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Some Properties of Generalized Laplace Transform. - I.

1. – Recently, a generalization of the classical LAPLACE transform

$$\varphi(s) = s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t, \tag{1.1}$$

has been introduced by VARMA [**1**, p. 209] in the form

$$\varphi(s) = s\int_0^\infty (st)^{m-1/2} e^{-st/2}\, W_{k,\,m}(st)\, f(t)\,\mathrm{d}t, \tag{1.2}$$

where $W_{k,m}(z)$ denotes the WHITTAKER function. When $k+m=1/2$, (1.2) reduces to (1.1) by virtue of the identity

$$z^{m-(1/2)}\, W_{(1/2)-m,\,m}(z) = e^{-(1/2)z}.$$

The relation (1.1) is symbolically denoted by

$$\varphi(s) \doteqdot f(t)$$

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and we denote the relation (1.2) symbolically by

$$\varphi(s:\ k,\ m) = W[f(t);\ k,\ m].$$

As in the case of LAPLACE transform, we call $\varphi(s:\ k,\ m)$, the *image* of $f(t)$ and $f(t)$ the *original* of $\varphi(s:\ k,\ m)$.

The object of this paper is to extend certain results of LAPLACE transform to this generalized LAPLACE transform. The results have been given in the form of theorems.

2. – It is possible to obtain the original of certain functions, by interpretation, which can be expanded in descending powers of s. Whe shall illustrate this by the following examples.

(a) If, in (1.2) (1),

$$\varphi(s:\ k,\ m) = s^{-\varrho}{}_1F_1\left\{\begin{matrix}\mu \\ \nu\end{matrix};\ -bs^{-\beta}\right\} = \sum_{r=0}^{\infty}\frac{(\mu)_r}{(\nu)_r}\frac{(-b)^r}{r!}\,s^{-(\varrho+\beta r)},$$

then, on interpreting the right hand side term by term with the help of the known generalized LAPLACE transform [**2**, p. 382]

$$W[t^{\nu};\ k,\ m] = \frac{\Gamma_*(\nu+1+m\pm m)}{\Gamma(\nu+3/2+m-k)}\,s^{-\nu}, \tag{2.1}$$

$$\Re(\nu+1+m\pm m)>0 \qquad \text{and} \qquad \Re(s)>0,$$

we get

$$W\left[t^{\varrho}\sum_{r=0}^{\infty}\frac{(\mu)_r\,\Gamma(\varrho+3/2+\beta r+m-k)\,(-bt^{\beta})^r}{(\nu)_r\,\Gamma_*(\varrho+1+\beta r+m\pm m)\ r!};\ k,\ m\right] = s^{-\varrho}{}_1F_1\left\{\begin{matrix}\mu \\ \nu\end{matrix};-bs^{-\beta}\right\}, \tag{2.2}$$

$$\beta>0,\quad \Re(\varrho+1+m\pm m)>0 \qquad \text{and} \qquad \Re(s)>0.$$

(b) If, in (1.2),

$$\varphi(s:\ k,\ m) = s^{-\varrho}\,e^{-(b/2)s^{-\beta}}\cdot W_{\lambda,\mu}(bs^{-\beta}),$$

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(1) The symbol $(\alpha)_r$ denotes $\Gamma(\alpha+r)/\Gamma(\alpha)$, and $\Gamma_*(\alpha\pm\beta)$ denotes $\Gamma(\alpha+\beta)\cdot\Gamma(\alpha-\beta)$.

then, since [**3**, §§ 16.1, 16.11 and 16.41]

$$s^{-\varrho}\, e^{-(b/2)s^{-\beta}} \cdot W_{\lambda,\mu}(bs^{-\beta}) =$$

$$= \sum_{\mu,-\mu} \frac{\Gamma(2\mu)\cdot(bs^{-\beta})^{1/2-\mu}}{\Gamma(1/2+\mu-\lambda)}\, s^{-\varrho}\,{}_1F_1\left\{\begin{matrix}1/2+\lambda-\mu\\ 1-2\mu\end{matrix};\ -bs^{-\beta}\right\},$$

where the symbol $\sum_{\mu,-\mu}$ denotes that to the expression following it a similar expression with μ and $-\mu$ interchanged is to be added, we get, on interpreting the right hand side with the help of (2.2),

$$\text{(2.3)}\qquad W\left[t^{\varrho}\sum_{\mu,-\mu}\left(\frac{\Gamma(2\mu)\cdot(bt^{\beta})^{1/2-\mu}}{\Gamma(1/2-\lambda+\mu)}\cdot\right.\right.$$

$$\left.\left.\cdot\sum_{r=0}^{\infty}\frac{(1/2+\lambda-\mu)_r\cdot\Gamma(\varrho+3/2-\beta\mu+\beta/2+\beta r+m-k)}{(1-2\mu)_r\cdot\Gamma_*(\varrho+1-\beta\mu+\beta/2+\beta r+m\pm m)}\,\frac{(-bt^{\beta})^r}{r!}\right);\ k,\ m\right]=$$

$$= s^{-\varrho}\, e^{-(b/2)s^{-\beta}}\cdot W_{\lambda,\mu}(bs^{-\beta}),$$

$\Re(\varrho+1+\beta/2+m\pm m\pm\beta\mu)>0$, $\Re(s)>0$ and 2μ is not an integer or zero.

3. - Theorem 1.

If

$$\varphi(s) = W[t^{\nu}\cdot h(t);\ k,\ m], \qquad h(s) = W[f(t);\ l,\ n]$$

and

$$s^{1-\lambda+\mu}\cdot f(s^{\mu}) \doteqdot g(t), \qquad \mu>0,$$

then

$$\text{(3.1)}\qquad \varphi(s) = \int_0^{\infty} g(t)\cdot\tilde{\omega}(s,\ t)\ dt,$$

where (²)

$$(3.2)\qquad \tilde{\omega}(s,\ t) = \mu \int_0^\infty e^{-zt}\, z^{\lambda-\mu-\mu\nu-1} \cdot$$

$$\cdot G_{3,3}^{2,2}\left(\frac{z^\mu}{s}\,\middle|\,\begin{matrix} 0,\ -2m,\ \nu+3/2+n-l \\ \nu+1,\ \nu+1+2n,\ k-m-1/2 \end{matrix}\right) dz\,,$$

provided $\Re(\lambda+\mu n \pm \mu n) > 0$, $\Re(\nu+2+m+n \pm m \pm n) > 0$, $\Re(s) \geqslant s_0 > 0$, *the Laplace transform of* $|g(t)|$ *exists, the generalized Laplace transform of* $|f(t)|$ *exists and the integral in* (3.1) *is absolutely convergent.*

Proof.

