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Approximations of Φ-integrals by Riemann and Darboux sums, and other contributions to the theory of Φ-integrals in general spaces. (**)

Introduction.

This article is based directly upon a portion of the work on set functions by H. HAHN and A. ROSENTHAL (1). Consequently constant reference will be made to this work. The symbol HR will be used to denote such reference.

As in HR, § 12.1, let E be a general space, without particular metric or topological structure. Let Φ be a totally additive set function defined in a σ-field $\mathfrak{M}$ of subsets of E, such that $\mathfrak{M}$ is complete for Φ. Denote by Φ^+, Φ^-, and $\overline{\Phi}$ the positive-function, negative-function, and absolute-function of Φ, respectively (2).

Let $A \in \mathfrak{M}$, and denote by $\mathfrak{A}$ the σ-field of the subsets of A that are contained in $\mathfrak{M}$. Let f be a point function Φ-measurable on A. The function f is said to be *Φ-integrable on A* if for every $M \in \mathfrak{A}$ there exists a set function $\lambda(M)$ possessing the following properties (3):

1) $\lambda(M)$ is totally additive in $\mathfrak{A}$.

2) If $M \in \mathfrak{A}$ and if, for $x \in M$, $c \leqq f(x) \leqq c'$, then
$c \cdot \Phi(M) \leqq \lambda(M) \leqq c' \cdot \Phi(M)$ if $\Phi^-(M) = 0$ (4),
$c' \cdot \Phi(M) \leqq \lambda(M) \leqq c \cdot \Phi(M)$ if $\Phi^+(M) = 0$ (5).

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(1) H. HAHN and A. ROSENTHAL, *Set Functions* (Albuquerque, The University of New Mexico Press, 1948).

(2) HR, § 3.4.

(3) HR, § 12.1.

(4) That is, Φ is monotone increasing on M.

(5) That is, Φ is monotone decreasing on M.

If here $c = \pm\infty$ (or $c' = \pm\infty$) and $\Phi(M) = 0$, then it must be assumed that $c \cdot \Phi(M) = 0$ (or $c' \cdot \Phi(M) = 0$).

Since it can be proved readily [6] that if $\lambda(A)$ exists its value is uniquely determined by the conditions 1) and 2), the value of $\lambda(A)$ is called the *Φ-integral* of f on A, denoted [6] by $\lambda(A) = (A)\int f\,\mathrm{d}\Phi$.

In case $\Phi(A)$ is finite, the integral $(A)\int f\,\mathrm{d}\Phi$ is said to be proper. (It is to be noted that herein the case of infinite f is included). If $\Phi(A)$ is infinite, however, the integral $(A)\int f\,\mathrm{d}\Phi$ is said to be improper. The improper integral is here, as in HR, restricted to the case in which $A = \underset{i}{S} A_i$, where the A_i's are disjoint and every $\Phi(A_i)$ is finite.

The theory of the proper Φ-integral has been extensively developed in HR Chapter IV. In HR, § 12.7, a number of theorems have been extended to the improper case. Furthermore, P. T. Mielke, in his Thesis on improper integrals [7], has extended to the improper case many other theorems of HR, § 12, and in particular has discussed thoroughly the approximation of the improper Φ-integral by means of Lebesgue sums.

It is the purpose of the present article further to develop the theory of the improper Φ-integral.

In Chapter I the approximation of the Φ-integral by means of Riemann and Darboux sums will be discussed. In considering Riemann sums, the theorems of HR, §§ 13.2 and 13.3, will be examined and, wherever possible, extended to the case of infinite $\Phi(A)$. It will be seen, however, that important theorems concerning approximation of the Φ-integral by a sequence of Riemann sums cannot be extended directly. Nevertheless, valid extensions will be produced by the introduction of certain additional restrictions, namely, that all Riemann sums be finite and all bounded Φ-measurable sets be of finite Φ-measure, or in another case that all decompositions of A be subdivisions of a fixed decomposition.

In considering Darboux sums [8], the discussion will consist of three parts: the first concerning the extension of the theory to the case of infinite $\Phi(A)$ with monotone increasing Φ; the second extending the theory to the case where Φ is general (non-monotone) but $\Phi(A)$ is finite; and the third considering the case of a general Φ and an infinite $\Phi(A)$. In the first and third cases, certain restrictions must be introduced.

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(6) HR, § 12.1 and § 12.7.

(7) P. T. Mielke, *Improper integrals in abstract spaces*, Thesis, Purdue University, 1951.

(8) K. Krickeberg: Math. Nachr. 9, 86-128 (1953); Arch. Math. 6, 432-436 (1953), recently used the Darboux sums to define upper and lower integrals and, if both are equal, the integral. His results hold even without the above assumption that the set A of integration is the sum of countably many sets A_i with finite $\Phi(A_i)$.

In Chapter II a concept will be discussed which is not based so directly upon HR as the rest of this paper. This concept, denoted by $(A)\int_i {}^* f\,d\Phi$, is a limit, namely $\lim_i (B_i)\int f\,d\Phi$, where $B_i = \underset{j=1}{\overset{i}{S}} A_j$. However, if no restrictions were placed upon the space E and on the decompositions $A = \underset{i}{S} A_i$, then $(A)\int^* f\,d\Phi$, in the finite case or even under somewhat wider conditions, would not be different from $(A)\int f\,d\Phi$. In order, therefore, to produce an integral $(A)\int^* f\,d\Phi$ which is essentially more general than the corresponding integral $(A)\int f\,d\Phi$, a metric will be imposed upon the space E, and the decompositions $A = \underset{i}{S} A_i$ will be obtained by means of concentric spheres; furthermore, Φ will be assumed to be finite for bounded Φ-measurable sets.

