\tau}^{c+i\tau} \frac{1}{\Gamma(l)\,\Gamma(1-l)}\,t^{-l}\,\vartheta_1(l)\,dl, \tag{7}$$

where

$$\vartheta_1(l) = \int_0^\infty s^{l-1}\,\chi(s)\,ds$$

provided that conditions (i) to (iv) of Theorem A hold and $1 > \mathrm{Re}(l) > 0$.

If $\varphi(t)$ is continuous (7) takes the form

$$\varphi(t) = \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{c-i\tau}^{c+i\tau} \frac{1}{\Gamma(l)\,\Gamma(1-l)}\,t^{-l}\,\vartheta_1(l)\,dl.$$

(2) If $k - m = 1/2$, $2m + 1 = \sigma$ we have:

Theorem C: If

$$f_1(s) = \Gamma(\sigma)\,s^{\sigma-1}\int_0^\infty \frac{1}{(s+t)^\sigma}\,\varphi(t)\,dt,$$

then

$$\frac{1}{2}\left[\varphi(t+) + \varphi(t-)\right] = \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{c-i\tau}^{c+i\tau} \frac{1}{\Gamma(\sigma-1+l)\,\Gamma(1-l)}\,t^{-l}\,\vartheta_2(l)\,dl, \tag{8}$$

where

$$\vartheta_2(l) = \int_0^\infty s^{l-1} f_1(s)\, ds,$$

provided that conditions (i) to (iv) of Theorem A hold and

(v) $1 > \operatorname{Re}(l) > 0, \quad \operatorname{Re}(l + \sigma - 1) > 0 .$

If $\varphi(t)$ is continuous, result (8) becomes

$$\varphi(t) = \frac{1}{2\pi i} \lim_{\tau\to\infty} \int_{c-i\tau}^{c+i\tau} \frac{1}{\Gamma(\sigma - 1 + l)\ \Gamma(1-l)}\, t^{-l}\, \vartheta_2(l)\, dl .$$

4. – We vill now give an example each for the verification of the theorems given above.

Example A. Let us take

$$\varphi(t) = t^{\mu/2}\, Y_\nu(t^{1/2}) .$$

Then [**3**, p. 118, (52)]

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty F\left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix};\ -\frac{t}{s}\right] \varphi(t)\, dt =$$

$$= 2^{1/2}\, s^{(\mu-1/2)/2}\, G_{35}^{41}\left(\frac{s}{4}\,\middle|\, \begin{matrix} 1/4-\mu/2,\ m-k+3/4-\mu/2,\ -1/4-\nu/2 \\ 1/4+\nu/2,\ 1/4-\nu/2,\ 2m+1/4-\mu/2,\ 1/4-\mu/2,\ -1/4-\nu/2 \end{matrix}\right),$$

provided that $\operatorname{Re}(s) > 0$, $\operatorname{Re}(\mu + 1/2) > |\operatorname{Re}(\nu)| - 3/2$, $\operatorname{Re}(\mu + 1/2) < 2\cdot\operatorname{Re}(2m+1)$, $\operatorname{Re}(\mu + 1/2) < 2$; and [**2**, p. 337, (14)]

$$\vartheta(l) = \int_0^\infty s^{l-1} f(s)\, ds = 2^{2l+\mu} \int_0^\infty u^{l+\mu/2-5/4} \cdot$$

$$\cdot G_{35}^{41}\left(u\,\middle|\, \begin{matrix} 1/4-\mu/2,\ m-k+3/4-\mu/2,\ -1/4-\nu/2 \\ 1/4+\nu/2,\ 1/4-\nu/2,\ 2m+1/4-\mu/2,\ 1/4-\mu/2,\ -1/4-\nu/2 \end{matrix}\right) du =$$

$$= 2^{2l+\mu} \frac{\Gamma(l+\mu/2+\nu/2)\ \Gamma(l+\mu/2-\nu/2)\ \Gamma(2m+l)\ \Gamma(l)\ \Gamma(1-l)}{\Gamma(-l+\nu/2-\mu/2+3/2)\ \Gamma(m-k+l+1/2)\ \Gamma(l-\nu/2+\mu/2-1/2)},$$

provided that $-\min \operatorname{Re}(1/4+\nu/2,\ 1/4-\nu/2,\ 2m+1/4-\mu/2,\ 1/4-\mu/2) < \operatorname{Re}(l+\mu/2-1/4) < 1-\max \operatorname{Re}(1/4-\mu/2,\ m+k+3/4-\mu/2,\ -1/4-\nu/2)$.

Hence

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(m-k+l+1/2)}{\Gamma(2m+l)\ \Gamma(l)\ \Gamma(1-l)}\ t^{-l}\,\vartheta(l)\,\mathrm{d}l =$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{2l+\mu}\ \frac{\Gamma(l+\mu/2+\nu/2)\ \Gamma(l+\mu/2-\nu/2)}{\Gamma(-l+\nu/2-\mu/2+3/2)\ \Gamma(l-\nu/2+\mu/2-1/2)}\ t^{-l}\,\mathrm{d}l = t^{\mu/2}\ Y_\nu(t^{1/2})$$

since [1, p. 219, (46)]

$$x^{\mu/2}\ Y_\nu(x^{1/2}) = 2^\mu\ G_{13}^{20}\left(\frac{x}{4}\,\middle|\, \begin{matrix} \mu/2-\nu/2-1/2 \\ \mu/2-\nu/2,\ \mu/2+\nu/2,\ \mu/2-\nu/2-1/2 \end{matrix}\right) =$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{2l+\mu}\ \frac{\Gamma(l+\mu/2+\nu/2)\ \Gamma(l+\mu/2-\nu/2)}{\Gamma(-l+\nu/2-\mu/2+3/2)\ \Gamma(l-\nu/2+\mu/2-1/2)}\ x^{-l}\,\mathrm{d}l,$$

by definition of G function.