We have

$$(3.3)\qquad \varphi(s) = s \int_0^\infty (sx)^{m-1/2}\, e^{-sx/2} \cdot W_{k,m}(sx) \cdot x^\nu \cdot h(x)\, dx,$$

$$(3.4)\qquad h(x) = x \int_0^\infty (zx)^{n-1/2}\, e^{-zx/2} \cdot W_{l,n}(zx) \cdot f(z)\, dz$$

and

$$(3.5)\qquad z^{1-\lambda+\mu} \cdot f(z^\mu) = z \int_0^\infty e^{-zt} \cdot g(t)\, dt\,.$$

Substituting the value of $f(z^\mu)$ from (3.5) in (3.4) after replacing z by z^μ, we have

$$h(x) = \mu x \int_0^\infty (z^\mu x)^{n-1/2}\, e^{-z^\mu x/2} \cdot W_{l,n}(z^\mu x) \cdot z^{\lambda-1} \int_0^\infty e^{-zt} \cdot g(t)\, dt\, dz\,.$$

Changing the order of integration, we get

$$(3.6)\qquad h(x) = \mu x^{n+1/2} \int_0^\infty g(t) \int_0^\infty z^{\mu n-(\mu/2)+\lambda-1}\, e^{-zt-z^\mu x/2} \cdot W_{l,n}(z^\mu x)\, dz\, dt\,.$$

(²) For the behaviour of $G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$, see [**4**, p. 212].

Substituting the value of $h(x)$ from (3.6) in (3.3), we obtain

$$\varphi(s) = \mu s \int_0^\infty (sx)^{m-1/2} e^{-2^{-1}sx} \cdot W_{k,m}(sx) \cdot x^{\nu+n+1/2} \cdot$$

$$\cdot \int_0^\infty g(t) \int_0^\infty z^{\mu n-(\mu/2)+\lambda-1} e^{-zt-z^\mu x/2} \cdot W_{l,n}(z^\mu x) \, dz \, dt \, dx$$

and on changing the order of integration, this becomes

$$(3.7) \qquad \varphi(s) = \mu s^{m+1/2} \int_0^\infty g(t) \int_0^\infty z^{\mu n-(\mu/2)+\lambda-1} e^{-zt} \cdot$$

$$\cdot \int_0^\infty x^{m+n+\nu} \cdot \exp[-(s+z^\mu)x/2] \cdot W_{k,m}(sx) \cdot W_{l,n}(z^\mu x) \, dx \, dz \, dt.$$

Now, evaluating the x-integral with the help of [**5**, p. 411]

$$\int_0^\infty x^{\varrho-1} \cdot \exp[-(\alpha+\beta)x/2] \cdot W_{k,m}(\alpha x) \cdot W_{l,n}(\beta x) \, dx =$$

$$= \beta^{-\varrho} \cdot G^{2,2}_{3,3}\left(\frac{\beta}{\alpha} \middle| \begin{matrix} 1/2+m, \ 1/2-m, \ 1-l+\varrho \\ 1/2+n+\varrho, \ 1/2-n+\varrho, \ k \end{matrix}\right),$$

$$\Re(\alpha+\beta) > 0, \qquad \Re(\varrho+1 \pm m \pm n) > 0,$$

we obtain

$$\varphi(s) = \mu \int_0^\infty g(t) \int_0^\infty e^{-zt} z^{\lambda-\mu-\mu\nu-1} \cdot G^{2,2}_{3,3}\left(\frac{z^\mu}{s} \middle| \begin{matrix} 0, \ -2m, \ \nu+3/2+n-l \\ \nu+1+2n, \ \nu+1, \ k-m-1/2 \end{matrix}\right) dz \, dt,$$

on making use of the transformation [**4**, p. 209]

$$x^\sigma \cdot G^{m,n}_{p,q}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{m,n}_{p,q}\left(x \middle| \begin{matrix} a_r+\sigma \\ b_s+\sigma \end{matrix}\right).$$

The change of order of integration in (3.6) and (3.7) can be justified as follows.

In (3.6), the z-integral converges uniformly and absolutely if $x > 0$ and $\Re(\lambda + \mu n \pm \mu n) > 0$, the t-integral converges uniformly and absolutely if the LAPLACE transform of $|g(t)|$ exists and the repeated integral is absolutely convergent if the generalized LAPLACE transform of $|f(t)|$ exists. Hence the change of order of integration in (3.6) is justified by DE LA VALLÉE POUSSIN's Theorem [**6**, p. 504].

Now, we have to justify the change of order of the t-integral and x-integral in (3.7). Here the x-integral is uniformly and absolutely convergent if $\Re(\nu + 2 + m + n \pm m \pm n) > 0$ and $\Re(s) \geqslant s_0 > 0$, the t-integral is uniformly and absolutely convergent if the LAPLACE transform of $|g(t)|$ exists and the repeated integral is absolutely convergent due to the absolute convergence of (3.1). Therefore, the change of order of integration is justified.

Next, we have to justify the change of order of x-integral and z-integral in (3.7). The z-integral is uniformly and absolutely convergent if $\Re(\lambda + \mu n \pm \mu n) > 0$ and $t > 0$, the x-integral is uniformly and absolutely convergent if $\Re(\nu + 2 + m + n \pm m \pm n) > 0$, $\Re(s) \geqslant s_0 > 0$ and the repeated integral is absolutely convergent if $\Re(\lambda + \mu n \pm \mu n) > 0$ and $t > 0$. Hence the two changes of order of integration in (3.7) are justified.

3.1. – Asymptotic behaviour of $\tilde{\omega}(s,\ t)$.

It can easily be seen that $\tilde{\omega}(s,\ t) = O(1)$ for small t. To determine the asymptotic behaviour of $\tilde{\omega}(s,\ t)$ for large t, we make use of WATSON's lemma [**7**, p. 236]. We observe that

$$z^{\lambda-\mu-\mu\nu-1}\cdot G^{2,2}_{3,3}\left(\frac{z^\mu}{s}\,\middle|\,\begin{matrix}0,\ -2m,\ \nu+3/2+n-l\\ \nu+1,\ \nu+1+2n,\ k-m-1/2\end{matrix}\right)$$

is expressible [**4**, p. 208] as a sum of two hypergeometric functions of the type ${}_3F_2$, if $2n$ is not an integer, and hence is an analytic function of z save for a branch-point at the origin. Also it satisfies all the conditions of WATSON's lemma.