Chapter III will be a direct generalization of HR, § 14, which concerns mean value theorems and inequalities. Here, as in the case of the DARBOUX sums, since most of the theory already developed is for monotone increasing Φ only, the theorems will be generalized first for the case of monotone increasing Φ with $\Phi(A) = +\infty$, next for the case of general (i.e., non-monotone) Φ with $\Phi(A)$ finite, and finally for the case of general Φ with $\Phi(A) = \pm\infty$. The generalized definition of the polar metric $\Delta(g)$ (9) will have to be modified somewhat in order to yield a satisfactory theory.

In Chapter IV, the theory of integration of sequences of functions, as developed in HR, § 15, will be generalized directly. In extending HR, § 15.2, however, difficulties arise with respect to the definition of complete Φ-integrability. To clear up these difficulties, a modified (but quite natural) extension of the original definition will be presented, and the theory of HR, § 15.2, finally re-developed on the basis of this modification.

CHAPTER I. – Approximation of the improper Φ-integral.

1. – Riemann sums.

Throughout this article, unless otherwise stated, it will be assumed that A is a set of infinite Φ-measure, and further that $A = \underset{i}{S} A_i$, where the A_i's are disjoint and Φ-measurable, and $\Phi(A_i)$ is finite for $i = 1, 2, 3, \ldots$.

Definition. Let $A = \underset{i}{S} A_i$ denote a decomposition D of A into countably many subsets A_i as indicated above. Let $f(x)$ be Φ-defined on A, and

(9) HR, § 14.(4.2).

let $x_i \in A_i$. Then $S(f, \Phi, D) = \sum_i f(x_i)\Phi(A_i)$, provided this sum has meaning, is called a *Riemann sum* associated with the decomposition D (1).

Definition. Assume A to be a point set of a metric space E. The decomposition $A = \underset{i}{S} A_i$ is said to have the norm ϱ if for all A_i the diameters $d(A_i) \leqq \varrho$. If $\varrho_\nu \to 0$ and D_ν is a decomposition of A into Φ-measurable subsets of finite measure of norm ϱ_ν, then $\{D_\nu\}$ is said to be a *distinguished sequence of decompositions* of A (2). In every case where the notion of distinguished sequences is used, it must be assumed that E is a metric space.

The theorems HR, 13.2.1, 13.2.3 and 13.3.1, and their proofs can be extended immediately. We state here only the last of these three theorems:

Theorem 1.1. *In order that there be a distinguished sequence of decompositions of A, it is necessary and, if the open sets are Φ-measurable, also sufficient that A be separable.*

On the other hand, theorems HR, 13.2.2, 13.2.21, 13.2.4 and 13.3.2, are not extendable. Counter-examples can be constructed easily. Hence essential modifications of these theorems are formed in the following discussion. Some lemmas are needed for this purpose.

Lemma I. If f is bounded and Φ-continuous on A, if the open sets are Φ-measurable, and if for all decompositions of A all associated Riemann sums are finite, then 1) all sums of the form $\sum_i f(x_i)\Phi(A_i)$ are Riemann sums, i.e., have meaning, 2) $(A)\int f d\Phi$ exists, and 3) $(A)\int f d\Phi$ is finite.

Proof. 1) Let $\Phi(A) = +\infty$. Assume there exists a sum $S = S(f, \Phi, D) = \sum_i f(x_i)\Phi(A_i)$ which is meaningless. S can be separated into two sums, S_1 and S_2, producing a separation of A into two disjoint non-empty subsets, A_1 and A_2, such that $S_1(f, \Phi, D(A_1)) = +\infty$, and $S_2(f, \Phi, D(A_2)) = -\infty$, where it can be assumed that all terms of S_1 are positive; of S_2, negative. D divides A_1 into denumerably many disjoint subsets $A_{1,i}$ and A_2 into denumerably many disjoint subsets $A_{2,i}$. One may neglect the terms of S_1 for which $\Phi(A_{1,i}) < 0$, and the terms of S_2 for which $\Phi(A_{2,i}) < 0$, since their respective sums either are finite or converge absolutely, and hence do not affect the behavior of S_1 and S_2. Denote these remaining sums by S_1' and S_2'. By the help of these sums an infinite Riemann sum on A can be formed, contrary to the hypothesis. A similar argument holds if $\Phi(A) = -\infty$.

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(1) HR, § 13.2.
(2) HR, § 13.3.

2) Assume $(A)\int f\,d\Phi$ does not exist. Then $(A^+A' + A^-A'')\int f\,d\Phi = +\infty$ and $(A^+A'' + A^-A')\int f\,d\Phi = -\infty$ (3). Hence $(A^+A' + A^-A'')$ can be divided into denumerably many disjoint subsets M_i, $(i=1, 2, 3,\ldots)$, such that $(M_i)\int f\,d\Phi > 1$, and $(A^+A'' + A^-A')$ can be divided into denumerably many disjoint subsets $\overline{M}_i$, $(i=1, 2, 3,\ldots)$, such that $(\overline{M}_i)\int f\,d\Phi < -1$. By HR, 13.3.2, decompositions and corresponding RIEMANN sums exist such that, for all i and for $0<\varepsilon<1$,

$$\text{(I)} \qquad |(M_i)\int f\,d\Phi - S(f, \Phi, D_{1,i}(M_i))| < \varepsilon ,$$

$$\text{(II)} \qquad |(\overline{M}_i)\int f\,d\Phi - S(f, \Phi, D_{2,i}(\overline{M}_i))| < \varepsilon .$$

Defining a decomposition of A by the totality of the decompositions in (I) and (II), one may obtain a meaningless sum of the form $\sum_i f(x_i)\Phi(Q_i)$, contrary to part 1) of the lemma.