Example B. Let us take $\varphi(t) = t^\mu\, J_\nu(t^{1/2})$. Then

$$\chi(s) = \int_0^\infty \frac{1}{s+t}\,\varphi(t)\,\mathrm{d}t = \int_0^\infty \frac{1}{s+t}\ t^\mu\, J_\nu(t^{1/2})\,\mathrm{d}t =$$

$$= 2^{2\mu+2}\, s^{-1}\ G_{13}^{21}\left(\frac{s}{4}\,\middle|\, \begin{matrix} 1 \\ \mu+1+\nu/2,\ 1,\ \ \mu+1-\nu/2 \end{matrix}\right),$$

taking a particular case of the result [3, p. 81, (6)]

$$\int_0^\infty x^{\delta-\frac{1}{2}}\ {}_2F_1(\alpha,\ \beta;\ \gamma;\ -\lambda^2 x^2)\ J_\nu(xy)\ (xy)^{1/2}\,\mathrm{d}x =$$

$$= \frac{2^\delta\ \Gamma(\gamma)}{\Gamma(\alpha)\ \Gamma(\beta)}\ y^{-\delta-\frac{1}{2}}\ G_{24}^{31}\left(\frac{y^2}{4\lambda^2}\,\middle|\, \begin{matrix} 1,\ \ \gamma \\ (1+\delta+\nu)/2,\ \alpha,\ \beta,\ (1+\delta-\nu)/2 \end{matrix}\right)$$

provided that $\operatorname{Re}(2\mu + \nu + 2) > 0$ and $\operatorname{Re}(s) > 0$; and [**2**, p. 337, (14)]

$$\vartheta_1(l) = \int_0^\infty s^{l-1}\chi(s)\,ds = 2^{2l+2\mu}\frac{\Gamma(\mu+\nu/2+l)\,\Gamma(l)\,\Gamma(1-l)}{\Gamma(-\mu+\nu/2-l+1)}$$

provided that $\min \operatorname{Re}(\mu + 1 + \nu/2, 1) > \operatorname{Re}(1-l) > 0$.

Hence

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\vartheta_1(l)}{\Gamma(l)\,\Gamma(1-l)}\,t^{-l}\,dl = \frac{2^{2\mu}}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{2l}\frac{\Gamma(\mu+\nu/2+l)}{\Gamma(-\mu+\nu/2-l+1)}\,t^{-l}\,dl = t^{\mu}J_\nu(t^{1/2}),$$

since [**1**, p. 219, (44)]

$$t^{\mu}J_\nu(t^{1/2}) = 2^{2\mu}G^{10}_{02}\left(\frac{t}{4}\,\middle|\,\frac{1}{2}\nu+\mu,\ \mu-\frac{1}{2}\nu\right) = \frac{2^{2\mu}}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{2l}\frac{\Gamma(\mu+\nu/2+l)}{\Gamma(-\mu+\nu/2-l+1)}\,t^{-l}\,dl$$

by definition of G function.

Example C. Let us take

$$\varphi(t) = e^{-pt}, \qquad \operatorname{Re}(p) > 0.$$

Then

$$f_1(s) = \Gamma(\sigma)\,s^{\sigma-1}\int_0^\infty\frac{1}{(s+t)^\sigma}\varphi(t)\,dt = \Gamma(\sigma)\,s^{\sigma-1}\int_0^\infty\frac{1}{(s+t)^\sigma}e^{-pt}\,dt =$$

$$= \Gamma(\sigma)\,(ps)^{\sigma/2-1}\,e^{ps/2}\,W_{\sigma/2,(\sigma-1)/2}(ps)$$

taking a particular case of the result [**5**]

$$W_{k,m}(z) = \frac{1}{\Gamma(c)}e^{-z/2}z^k\int_0^\infty e^{-t}\,t^{c-1}\,{}_2F_1\left[\begin{matrix}1/2-k+m, & 1/2-k-m\\ & c\end{matrix};\ -\frac{t}{2}\right]dt$$

provided that $|\arg s| < \pi$; and [4]

$$\vartheta_2(l) = \int_0^\infty s^{l-1} f_1(s)\, ds = \Gamma(\sigma) \int_0^\infty s^{l-1} (ps)^{\sigma/2-1} e^{ps/2}\, W_{\sigma/2,(\sigma-1)/2}(ps)\, ds =$$

$$= p^{-l}\, \Gamma(l)\, \Gamma(1-l)\, \Gamma(\sigma - 1 + l)$$

provided that $\mathrm{Re}\,(2l + \sigma) > 0$ and $\mathrm{Re}\,(2l + 1 - \sigma) > 0$.

Hence

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{t^{-l}\, \vartheta_2(l)}{\Gamma(\sigma - 1 + l)\, \Gamma(1-l)}\, dl = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(l)\, (pt)^{-l}\, dl = e^{-pt}.$$

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