Further [**4**, p. 212]

$$z^{\lambda-\mu-\mu\nu-1}\cdot G^{2,2}_{3,3}\left(\frac{z^\mu}{s}\,\middle|\,\begin{matrix}0,\ -2m,\ \nu+3/2+n-l\\ \nu+1,\ \nu+1+2n,\ k-m-1/2\end{matrix}\right) = O(z^{\lambda-1+\mu n\pm\mu n})$$

for small z. Hence, by the lemma,

$$\tilde{\omega}(s,\ t) = O(t^{-\lambda-\mu n\pm\mu n})$$

for large t.

3.2. - Corollary.

Putting $k+m=1/2$ and $l+n=1/2$ in the theorem, we arrive at a known result of LAPLACE transform due to BOSE [**8**, p. 127].

If

$$\varphi(s) \doteqdot t^{\nu}\cdot h(t), \qquad h(s) \doteqdot f(t)$$

and

$$s^{1-\lambda+\mu}\cdot f(s^{\mu}) \doteqdot g(t), \qquad \mu>0,$$

then

$$\varphi(s) = \int_0^{\infty} g(t)\cdot\tilde{\omega}(s,\ t)\,\mathrm{d}t, \tag{3.9}$$

where

$$\tilde{\omega}(s,\ t) = \mu s\cdot\Gamma(\nu+2)\int_0^{\infty} e^{-zt}\,(s+z^{\mu})^{-\nu-2}\,z^{\lambda-1}\,\mathrm{d}z, \tag{3.10}$$

provided $\mathfrak{R}(s)\geqslant s_0>0$, $\mathfrak{R}(\lambda)>0$, $\mathfrak{R}(\nu+2)>0$, *the Laplace transforms of* $|f(t)|$ *and* $|g(t)|$ *exist and the integral in* (3.9) *converges absolutely.*

3.3. - Particular cases.

(i) *If*

$$\varphi(s) = W[t^{\nu}\cdot h(t);\ k,\ m], \qquad h(s) = W[f(t);\ l,\ n]$$

and

$$s^{2-\lambda}f(s) \doteqdot g(t),$$

then

$$\varphi(s) = \int_0^{\infty} t^{\nu+1-\lambda}\cdot G_{3,4}^{3,2}\left(st \,\middle|\, \begin{matrix} -\nu-2n,\ \ -\nu,\ \ (3/2)-k+m \\ \lambda-\nu-1,\ 1,\ 1+2m,\ -\nu-(1/2)+l-n \end{matrix}\right)\cdot g(t)\,\mathrm{d}t,$$

under the conditions of the theorem.

This follows from Theorem 1 with $\mu = 1$, the known integral [**5**, p. 419]

$$\int_0^\infty x^{-\varrho}\, e^{-\beta x}\cdot G^{m,n}_{p,q}\left(\alpha x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \mathrm{d}x = \beta^{\varrho-1}\cdot G^{m,n+1}_{p+1,q}\left(\frac{\alpha}{\beta} \,\middle|\, \begin{matrix} \varrho, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right),$$

$$p + q < 2(m + n), \qquad |\arg \alpha| < \{m + n - p/2 - q/2\}\pi, \qquad |\arg \beta| < \pi/2,$$

$$\Re(b_j - \varrho + 1) > 0 \qquad (j = 1, 2, \dots, m),$$

and the transformation [**4**, p. 209]

$$G^{m,n}_{p,q}\left(x^{-1} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{n,m}_{q,p}\left(x \,\middle|\, \begin{matrix} 1 - b_s \\ 1 - a_r \end{matrix}\right).$$

(ii) Putting $l + n = 1/2$, the theorem reduces to:

If

$$\varphi(s) = W[t^{\nu}\cdot h(t);\quad k,\ m], \qquad h(s) \doteqdot f(t)$$

and

$$s^{1-\lambda+\mu}\cdot f(s^{\mu}) \doteqdot g(t), \qquad \mu > 0,$$

then

$$\varphi(s) = \int_0^\infty g(t)\cdot\widetilde{\omega}(s, t)\,\mathrm{d}t,$$

where

$$\widetilde{\omega}(s, t) = \frac{\mu\cdot\Gamma_*(\nu + 2 + m \pm m)}{\Gamma(\nu + 5/2 + m - k)s^{\nu+1}} \int_0^\infty e^{-zt}\, z^{\lambda-1}\cdot {}_2F_1\left\{ \begin{matrix} \nu + 2,\ \nu + 2 + 2m \\ \nu + 5/2 + m - k \end{matrix} ;\ -\frac{z^{\mu}}{s} \right\} \mathrm{d}z,$$

under the conditions of the theorem.

(iii) Putting $\mu = 1$, the above case reduces to a result of RATHIE [**9**, p. 134]:

If

$$\varphi(s) = W[t^{\nu} \cdot h(t); \quad k, \quad m], \qquad h(s) \doteqdot f(t)$$

and

$$s^{2-\lambda} \cdot f(s) \doteqdot g(t),$$

then

$$\varphi(s) = s^{-\nu-1} \int_0^{\infty} g(t) \cdot E(\lambda, \ \nu + 2, \ \nu + 2 + 2m; \ \nu + 5/2 + m - k; \ st) t^{-\lambda} \, \mathrm{d}t,$$

under the conditions of the theorem.

(iv) Putting $k + m = 1/2$, the theorem reduces to:

If

$$\varphi(s) \doteqdot t^{\nu} \cdot h(t), \qquad h(s) = W[f(t); \ l, \ n]$$

and

$$s^{1-\lambda+\mu} \cdot f(s^{\mu}) \doteqdot g(t), \qquad \mu > 0,$$

then

$$\varphi(s) = \int_0^{\infty} g(t) \cdot \widetilde{\omega}(s, \ t) \, \mathrm{d}t,$$

where

$$\widetilde{\omega}(s, \ t) = \mu s \int_0^{\infty} e^{-zt} z^{\lambda-1-2\mu-\mu\nu} \cdot E(\nu + 2, \ \nu + 2 + 2n; \nu + 5/2 + n - l; \ z^{\mu}/s) \, \mathrm{d}z,$$

under the conditions of the theorem.

(v) With $\mu = 1$, the above case reduces to another result of RATHIE [**2**, p. 388]:

If

$$\varphi(s) \doteqdot t^{\nu}\cdot h(t), \qquad h(s) = W[f(t);\ \ l, n]$$

and

$$s^{2-\lambda}\cdot f(s) \doteqdot g(t),$$

then

$$\varphi(s) = \int_0^\infty t^{\nu+1-\lambda}\cdot G_{2,3}^{2,2}\left(st \,\middle|\, \begin{matrix} -\nu, & -\nu-2n \\ -\nu-1+\lambda, \ 1, & -\nu-1/2+l-n \end{matrix}\right)\cdot g(t)\,\mathrm{d}t,$$

under the conditions of the theorem.

3.4. – Example.