3) By part 2), $(A)\int f\,d\Phi$ exists. Assume $(A)\int f\,d\Phi = +\infty$. Then by the help of HR, 13.3.2, an infinite RIEMANN sum can be constructed, contrary to the hypothesis. A similar argument holds in case $(A)\int f\,d\Phi = -\infty$.

Lemma II. If $\Phi(A)$ is finite, if f is bounded and Φ-continuous on A, if $\{D_\nu\}$ is a given distinguished sequence of decompositions of A, and if M is any Φ-measurable subset of A, then for $\varepsilon > 0$ there exists $\bar{\nu}(\varepsilon)$, independent of M, such that for $\nu > \bar{\nu}$ and for all associated RIEMANN sums, $|S(f, \Phi, D_\nu(M)) - (M)\int f\,d\Phi| < \varepsilon$, where $D_\nu(M)$ denotes the decomposition of M induced by the decomposition D_ν of A.

Proof. The sequence $\{D_\nu\}$ is adapted (4) to the function f (5). Then for $\delta > 0$ there exists a sequence of positive constants $\eta_\nu \to 0$ such that for any RIEMANN sum associated with D_ν, $|S(f, \Phi, D_\nu) - (A)\int f\,d\Phi| < \delta\cdot\overline{\Phi}(A) + 2B\eta_\nu$, where B denotes $\sup_{x\in A}|f(x)|$. Choose δ so that $\delta\cdot\overline{\Phi}(A) < \varepsilon/2$. For $\nu > \bar{\nu}$ we have $2B\eta_\nu < \varepsilon/2$ and hence $|S(f, \Phi, D_\nu) - (A)\int f\,d\Phi| < \varepsilon$. Furthermore, there is a sequence of positive constants ξ_ν with $\xi_\nu \leqq \eta_\nu$, such that for $\nu > \bar{\nu}$, and for any RIEMANN sum associated with $D_\nu(M)$,

$$|S(f, \Phi, D_\nu(M)) - (M)\int f\,d\Phi| < \delta\cdot\overline{\Phi}(M) + 2B\xi_\nu \leqq \delta\cdot\overline{\Phi}(A) + 2B\eta_\nu < \varepsilon .$$

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(3) For every Φ-measurable set A there exists a decomposition $A=A'+A''$ where A' and A'' are disjoint and where $\Phi^-(A')=0$ and $\Phi^+(A'')=0$ (HR, 3.4.71). $A^+=A[f(x)\geqq 0]$ and $A^-=A[f(x)<0]$ (HR, p. 156).

(4) A sequence $\{D_\nu\}$ of decompositions is said to be *adapted to the function* f if for every $\delta > 0$ the sum B_ν of all those $A_{\nu,i}$ on which $\sup_{x\in A_{\nu,i}} f(x) - \inf_{x\in A_{\nu,i}} f(x) \geqslant \delta$ satisfies the condition $\overline{\Phi}(B_\nu) \to 0$ (HR, § 13.2).

(5) See the proof of HR, 13.3.2.

Lemma III. If f is Φ-summable on A and if $\Phi(A)$ is finite, there exists $x \in A$ such that

$$|f(x)\cdot\Phi(A)| \leqq (A)\int |f|\,\mathrm{d}\overline{\Phi} \quad (^6).$$

Proof. Let l be $\inf_{x\in A}|f|$. If f assumes the value l at $x' \in A$, then $|f(x')\cdot\Phi(A)| \leqq |f(x')|\cdot\overline{\Phi}(A) \leqq (A)\int|f|\,\mathrm{d}\overline{\Phi}$. Otherwise $l\cdot\overline{\Phi}(A) < (A)\int|f|\,\mathrm{d}\overline{\Phi}$; i.e., $(A)\int|f|\,\mathrm{d}\overline{\Phi} = l\cdot\overline{\Phi}(A) + R$ where $R > 0$. From the definition of l, there exists $x'' \in A$ such that $l < |f(x'')| < l + R/\Phi(A)$. Hence $|f(x'')\Phi(A)| \leqq |f(x'')|\overline{\Phi}(A) < l\overline{\Phi}(A) + R = (A)\int|f|\,\mathrm{d}\overline{\Phi}$.

Lemma IV. If f is Φ-summable on A and if $\Phi(A)$ is finite, there exists a Riemann sum $S(f, \Phi, D)$ on A such that $(A)\int f\,\mathrm{d}\Phi \leqq S(f, \Phi, D)$.

Proof. Set $(A)\int f\,\mathrm{d}\Phi = (A')\int f\,\mathrm{d}\Phi + (A'')\int f\,\mathrm{d}\Phi$. Let L designate $\sup_{x\in A'} f(x)$. It may be that $\Phi(A') = 0$. Then $(A')\int f\,\mathrm{d}\Phi = 0 = f(x')\Phi(A')$ for all $x' \in A'$. Otherwise $\Phi(A') \neq 0$. If f attains the value L at a point $x' \in A'$, then $(A')\int f\,\mathrm{d}\Phi \leqq f(x')\Phi(A')$. Otherwise $(A')\int f\,\mathrm{d}\Phi < L\cdot\Phi(A')$. We may assume $L < +\infty$; and then $(A')\int f\,\mathrm{d}\Phi = L\cdot\Phi(A') - R$ where $R > 0$. From the definition of L, there exists $x' \in A'$ such that $L - R/\Phi(A') < f(x') < L$. Hence $(A')\int f\,\mathrm{d}\Phi = L\cdot\Phi(A') - R < f(x')\Phi(A')$. Thus in every case there exists $x' \in A'$ such that

$$\text{(I)} \qquad (A')\int f\,\mathrm{d}\Phi \leqq f(x')\Phi(A')\,.$$

Analogously, using $\inf_{x\in A''} f(x)$ instead of $\sup_{x\in A'} f(x)$, it can be proved that there exists $x'' \in A''$ such that

$$\text{(II)} \qquad (A'')\int f\,\mathrm{d}\Phi \leqq f(x'')\Phi(A'')\,.$$

The conclusion follows upon combining (I) and (II).

The following theorem is a modified extension of HR, 13.3.21.

Theorem 1.2. *If f is bounded and Φ-continuous on A, if the open sets are Φ-measurable, if all bounded Φ-measurable sets are of finite measure, and if all Riemann sums are finite* (^{7}), *then for every distinguished sequence of decompositions of A and all associated Riemann sums,* $\lim_{\nu} S(f, \Phi, D_\nu) = (A)\int f\,\mathrm{d}\Phi$.

(^{6}) By HR, 12.3.21, $(A)\int|f|\,\mathrm{d}\overline{\Phi}$ is finite.
(^{7}) See Lemma I, part 1).

Proof. By Lemma I, part. 3), f is Φ-summable. Assume the theorem does not hold. Then there exists a distinguished sequence $\{D_\nu\}$ of decompositions of A and an $\varepsilon > 0$ such that for denumerably many ν's (ν_k where $k = 1, 2, 3, \ldots$) there are RIEMANN sums having the property:

1) $(A)\int f\,d\Phi - S(f, \Phi, D_{\nu_k}) > 2\varepsilon$, or, for denumerably many ν's;

2) $S(f, \Phi, D_{\nu_k}) - (A)\int f\,d\Phi > 2\varepsilon$.

Assume situation 1) exists. We replace the notation D_{ν_k} by D_k, since the sequence $\{D_{\nu_k}\}$ is also distinguished. For k fixed, there exists a set B_k (8), with $\Phi(B_k)$ finite, such that

$$(B_k)\int f\,d\Phi - S(f, \Phi, D_k(B_k)) > \varepsilon\,. \tag{I}$$

Set $B^* = \underset{k}{S} B_k$. Define a new sequence $\{M_j\}$ as follows: Let $M_1 = B_1$. It may be that there exists a $k_2 > 1$ with the following property: $B^* - M_1$ contains so many parts of B_{k_2}, determined by D_{k_2}, that, if M_2 denotes the sum of these parts, there is a RIEMANN sum satisfying the relation $(M_2)\int f\,d\Phi - S(f, \Phi, D_{k_2}(M_2)) > \varepsilon/2$. If so, M_2 is defined. Continue this process; i.e., assume that M_{j-1} has been defined. If there exists a decomposition D_{k_j} with index higher than any already used, and a subset M_j of the set $B^* - (M_1 + M_2 + \ldots + M_{j-1})$ where M_j consists of parts of B_{k_j}, determined by D_{k_j}, with a RIEMANN sum satisfying the relation

$$(M_j)\int f\,d\Phi - S(f, \Phi, D_{k_j}(M_j)) > \varepsilon/2\,, \tag{II}$$

then M_j is defined.

Either this process terminates or it does not. In the former case, let $M = \underset{j}{S} M_j$. Since M is bounded, $\Phi(M)$ is finite. Let $M^* \supseteq M$ be the set consisting of all points x of A such that there are points m of M with $\overline{xm} < d$, where $d = \max\limits_{\nu} \varrho_\nu$ (9). By (I) and (II), for every decomposition D_k there is a Φ-measurable subset M_k^* of M^* and an associated RIEMANN sum such that

$$(M_k^*)\int f\,d\Phi - S(f, \Phi, D_k(M_k^*)) \geqq \varepsilon/2\,. \tag{III}$$

(8) $B_k = \overset{i_k}{\underset{j=1}{S}} A_{k,j}$.

(9) Here ϱ_ν denotes the norm of the decomposition D_ν.

For let D_k be any decomposition of our distinguished sequence. If $\underset{i}{S} A_{k,i'}$ represents the sum of parts of D_k contained in $B^* - M$, the relation

$$(\underset{i'}{S} A_{k,i'})\int f\,\mathrm{d}\Phi - S(f,\,\Phi,\,D_k(\underset{i'}{S} A_{k,i'})) \leqq \varepsilon/2$$

holds, and consequently, by (I),

$$(B_k - \underset{i'}{S} A_{k,i'})\int f\,\mathrm{d}\Phi - S(f,\,\Phi,\,D_k(B_k - \underset{i'}{S} A_{k,i'})) > \varepsilon/2,$$

where $B_k - \underset{i'}{S} A_{k,i'}$ consists of parts of D_k lying wholly within M and parts lying partly within M and partly within $B_k - M$. But $B_k - \underset{i'}{S} A_{k,i'}$ lies wholly within M^*; in particular it is the above-mentioned M_k^*. But (III) contradicts Lemma II.