Let, in Theorem 1, $g(t) = t^{\alpha}$, then

$$f(t) = \Gamma(\alpha+1)\cdot t^{\mu^{-1}\cdot(\lambda-\alpha-1)-1}, \qquad \Re(\alpha+1) > 0,$$

therefore,

$$t^{\nu}\cdot h(t) = \frac{\Gamma(\alpha+1)\cdot\Gamma_*\{\mu^{-1}\cdot(\lambda-\alpha-1)+n\pm n\}}{\Gamma\{\mu^{-1}\cdot(\lambda-\alpha-1)+1/2+n-l\}}\, t^{\nu+1-\mu^{-1}\cdot(\lambda-\alpha-1)},$$

$$\Re(\lambda-\alpha-1+\mu n\pm\mu n) > 0,$$

and then

$$\varphi(s) =$$

$$= \frac{\Gamma(\alpha+1)\cdot\Gamma_*\{\mu^{-1}\cdot(\lambda-\alpha-1)+n\pm n\}\cdot\Gamma_*\{\nu+2-\mu^{-1}\cdot(\lambda-\alpha-1)+m\pm m\}}{\Gamma\{\mu^{-1}\cdot(\lambda-\alpha-1)+1/2+n-l\}\cdot\Gamma\{\nu+5/2-\mu^{-1}\cdot(\lambda-\alpha-1)+m-k\}}\cdot$$

$$\cdot s^{\mu^{-1}\cdot(\lambda-\alpha-1)-\nu-1},$$

$$\Re\{\nu+2-\mu^{-1}(\lambda-\alpha-1)+m\pm m\} > 0\,.$$

Hence, by the theorem, we have

$$\int_0^\infty t^\alpha \cdot \widetilde{\omega}(s,\ t)\,\mathrm{d}t =$$

$$= \frac{\Gamma(\alpha+1)\cdot\Gamma_*\{\mu^{-1}\cdot(\lambda-\alpha-1)+n\pm n\}\cdot\Gamma_*\{\nu+2-\mu^{-1}\cdot(\lambda-\alpha-1)+m\pm m\}}{\Gamma\{\mu^{-1}\cdot(\lambda-\alpha-1)+(1/2)+n-l\}\cdot\Gamma\{\nu+(5/2)-\mu^{-1}\cdot(\lambda-\alpha-1)+m-k\}}\cdot$$

$$\cdot s^{\mu^{-1}\cdot(\lambda-\alpha-1)-\nu-1},$$

$$\mu>0, \qquad \mathfrak{R}(s)>0, \qquad \mathfrak{R}(\lambda+\mu n\pm\mu n)>\mathfrak{R}(\alpha+1)>0,$$

$\mathfrak{R}(\nu+2+m+n\pm m\pm n)>0$ and $\mathfrak{R}\{\nu+2-\mu^{-1}\cdot(\lambda-\alpha-1)+m\pm m\}>0$

$\widetilde{\omega}(s,\ t)$ being given by (3.2).

4. – Theorem 2.

If

$$\varphi(s) = W[f(t);\quad k,\ m]$$

and

$$x^{m-(1/2)}\,s^{\nu-(\mu/2)+\mu m}\,e^{-xs^\mu/2}W_{k,m}(xs^\mu) = W[g(x,\ t);\ k,\ m],$$

then

$$s^{\nu-\mu}\cdot\varphi(s^\mu) = W\Big[\int_0^\infty f(x)\cdot g(x,\ t)\,\mathrm{d}x;\ k,\ m\Big], \tag{4.1}$$

provided $\mathfrak{R}(\alpha+1+m\pm m)>0$, *where* $f(t)=O(t^\alpha)$ *for small* t, $f(t)$ *is continuous for* $t\geqslant 0$, $\mathfrak{R}(s)\geqslant s_0>0$ *and the integral*

$$\int_B^\infty \left| e^{-xs^\mu}\,x^{k+m-(1/2)}\cdot f(x)\right|\,\mathrm{d}x,$$

where B *is large, converges.*

Proof.

We have

$$(4.2)\qquad x^{m-(1/2)} s^{\nu-(\mu/2)+\mu m} e^{-xs^{\mu}/2}\cdot W_{k,m}(xs^{\mu}) = s\int_0^{\infty}(st)^{m-(1/2)} e^{-st/2}\cdot W_{k,m}(st)\cdot g(x,\ t)\ \mathrm{d}t$$

and

$$\varphi(s) = s\int_0^{\infty}(sx)^{m-(1/2)} e^{-sx/2}\cdot W_{k,m}(sx)\cdot f(x)\ \mathrm{d}x\,.$$

Replacing s by s^{μ} in the last equation, multiplying by $s^{\nu-\mu}$ and using (4.2), we get

$$s^{\nu-\mu}\cdot\varphi(s^{\mu}) = s\int_0^{\infty} f(x)\int_0^{\infty}(st)^{m-(1/2)} e^{-st/2}\cdot W_{k,m}(st)\cdot g(x,\ t)\ \mathrm{d}t\ \mathrm{d}x\,.$$

Changing the order of integration, we get (4.1).

The change of order of integration can by justified as follows. Let

$$\Theta(x) = f(x)\cdot\int_0^{\infty}(st)^{m-(1/2)} e^{-st/2}\cdot W_{k,m}(st)\cdot g(x,\ t)\ \mathrm{d}t = f(x)\cdot F(x),$$

say. From (4.2), $F(x) = O(x^{m\pm m})$ for small x. Hence $\Theta(x)$ is continuous for $x\geqslant 0$, if $\Re(\alpha+m\pm m)>0$, where $f(x)$ is $O(x^{\alpha})$ for small x and continuous for $x\geqslant 0$. The integral $\int_{\varepsilon}^{a}\Theta(x)\,\mathrm{d}x$, where $0<\varepsilon<a<\infty$, converges as $\varepsilon\to 0$, if $\Re(\alpha+1+m\pm m)>0$.

Further, from (4.2) and the asymptotic expansion of $W_{k,m}(x)$, we have

$$F(x) = O\left\{x^{k+m-(1/2)} e^{-xs^{\mu}}\right\}$$

for large values of x.

Again, consider the integral

$$I = \int_B^{\infty} f(x)\int_A^{\infty}(st)^{m-(1/2)} e^{-st/2}\cdot W_{k,m}(st)\cdot g(x,\ t)\ \mathrm{d}t\ \mathrm{d}x,$$

where A and B are large. For large values of t,

$$|I| \leqslant \int_B^\infty |f(x)| \left[\int_A^{B'} |(st)^{m-1/2} e^{-st/2} \cdot W_{k,m}(st) \cdot g(x,\ t)| \,\mathrm{d}t \right] \mathrm{d}x,$$

which tends to zero as B and B' tend to ∞, if $\Re(s) \geqslant s_0 > 0$, the integral

$$\int_B^\infty | e^{-xs^\mu} x^{k+m-1/2} \cdot f(x) | \,\mathrm{d}x$$

being supposed to be convergent.