Assume, on the other hand, that the process of forming M_j's does not terminate. Again let $M = \underset{j}{S} M_j$. On M construct a RIEMANN sum as follows: Consider the RIEMANN sums on M_j, $(j = 1, 2, 3, \ldots)$ existing according to (II), such that $(M_j)\int f\,\mathrm{d}\Phi - S(f,\,\Phi,\,D_{k_j}(M_j)) > \varepsilon/2$. It follows that $\sum_j (M_j)\int f\,\mathrm{d}\Phi - \sum_j S(f,\,\Phi,\,D_{k_j}(M_j)) = +\infty$ provided the indicated sums have meaning. The sum $\sum_j (M_j)\int f\,\mathrm{d}\Phi$ has meaning and also has a finite value as a result of the Φ-summability of f on A. Since for all j, $(M_j)\int f\mathrm{d}\Phi > S(f,\,\Phi,\,D_{k_j}(M_j))$, the sum $\sum_j S(f,\,\Phi,\,D_{k_j}(M_j))$ has meaning. Hence $\sum_j S(f,\,\Phi,\,D_{k_j}(M_j)) = -\infty$. But this sum is a RIEMANN sum on M. By the help of Lemma III and the Φ-summability of f on A, a RIEMANN sum of finite value can be constructed on $(A - M)$. Combining these RIEMANN sums on M and on $(A - M)$, one obtains a RIEMANN sum on A with $S(f,\,\Phi,\,D) = -\infty$, contrary to the hypothesis.

The case in which there exist denumerably many decompositions of A having the property that $S(f,\,\Phi,\,D_{\nu_k}) - (A)\int f\,\mathrm{d}\Phi > 2\varepsilon$ can be proved analogously. In this case Lemma IV must be used instead of Lemma III.

While Theorem 1.2 refers to a Φ-summable f, we should consider also the case where $(A)\int f\,\mathrm{d}\Phi = +\infty$. One then obtains the following theorem, whose proof is not presented here, but is contained in the author's Thesis, pp. 18-20.

Theorem 1.3. *If f bounded and Φ-continuous on A, if the open sets are Φ-measurable, if $(A)\int f\,\mathrm{d}\Phi = +\infty$, and if all sums of the form $\sum_i f(x_i)\Phi(A_i)$ have meaning, then for every distinguished sequence $\{D_\nu\}$ of decompositions of A such that D_ν $(\nu = 2, 3, 4, \ldots)$ is a subdivision of $D_{\nu-1}$, and for all associated Riemann sums, $S(f,\,\Phi,\,D_\nu) \to (A)\int f\,\mathrm{d}\Phi$.*

A modified extension of HR, 13.3.21, namely Theorem 1.4, will next be proved. A lemma is necessary, whose proof is rather obvious.

Lemma V. If for every distinguished sequence $\{D_\nu\}$ of decompositions of A there exists a finite $\lim_\nu S(f, \Phi, D_\nu)$, however the RIEMANN sums are chosen, such a limit also exists for every Φ-measurable subset M of A.

Theorem 1.4. *If A is separable, if the open sets are Φ-measurable, if f is bounded on A, and if all bounded Φ-measurable sets are of finite measure, then in order that for every distinguished sequence $\{D_\nu\}$ of decompositions of A there exists the finite $\lim_\nu S(f, \Phi, D_\nu)$, however the Riemann sums may be chosen, it is necessary that f be Φ-continuous.*

Proof. Assume f is not Φ-continuous on A. Then there exists a Φ-measurable subset M of A such that $\overline{\Phi}(M) > 0$, $\Phi(M) \neq \pm\infty$, on which f is not Φ-continuous. Because of Lemma V, this contradicts HR, 13.3.21.

Two modified extensions of HR, 13.3.22, can be proved:

Theorem 1.5. *If Φ is content-like* (10)*, if f is Φ-summable on A and if $\{D_\nu\}$ is a distinguished sequence of decompositions such that D_ν $(\nu=2, 3, 4,\ldots)$ is a subdivision of D_1, them there are associated Riemann sums such that $\lim_\nu S(f, \Phi, D_\nu) = (A)\int f\,\mathrm{d}\Phi$.*

Proof. Let D_1 be the decomposition $A = \underset{i}{S} A_i$, defined in the usual manner. Let $\sum_{j=1}^{i} A_j = B_i$, and define R_i by $A = B_i + R_i$. Since f is Φ-summable and hence $|f|$ is $\overline{\Phi}$-summable on A, for $\varepsilon > 0$ there exists $i_1(\varepsilon)$ such that for $i > i_1$, $(R_i)\int|f|\,\mathrm{d}\overline{\Phi} < \varepsilon$. Therefore for $i > i_1$, $|(R_i)\int f\,\mathrm{d}\Phi| \leq$ $\leq (R_i)\int|f|\,\mathrm{d}\overline{\Phi} < \varepsilon$. Choose a fixed $i > i_1$. Let $\{D_\nu\}$ be a distinguished sequence of decompositions which are subdivisions of D_1. By HR, 13.3.22, on B_i there exist RIEMANN sums such that $|S(f, \Phi, D_\nu(B_i)) - (B_i)\int f\,\mathrm{d}\Phi| < \varepsilon$ for $\nu > \nu_1(\varepsilon)$; while as a result of Lemma III, on R_i, for all ν, there exist RIEMANN sums such that $|S(f, \Phi, D_\nu(R_i))| < \varepsilon$. Hence for $\nu > \nu_1$, (i fixed $> i_1$), $|S(f, \Phi, D_\nu) -$ $- (A)\int f\,\mathrm{d}\Phi| \leq |S(f, \Phi, D_\nu(B_i)) - (B_i)\int f\,\mathrm{d}\Phi| + |S(f, \Phi, D_\nu(R_i))| + |(R_i)\int f\,\mathrm{d}\Phi| < 3\varepsilon$, that is, $S(f, \Phi, D_\nu) \to (A)\int f\,\mathrm{d}\Phi$.

Theorem 1.6. *If Φ is content-like, if every bounded Φ-measurable set is of finite Φ-measure, if f is Φ-summable on A, and if $\{D_\nu\}$ is a distinguished sequence of decompositions, then there exist associated Riemann sums such that $\lim_\nu S(f, \Phi, D_\nu) = (A)\int f\,\mathrm{d}\Phi$.*

Proof. Consider a system of concentric open spheres $\{S_{a_0,\varrho}\}$ centered at a fixed point a_0 with radius ϱ, $(0 < \varrho < \infty)$. By hypothesis each sphere is

(10) If E is metric, if $\mathfrak{M}$ contains all open sets, and if to every $M \in \mathfrak{M}$ there is a G_δ-set $B \supseteq M$ with $\overline{\Phi}(B) = \overline{\Phi}(M)$, then Φ is called *content-like* (HR, § 4).

of finite Φ-measure. Designate by $R_{a_0,\varrho}$ the surface of the sphere $S_{a_0,\varrho}$ (i.e., the set of points x for which $\overline{xa_0} = \varrho$). It is easy to show that there exist only countably many ϱ's for which $\bar{\Phi}(R_{a_0,\varrho}) > 0$. Hence there exists a monotone sequence $\{\varrho_i\}$ with $\varrho_i \to \infty$ and $\bar{\Phi}(R_{a_0,\varrho_i}) = 0$. Designate by S_i the corresponding open spheres S_{a_0,ϱ_i}. Now set $B_i = S_i A_i$; then $A = \lim_i B_i$. Moreover set $A_1 = B_1$, $A_i = B_i - B_{i-1}$ $(i > 1)$, and $R_i = A - B_i$.

By arguments analogous to those in the proof of Theorem 1.5, one obtains RIEMANN sums such that for fixed i $(i > \bar{i})$,

$$\text{(I)} \qquad \lim_\nu \left[S(f, \Phi, D_\nu(B_i)) + S(f, \Phi, D_\nu(R_i))\right] = (A)\int f \, d\Phi ,$$

where $D_\nu(B_i)$ and $D_\nu(R_i)$ denote the decompositions of B_i and R_i respectively induced by the decomposition D_ν of A.

It remains to be shown that for any distinguished sequence $\{D_\nu\}$ there exist sequences of RIEMANN sums, $\{S(f, \Phi, D_\nu)\}$, $\{S(f, \Phi, D_\nu(B_i))\}$, and $\{S(f, \Phi, D_\nu(R_i))\}$ such that

$$\text{(II)} \qquad \lim_\nu \left[(S(f, \Phi, D_\nu(B_i)) + S(f, \Phi, D_\nu(R_i))) - S(f, \Phi, D_\nu)\right] = 0 .$$

To any $S(f, \Phi, D_\nu)$, there exist RIEMANN sums $S(f, \Phi, D_\nu(B_i))$ and $S(f, \Phi, D_\nu(R_i))$ such that their sum differs from $S(f, \Phi, D_\nu)$ at most on a set $M_{i,\nu}$, consisting of those parts of D_ν which contain points of both B_i and R_i. That $\bar{\Phi}(M_{i,\nu}) \to 0$ as $\nu \to \infty$, i being fixed, is shown as follows:

Define $Q_{i,\nu}$ to be the set of all $a \in A$ such that there exists an $x \in B_i$ with $\overline{xa} < d_\nu$, where $d_\nu = \max(\delta_\nu, \delta_{\nu+1}, \delta_{\nu+2}, \ldots)$ and hence $d_\nu \to 0$ monotonically. By the triangle inequality $Q_{i,\nu}$ is bounded; hence $\bar{\Phi}(Q_{i,\nu})$ is finite. The sequence $\{Q_{i,\nu}\}$ is monotone decreasing, and every $Q_{i,\nu}$ is open. Hence $\mathop{D}_\nu Q_{i,\nu} = \bar{B}_i$ and $\lim_\nu \bar{\Phi}(Q_{i,\nu}) = \bar{\Phi}(\bar{B}_i)$. But $\bar{\Phi}(\bar{B}_i - B_i) \leqq \bar{\Phi}(R_{a_0,\varrho_i}) = 0$; and hence $\bar{\Phi}(\bar{B}_i) = \bar{\Phi}(B_i)$. Thus $\lim_\nu \bar{\Phi}(Q_{i,\nu} - B_i) = 0$.