Thus the inversion of the order of integration is justified under the conditions of the theorem.

4.1. – Corollary.

On putting $k + m = 1/2$, we get the well-known result of LAPLACE transform [**10**, p. 20].

If

$$\varphi(s) \doteqdot f(t) \qquad \textit{and} \qquad s^\nu e^{-xs^\mu} \doteqdot g(x,\ t),$$

then

$$s^{\nu-\mu} \cdot \varphi(s^\mu) \doteqdot \int_0^\infty f(x) \cdot g(x,\ t) \,\mathrm{d}x,$$

provided $\Re(\alpha + 1) > 0$, *where* $f(t) = O(t^\alpha)$ *for small* t, $f(t)$ *is continuous for* $t \geqslant 0$, $\Re(s) \geqslant s_0 > 0$ *and the integral*

$$\int_B^\infty | e^{-xs^\mu} \cdot f(x) | \,\mathrm{d}x,$$

where B *is large, converges.*

4.2. – A particular case.

When $\mu = -\beta$, where $\beta > 0$, $g(x,\ t)$ can be determined. With the help of (2.3) it can easily be seen that

$$g(x,\ t) = \frac{\Gamma(2m)}{\Gamma(1/2 - k + m)\cdot t^{\nu}} \sum_{r=0}^{\infty} \frac{(1/2 + k - m)_r \cdot \Gamma(3/2 - \nu + \beta r + m - k)}{(1 - 2m)_r \cdot \Gamma_*(1 - \nu + \beta r + m \pm m)} \frac{(-xt^{\beta})^r}{r!} +$$

$$+ \frac{\Gamma(-2m)\cdot(xt^{\beta})^{2m}}{\Gamma(1/2 - k - m)\cdot t^{\nu}} \sum_{r=0}^{\infty} \frac{(1/2 + k + m)_r \cdot \Gamma(3/2 - \nu - 2\beta m + \beta r + m - k)}{(1 + 2m)_r \cdot \Gamma_*(1 - \nu - 2\beta m + \beta r + m \pm m)} \frac{(-xt^{\beta})^r}{r!},$$

with $\Re(1 - \nu + m + \beta m \pm m \pm \beta m) > 0$ and $2m$ not an integer or zero.

5. – Theorem 3.

If

$$\varphi(s) = W[f(t);\ k,\ m], \tag{5.1}$$

$$x^{m-(1/2)} s^{\nu-(\mu/2)+\mu m} e^{-xs^{\mu}/2} \cdot W_{k,m}(xs^{\mu}) = W[g(x,\ t);\ k,\ m] \tag{5.2}$$

and

$$\psi(s) = W[\Theta(t);\ k,\ m], \tag{5.3}$$

then

$$\int_0^{\infty} f(x)\cdot G(x)\ dx = \int_0^{\infty} u^{\nu-\mu-1}\cdot\varphi(u^{\mu})\cdot\Theta(u)\ du, \tag{5.4}$$

where

$$G(x) = \int_0^{\infty} \psi(t)\cdot g(x,\ t)\cdot t^{-1}\ dt, \tag{5.5}$$

provided $\Re(s) \geqslant s_0 > 0$, $\Re(\alpha + 1 + m \pm m) > 0$, *where* $f(t)$ *is* $O(t^{\alpha})$ *for small* t, $f(t)$ *is continuous for* $t \geqslant 0$, *the integral*

$$\int_B^{\infty} |\ e^{-xs^{\mu}} x^{k+m-(1/2)}\cdot f(x)\ |\ dx,$$

where B is large, converges and the integrals in (5.4), (5.5) *and* (5.6) *are absolutely convergent.*

Proof.

By virtue of the relations (5.1) and (5.2), it follows from Theorem 2 that

$$s^{\nu-\mu}\cdot\varphi(s^{\mu}) = W[\int_0^\infty f(x)\cdot g(x,\ t)\,\mathrm{d}x;\ k,\ m], \tag{5.6}$$

under the conditions of Theorem 2.

We know that [**11**, p. 27] if

$$\varphi_1(s) = W[f_1(t);\ k,\ m] \qquad \text{and} \qquad \varphi_2(s) = W[f_2(t);\ k,\ m],$$

then

$$\int_0^\infty \varphi_1(u)\cdot f_2(u)\cdot u^{-1}\,\mathrm{d}u = \int_0^\infty \varphi_2(u)\cdot f_1(u)\cdot u^{-1}\,\mathrm{d}u\,. \tag{5.7}$$

Now, using the relations (5.3) and (5.6) in (5.7), we get

$$\int_0^\infty u^{\nu-\mu-1}\cdot\varphi(u^{\mu})\cdot\Theta(u)\,\mathrm{d}u = \int_0^\infty \psi(u)\left\{\int_0^\infty f(x)\cdot g(x,\ u)\,\mathrm{d}x\right\}u^{-1}\,\mathrm{d}u = \tag{5.8}$$

$$= \int_0^\infty f(x)\int_0^\infty \psi(u)\cdot g(x,\ u)\cdot u^{-1}\,\mathrm{d}u\,\mathrm{d}x = \int_0^\infty f(x)\cdot G(x)\,\mathrm{d}x\,.$$

The change of order of integration in (5.8) can be justified by DE LA VALLÉE POUSSIN's Theorem.

5.1. – Corollary.

On putting $k+m=1/2$, we arrive at a known result of LAPLACE transform due to BOSE [**12**, p. 91].

If

$$\varphi(s) \doteqdot f(t), \qquad s^{\nu}\,e^{-xs^{\mu}} \doteqdot g(x,\ t)$$

and

$$\psi(s) \doteqdot \Theta(t),$$

then

$$\int_0^\infty f(x)\cdot G(x)\,\mathrm{d}x = \int_0^\infty u^{\nu-\mu-1}\cdot\varphi(u^\mu)\cdot\Theta(u)\,\mathrm{d}u, \tag{5.9}$$

where $G(x)$ *is given by* (5.5), *provided* $\Re(s)\geqslant s_0>0$, $\Re(\alpha+1)>0$, *where* $f(t)$ *is* $O(t^\alpha)$ *for small* t *and is continuous for* $t\geqslant 0$, *the integral*

$$\int_B^\infty |\, e^{-xs^\mu}\cdot f(x)\,|\,\mathrm{d}x,$$

where B *is large, converges and the integrals in* (5.9), (5.5) *and* $\int_0^\infty f(x)\cdot g(x,\,t)\,\mathrm{d}x$ *are absolutely convergent.*

6. – Theorem 4.