Designate by $Q^*_{i,\nu}$ the set of all $a \in A$ such that there exists an $x \in A_{i+1}$ with $\overline{xa} < d_\nu$. Since $Q^*_{i,\nu}$ is bounded, $\bar{\Phi}(Q^*_{i,\nu})$ is finite. $\mathop{D}_\nu Q^*_{i,\nu} = \bar{A}_{i+1}$ implies $\lim_\nu \bar{\Phi}(Q^*_{i,\nu}) = \bar{\Phi}(\bar{A}_{i+1})$. But $\bar{\Phi}(\bar{A}_{i+1} - A_{i+1}) \leqq \bar{\Phi}(R_{a_0,\varrho_{i+1}} + R_{a_0,\varrho_i}) = 0$. Hence $\bar{\Phi}(\bar{A}_{i+1}) = \bar{\Phi}(A_{i+1})$. Thus $\lim_\nu \bar{\Phi}(Q^*_{i,\nu} - A_{i+1}) = 0$. Now consider the set $Q^{**}_{i,\nu} = Q_{i,\nu} \cdot Q^*_{i,\nu}$. For $d_\nu < (\varrho_{i+1} - \varrho_i)$, the set $M_{i,\nu}$ is contained in $Q^{**}_{i,\nu}$. Moreover, $Q^{**}_{i,\nu} \subseteq (Q_{i,\nu} - B_i) + (Q^*_{i,\nu} - A_{i+1})$. Hence $\bar{\Phi}(Q^{**}_{i,\nu}) \leqq \bar{\Phi}(Q_{i,\nu} - B_i) + \bar{\Phi}(Q^*_{i,\nu} - A_{i+1})$ and $\bar{\Phi}(Q^{**}) \to 0$ as $\nu \to \infty$. Thus $\bar{\Phi}(M_{i,\nu}) \to 0$.

Since f if Φ-summable, for $\varepsilon > 0$ there exists $\nu_0(\varepsilon)$ such that for $\nu > \nu_0$, $(M_{i,\nu})\int |f| \, d\bar{\Phi} < \varepsilon$. Hence by Lemma III, for any decomposition D_ν of $M_{i,\nu}$ $(\nu > \nu_0)$ there exists a RIEMANN sum $S(f, \Phi, D_\nu(M_{i,\nu}))$ such that $S(f, \Phi, D_\nu(M_{i,\nu})) < \varepsilon$. Thus relation (II) holds, and from (I) and (II) the Theorem follows.

2. – Darboux sums.

A) Φ monotone increasing and $\Phi(A) = +\infty$.

Definition. Let $\Phi \geqq 0$. Let $\Phi(A) = +\infty$. Let $A = \underset{i}{S} A_i$ where the A_i's are disjoint and $\Phi(A_i)$ is finite for $i=1, 2, 3, \ldots$. Let f be Φ-defined but not necessarily Φ-measurable on A. Let $g_i = \sup_{x \in A_i} f(x)$ and $h_i = \inf_{x \in A_i} f(x)$, and form the sums $\bar{S}(f, \Phi, D) = \sum_i g_i \Phi(A_i)$ and $\underline{S}(f, \Phi, D) = \sum_i h_i \Phi(A_i)$.

If they have meaning, call them the *Darboux sums* (11) associated with the decomposition D; in particular, call the first the *upper sum* and the second the *lower sum*.

By HR, 12.7.2, and HR, § 12.5, (if the sums under consideration have meaning),

$$\text{(I)} \qquad \underline{S}(f, \Phi, D) \leqq (A)\underline{\int} f \mathrm{d}\Phi \leqq (A)\overline{\int} f \mathrm{d}\Phi \leqq \bar{S}(f, \Phi, D) \quad (12).$$

Theorem 1.7. *If* $(A)\underline{\int} f \mathrm{d}\Phi > -\infty$ [*or* $(A)\overline{\int} f \mathrm{d}\Phi < +\infty$], *then all sums* $\bar{S}(f, \Phi, D)$ [*or* $\underline{S}(f, \Phi, D)$] *have meaning.*

This theorem, a direct extension of HR, 13.4.1, holds by the proof of that theorem.

Lemma VI. If $(A)\overline{\int} f \mathrm{d}\Phi < +\infty$, and if C is any Φ-measurable subset of A, then $(C)\overline{\int} f \mathrm{d}\Phi < +\infty$. This lemma follows directly from the extension of HR, 12.5.211. Similarly one obtains the following:

Lemma VII. If $(A)\overline{\int} f \mathrm{d}\Phi > -\infty$ and if C is any Φ-measurable subset of A, then $(C)\underline{\int} f \mathrm{d}\Phi > -\infty$.

Theorem 1.8. *If* $(A)\underline{\int} f \mathrm{d}\Phi > -\infty$ [*or* $(A)\overline{\int} f \mathrm{d}\Phi < +\infty$], *then there is a sequence* $\{D_\nu\}$ *of decompositions of* A *such that* $\bar{S}(f, \Phi, D_\nu) \to (A)\overline{\int} f \mathrm{d}\Phi$ [*or* $\underline{S}(f, \Phi, D_\nu) \to (A)\underline{\int} f \mathrm{d}\Phi$].

This theorem is a direct extension of HR, 13.4.2.