If

$$\varphi(s:\,k,\,m) = W[f(t);\,k,\,m] \qquad \textit{and} \qquad \sqrt{s}\cdot f(1/s) = W[g(t);\,k,\,m],$$

then

$$\varphi(s^2:\,k/2+1/4,\,m/2) = W[\sqrt{\pi}\,2^{-(1/2)-k-m}\,t\cdot g(t^2/4);\,k,\,m],$$

provided $\Re(s)\geqslant s_0>0$, *and the generalized* L a p l a c e *transforms of* $|\,g(t)\,|$ *and* $|\,t\cdot g(t^2/4)\,|$ *exist.*

P r o o f.

We have

$$\varphi(s^2:\,k,\,m) = s^2\int_0^\infty (s^2x)^{m-1/2}\,e^{-s^2x/2}\cdot W_{k,m}(s^2x)\cdot f(x)\,\mathrm{d}x. \tag{6.1}$$

Also,

$$\sqrt{s}\cdot f(1/s) = s\int_0^\infty (st)^{m-1/2}\,e^{-st/2}\cdot W_{k,m}(st)\cdot g(t)\,\mathrm{d}t.$$

Replacing s by $1/x$ and substituting the value of $f(x)$ in (6.1), we get

$$\varphi(s^2:\ k,\ m) =$$

$$= s^2 \int_0^\infty (s^2 x)^{m-(1/2)} e^{-s^2 x/2} \cdot W_{k,m}(s^2 x)\, x^{-1/2} \int_0^\infty (t/x)^{m-(1/2)} e^{-t/(2x)} \cdot W_{k,m}(t/x) \cdot g(t)\, dt\, dx .$$

Changing the order of integration and evaluating the x-integral by a known integral [**5**, p. 412]

$$\int_0^\infty x^{-1/2} \cdot \exp\left[-\left\{(x/\alpha) + (\beta/x)\right\}/2\right] \cdot W_{k,m}(x/\alpha) \cdot W_{k,m}(\beta/x)\, dx =$$

$$= \sqrt{\pi}\, 2^{(1/2)-2k} (\alpha\beta)^{1/4} \cdot \exp\left[-(\beta/\alpha)^{1/2}\right] \cdot W_{2k-(1/2),2m}\left[2(\beta/\alpha)^{1/2}\right],$$

$$\Re(\alpha) > 0, \qquad \Re(\beta) > 0,$$

we get

$$\varphi(s^2:\ k,\ m) = \sqrt{\pi}\, s\, 2^{-2(k+m)} \int_0^\infty (st)^{2m-(1/2)} e^{-st/2} \cdot W_{2k-(1/2),2m}(st) \cdot t \cdot g(t^2/4)\, dt.$$

Replacing k by $k/2 + 1/4$ and m by $m/2$, we get the result.

The change of order of integration is justified by DE LA VALLÉE POUSSIN'S Theorem.

6.1. – Corollary.

On putting $k + m = 1/2$ in Theorem 4, we obtain a known result of LAPLACE transform [**10**, p. 18].

If

$$\varphi(s) \doteqdot f(t) \qquad \textit{and} \qquad \sqrt{s} \cdot f(1/s) \doteqdot g(t),$$

then

$$\varphi(s^2) \doteqdot (\sqrt{\pi}/2) t \cdot g(t^2/4),$$

provided $\Re(s) \geqslant s_0 > 0$ *and the Laplace transforms of* $|g(t)|$ *and* $|t \cdot g(t^2/4)|$ *exist.*

7. – Theorem 5.

If

$$W[f(t);\ k,\ m] = \varphi(s:\ k,\ m)$$

and

$$W[(1/\sqrt{t})\cdot\varphi((1/t) : k,\ m);\ k,\ m] = g(s : k,\ m),$$

then

$$W[f(t^2);\ k,\ m] = 2^{(1/2)+k+m}(\sqrt{\pi}\cdot s)^{-1}\cdot g((s^2/4):\ k/2 + 1/4,\ m/2),$$

provided $\mathfrak{R}(s) \geqslant s_0 > 0$ *and the generalized Laplace transforms of* $|f(t)|$ *and* $|f(t^2)|$ *exist.*

The proof of this follows exactly on the same lines as those of Theorem 4.

7.1. – Corollary.

On putting $k + m = 1/2$, the theorem reduces to a known result of LAPLACE transform [**10**, p. 19].

If

$$f(t) \doteqdot \varphi(s) \qquad \text{and} \qquad (1/\sqrt{t})\cdot\varphi(1/t) \doteqdot g(s),$$

then

$$f(t^2) \doteqdot 2\cdot(\sqrt{\pi}s)^{-1}\cdot g(s^2/4),$$

provided $\mathfrak{R}(s) \geqslant s_0 > 0$ *and the Laplace transforms of* $|f(t)|$ *and* $|f(t^2)|$ *exist.*

7.2. – Example.

If, in Theorem 5, $f(t) = t^{-\varrho}\cdot G^{l,n}_{p,q}\left(bt \,\middle|\, \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right)$, then [**13**, p. 275]

$$\varphi(s:\ k,\ m) = s^{\varrho}\cdot G^{l,n+2}_{p+2,q+1}\left(\frac{b}{s} \,\middle|\, \begin{matrix}\varrho - 2m,\ \varrho,\ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q,\ \varrho + k - m - 1/2\end{matrix}\right),$$

$\Re(\beta_j - \varrho + 1 + m \pm m) > 0$ $(j = 1, \ldots, l)$, $|\arg(s/b)| < (l + n - p/2 - q/2 - 1/2)\pi$ and $p + q + 1 < 2(l + n)$. Hence

$$\frac{1}{\sqrt{t}}\,\varphi\left(\frac{1}{t} : k,\ m\right) = t^{-\varrho-(1/2)} \cdot G^{l,n+2}_{p+2,q+1}\left(bt \,\middle|\, \begin{matrix} \varrho - 2m,\ \varrho,\ \alpha_1,\ \ldots,\ \alpha_p \\ \beta_1,\ \ldots,\ \beta_q,\ \varrho + k - m - 1/2 \end{matrix}\right),$$

therefore, again

$$g(s:\ k,\ m) = s^{\varrho+(1/2)} \cdot G^{l,n+4}_{p+4,q+2}\left(\frac{b}{s} \,\middle|\, \begin{matrix} \varrho + 1/2 - 2m,\ \varrho + 1/2,\ \varrho - 2m,\ \varrho,\ \alpha_1,\ \ldots,\ \alpha_p \\ \beta_1,\ \ldots,\ \beta_q,\ \varrho + k - m - 1/2,\ \varrho + k - m \end{matrix}\right),$$

$\Re(\beta_j - \varrho + 1/2 + m \pm m) > 0$ $(j = 1, \ldots, l)$, $|\arg(s/b)| < (l + n - p/2 - q/2)\pi$ and $p + q < 2(l + n)$. Using these in Theorem 5, we get

$$W\left[t^{-2\varrho} \cdot G^{l,n}_{p,q}\left(bt^2 \,\middle|\, \begin{matrix} \alpha_1,\ \ldots,\ \alpha_p \\ \beta_1,\ \ldots,\ \beta_q \end{matrix}\right);\ k,\ m\right] = \frac{s^{2\varrho}}{2^{2\varrho+(1/2)-k-m}\sqrt{\pi}} \cdot$$

$$\cdot\, G^{l,n+4}_{p+4,q+2}\left(\frac{4b}{s^2} \,\middle|\, \begin{matrix} \varrho,\ \varrho + 1/2,\ \varrho - m,\ \varrho + 1/2 - m,\ \alpha_1,\ \ldots,\ \alpha_p \\ \beta_1,\ \ldots,\ \beta_q,\ \varrho + k/2 - m/2 - 1/4,\ \varrho + k/2 - m/2 + 1/4 \end{matrix}\right),$$

$\Re(\beta_j - \varrho + 1/2 + m \pm m) > 0$ $(j = 1, \ldots, l)$, $|\arg(s/b)| < (l + n - p/2 - q/2 - 1/2)\pi$, $\Re(s) > 0$ and $p + q + 1 < 2(l + n)$.

8. - Theorem 6.

If

(8.1) $$\varphi(s:\ k,\ m) = W[f(t);\ k,\ m],$$

(8.2) $$\sqrt{s} \cdot f(1/s) = W[g(t);\ k,\ m]$$

and

(8.3) $$\psi(s:\ k,\ m) = W[\Theta(t);\ k,\ m],$$

then, provided the integrals involved converge absolutely,

(8.4) $$\int_0^\infty \varphi(t^2:\ k/2 + 1/4,\ m/2) \cdot \Theta(t) \cdot t^{-1}\, dt = \sqrt{\pi}\, 2^{-k-m-(1/2)} \int_0^\infty \psi(t:\ k,\ m) \cdot g(t^2/4)\, dt\,.$$

Proof.

By virtue of the relations (8.1) and (8.2), it follows from Theorem 4 that

$$(8.5)\qquad \varphi(s^2:\ k/2+1/4,\ m/2) = W[\sqrt{\pi}\, 2^{-k-m-(1/2)}\, t\cdot g(t^2/4);\ k,\ m].$$

Using the relations (8.3) and (8.5) in (5.7), we get the result.

8.1. - A particular case.

Let, in Theorem 6, $g(t) = t^{\nu/2}\cdot J_\nu(2a\sqrt{t})$, so that [**11**, p. 31]

$$\sqrt{s}\cdot f(1/s) = \frac{\Gamma(\nu+1+2m)}{\Gamma(\nu+3/2+m-k)}\left(\frac{a}{s}\right)^\nu\cdot {}_1F_1\left\{\begin{matrix}\nu+1+2m\\ \nu+3/2+m-k\end{matrix};\ -\frac{a^2}{s}\right\},$$

$$\Re(\nu+1+2m)>0,\qquad \Re(s)>0,$$

therefore,

$$f(t) = \frac{a^\nu\cdot\Gamma(\nu+1+2m)}{\Gamma(\nu+3/2+m-k)}\, t^{\nu+(1/2)}\cdot {}_1F_1\left\{\begin{matrix}\nu+1+2m\\ \nu+3/2+m-k\end{matrix};\ -a^2t\right\}$$

and [**2**, p. 383]

$$\varphi(s:\ k,\ m) = \frac{a^\nu\cdot\Gamma(\nu+1+2m)\cdot\Gamma_*(\nu+(3/2)+m\pm m)}{s^{\nu+(1/2)}\cdot\Gamma_*(\nu+7/4+m-k\pm 1/4)}\cdot$$

$$\cdot {}_3F_2\left\{\begin{matrix}\nu+1+2m,\ \nu+3/2,\ \nu+3/2+2m\\ \nu+3/2+m-k,\ \nu+2+m-k\end{matrix};\ -\frac{a^2}{s}\right\},$$

$$\Re(\nu+3/2+m\pm m)>0,\qquad \Re(s)>0.$$

Theorem 6, then, gives

$$(8.6)\qquad \int_0^\infty t^{-2\nu-2}\cdot {}_3F_2\left\{\begin{matrix}\nu+1+m,\ \nu+3/2,\ \nu+3/2+m\\ \nu+5/4+m/2-k/2,\ \nu+7/4+m/2-k/2\end{matrix};\ -\frac{a^2}{t^2}\right\}\Theta(t)\,dt=$$

$$=\frac{\sqrt{\pi}\cdot\Gamma(2\nu+5/2+m-k)}{2(2a)^\nu\cdot\Gamma(\nu+3/2)\cdot\Gamma(2\nu+2+2m)}\int_0^\infty t^\nu\cdot J_\nu(at)\cdot\psi(t:\ k,\ m)\,dt,$$

prodived $\Re(\nu+1+m\pm m)>0$ and the integrals converge absolutely.

8.2. – Examples.

(i) Let, in (8.6), $\varphi(s:\ k,\ m) = s^{-\nu-1}\, e^{-2b/s}$, so that [**14**, p. 313]

$$\Theta(t) = \frac{\Gamma(\nu + 5/2 + m - k)}{\Gamma_*(\nu + 2 + m \pm m)}\, t^{\nu+1}\cdot {}_1F_2\left\{\begin{matrix} \nu + 5/2 + m - k \\ \nu + 2,\ \nu + 2 + 2m \end{matrix};\ -2bt\right\},$$

$$\Re(\nu + 2 + m \pm m) > 0 \qquad \text{and} \qquad \Re(s) > 0.$$

The equation (8.6), then, reduces to

$$\int_0^\infty t^{-\nu-1}\cdot {}_3F_2\left\{\begin{matrix} \nu + 1 + m,\ \nu + 3/2,\ \nu + 3/2 + m \\ \nu + 5/4 + m/2 - k/2,\ \nu + 7/4 + m/2 - k/2 \end{matrix};\ -\frac{a^2}{t^2}\right\}\cdot$$

$$\cdot\, {}_1F_2\left\{\begin{matrix} \nu + 5/2 + m - k \\ \nu + 2,\ \nu + 2 + 2m \end{matrix};\ -2bt\right\} dt =$$

$$= \frac{\sqrt{\pi}\cdot\Gamma(2\nu + 5/2 + m - k)\cdot\Gamma_*(\nu + 2 + m \pm m)}{2(2a)^\nu\cdot\Gamma(\nu + 3/2)\cdot\Gamma(\nu + 5/2 + m - k)\cdot\Gamma(2\nu + 2 + 2m)}\cdot$$

$$\cdot\int_0^\infty t^{-1}\cdot\exp\,(-2b/t)\cdot J_\nu(at)\, dt\,.$$

Evaluating the right hand integral by a known integral [**5**, p. 340]

$$\int_0^\infty x^{-1}\cdot\exp\,(-2b/x)\cdot J_\nu(x)\, dx = 2\cdot J_\nu(2\sqrt{b})\cdot K_\nu\,(2\sqrt{b}), \qquad \Re(b) > 0,$$

and replacing $\nu + 1 + m$ by α and $\nu + 5/4 + m/2 - k/2$ by β, we get

$$\int_0^\infty t^{-\nu-1}\cdot {}_3F_2\left\{\begin{matrix} \alpha,\ \alpha + 1/2,\ \nu + 3/2 \\ \beta,\ \beta + 1/2 \end{matrix};\ -\frac{a^2}{t^2}\right\}\cdot {}_1F_2\left\{\begin{matrix} 2\beta - \nu \\ 2\alpha - \nu,\ \nu + 2 \end{matrix};\ -2bt\right\} dt =$$

$$= \frac{\sqrt{\pi}\cdot\Gamma(2\alpha - \nu)\cdot\Gamma(2\beta)\cdot\Gamma(\nu + 2)}{(2a)^\nu\cdot\Gamma(2\alpha)\cdot\Gamma(2\beta - \nu)\cdot\Gamma(\nu + (3/2))}\, J_\nu(2\sqrt{ab})\cdot K_\nu(2\sqrt{ab}),$$

$$\Re(\nu + 1) > 0, \qquad \Re(2\alpha - \nu) > 0,$$

$$\Re(3\nu + 2\alpha + (3/2)) > \Re(2\beta) > 0 \qquad \text{and} \qquad \Re(b) > 0\,.$$

(ii) Let, in (8.6), $\psi(s : k,\ m) = \dfrac{s}{(s+b)^{\mu+1}}$, so that [**14**, p. 316],

$$\Theta(t) = \frac{\Gamma(\mu + 3/2 + m - k)}{\Gamma_*(\mu + 1 + m \pm m)}\, t^{\mu} \cdot {}_1F_1 \left\{ \begin{matrix} \mu + 3/2 + m - k \\ \mu + 1 + 2m \end{matrix} ;\ -bt \right\},$$

$$\Re(\mu + 1 + m \pm m) > 0 \qquad \text{and} \qquad \Re(s) > 0.$$

The equation (8.6), then, reduces to

$$\int_0^\infty t^{\mu-2\nu-2} \cdot {}_3F_2 \left\{ \begin{matrix} \nu + 1 + m,\ \nu + 3/2,\ \nu + 3/2 + m \\ \nu + 5/4 + m/2 - k/2,\ \nu + 7/4 + m/2 - k/2 \end{matrix} ;\ -\frac{a^2}{t^2} \right\} \cdot$$

$$\cdot {}_1F_1 \left\{ \begin{matrix} \mu + 3/2 + m - k \\ \mu + 1 + 2m \end{matrix} ;\ -bt \right\} dt =$$

$$= \frac{\sqrt{\pi} \cdot \Gamma(2\nu + 5/2 + m - k) \cdot \Gamma_*(\mu + 1 + m \pm m)}{2(2a)^{\nu} \cdot \Gamma(\nu + 3/2) \cdot \Gamma(2\nu + 2 + 2m) \cdot \Gamma(\mu + 3/2 + m - k)} \int_0^\infty \frac{t^{\nu+1} \cdot J_\nu(at)}{(t+b)^{\mu+1}}\, dt .$$

Evaluating the integral on the right hand side by a known integral [**7**, p. 436] and replacing $\nu + 1 + m$ by α, $\nu + 5/4 + m/2 - k/2$ by β, $\nu + 3/2$ by γ and $\mu - 2\nu - 1$ by 2δ, we get

$$\int_0^\infty t^{2\delta-1} \cdot {}_3F_2 \left\{ \begin{matrix} \gamma,\ \alpha,\ \alpha + 1/2 \\ \beta,\ \beta + 1/2 \end{matrix} ;\ -\frac{a^2}{t^2} \right\} \cdot {}_1F_1 \left\{ \begin{matrix} 2(\beta + \delta) \\ 2(\alpha + \delta) \end{matrix} ;\ -bt \right\} dt =$$

$$= \frac{\pi \cdot \Gamma(2\beta) \cdot \Gamma(2\alpha + 2\delta)}{b^{2\delta} \cdot \Gamma(2\alpha) \cdot \Gamma(\gamma) \cdot \Gamma(2\beta + 2\delta) \cdot \sin(2\delta\pi)} \cdot$$

$$\cdot \left[\sum_{r=0}^{\infty} \frac{(-a^2b^2)^r \cdot \Gamma(\gamma + r)}{r!\ \Gamma(1 - 2\delta + 2r)} - \sum_{r=0}^{\infty} \frac{(ab)^{2\delta+r} \cdot \Gamma(\delta + \gamma + r/2) \cdot \cos\{(2\delta + r)\pi/2\}}{r!\ \Gamma(\delta + 1 + r/2)} \right],$$

$$\Re(\alpha + \delta) > 0, \qquad \Re(2\alpha - \gamma + 3/2) > 0, \qquad \Re(\gamma + \delta - 1/2) > 0,$$

$$\Re(\delta) > 0, \qquad \Re(\nu) > 0, \qquad \Re(\beta) > 0, \qquad \Re(a) > 0 \qquad \text{and} \qquad \Re(b) > 0 .$$

9. - Theorem 7.

If

$$\varphi(s:k,\ m) = W[f(t);\ k,\ m], \qquad g(s:k,\ m) = W[\varphi(1/t:k,\ m)/\sqrt{t}\,;\ k,\ m]$$

and

$$\psi(s:k,\ m) = W[\Theta(t);\ k,\ m],$$

then, provided the integrals involved converge absolutely,

$$\int_0^\infty g(t^2/4:(k/2)+(1/4),\ m/2)\cdot\Theta(t)\cdot t^{-2}\,dt = \sqrt{\pi}\,2^{-k-m-(1/2)}\int_0^\infty \psi(t:k,\ m)\cdot f(t^2)t^{-1}\,dt.$$

The theorem follows if we proceed as in Theorem 6 and use Theorem 5 instead of Theorem 4.

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