Proof. Case I: *$(A)\overline{\int} f \mathrm{d}\Phi$ is finite.* By Lemma VII, $(A_i)\underline{\int} f \mathrm{d}\Phi > -\infty$

(11) HR, § 13.4.
(12) HR, § 13.(4.1).

for $i=1, 2, 3, \ldots$. Thus by HR, 13.4.2, one can form the decompositions $D_{1,i}$ of A_i $(i=1, 2, 3, \ldots)$ such that $|\bar{S}(f, \Phi, D_{1,i}) - (A_i)\bar{\int} f\,d\Phi| < 1/2^i$. It follows that $|\sum_i \bar{S}(f, \Phi, D_{1,i}) - \sum_i (A_i)\bar{\int} f\,d\Phi| < 1$ since the sum $\sum_i (A_i)\bar{\int} f d\Phi$ has meaning and is finite as a result of the extension of HR, 12.5.211. This inequality can be rewritten in the form $|\bar{S}(f, \Phi, D_1) - (A)\bar{\int} f\,d\Phi| < 1$, where D_1 denotes the decomposition of A generated by the $D_{1,i}$'s $(i=1, 2, 3, \ldots)$. Similarly there exist decompositions D_ν of A $(\nu=2, 3, \ldots)$ such that $|\bar{S}(f, \Phi, D_\nu) - (A)\bar{\int} f\,d\Phi| < 1/\nu$. Hence $\lim_\nu \bar{S}(f, \Phi, D_\nu) = (A)\bar{\int} f\,d\Phi$.

Case II: $(A)\bar{\int} f\,d\Phi = +\infty$. From relation (I), it follows for all upper Darboux sums that $\bar{S}(f, \Phi, D) = +\infty$. Thus the Theorem 1.8 holds.

The following two theorems, direct extensions of HR, 13.4.21 and 13.4.22 respectively, hold as immediate consequences of Theorem 1.8.

Theorem 1.9. *If* $(A)\underline{\int} f\,d\Phi > -\infty$ [*or* $(A)\bar{\int} f\,d\Phi < +\infty$], *then* $(A)\bar{\int} f\,d\Phi = \inf_D \bar{S}(f, \Phi, D)$ [*or* $(A)\underline{\int} f\,d\Phi = \sup_D \underline{S}(f, \Phi, D)$].

Theorem 1.10. *If f is Φ-summable on A, then* $(A)\int f\,d\Phi = \inf_D \bar{S}(f, \Phi, D) = \sup_D \underline{S}(f, \Phi, D)$.

HR, 13.4.3, is not directly extendable. However, a modified extension, namely Theorem 1.11, can be proved. One needs the following lemmas:

Lemma VIII. If all Darboux sums are finite, then all sums of the form $\bar{S}(f, \Phi, D) = \sum_i g_i \Phi(A_i)$ and $\underline{S}(f, \Phi, D) = \sum_i h_i \Phi(A_i)$ are Darboux sums, i.e., have meaning. This lemma follows as an immediate consequence of relation (I) and Theorem 1.7.

Lemma IX. In order that all Darboux sums (both upper and lower) be bounded, it is necessary and sufficient that all Riemann sums be bounded; furthermore, the sums $\bar{S}(f, \Phi, D)$ and $S(f, \Phi, D)$ have the same upper bound, while the sums $\underline{S}(f, \Phi, D)$ and $S(f, \Phi, D)$ have the same lower bound.

Lemma X. If all Darboux sums (both upper and lower) are finite, then all Riemann sums are also finite.

The proofs of Lemmas IX and X are straightforward. In both cases Lemma VIII is used.

On the other hand, if all Riemann sums are finite, it does not follow in general that all Darboux sums are finite. This is shown by the following example:

Example I. Let $A = (0, 1)$. Let $f(x) = q$ for $x = p/q$ (reduced), $f(x)=0$ or irrational x. Let $\Phi(M) = 1$ if $\frac{1}{2} \in M$, $\Phi(M) = 0$ if $\frac{1}{2} \sim\in M$. Then all

RIEMANN sums are finite, but the upper DARBOUX sum is $+\infty$, e.g., for the subdivision $(0, \frac{1}{3})$, $[\frac{1}{3}, \frac{2}{3}]$, $(\frac{2}{3}, 1)$.

Theorem 1.11. *If f is bounded and Φ-continuous on A, if the open sets are Φ-measurable, if all bounded Φ-measurable sets of finite Φ-measure, and if the Darboux sums are finite, then for every distinguished sequence $\{D_\nu\}$ of decompositions of A, $\overline{S}(f, \Phi, D_\nu) \to (A)\int f\,d\Phi$ and $\underline{S}(f, \Phi, D_\nu) \to (A)\int f\,d\Phi$* [13].

Proof. (The proof of the first part of the conclusion is presented here. The proof of the second part is analogous.) Let $\{D_\nu\}$ be a distinguished sequence of decompositions of A. Let $\{\varepsilon_\nu\}$ be a monotone null sequence of positive constants. For all ν there exist RIEMANN sums such that $0 < \overline{S}(f, \Phi, D_\nu) - S(f, \Phi, D_\nu) < \varepsilon_\nu$. Hence it follows that, for these sums, $\lim_\nu \overline{S}(f, \Phi, D_\nu) = \lim_\nu S(f, \Phi, D_\nu)$ provided at least one of these limits exists. By Lemma X all the RIEMANN sums are finite. Hence, by Theorem 1.2, $\lim_\nu S(f, \Phi, D_\nu) = (A)\int f\,d\Phi$, from which $\lim_\nu \overline{S}(f, \Phi, D_\nu) = (A)\int f\,d\Phi$.

Combining the above theorem with Lemma IX one obtains another modified extension of HR, 13.4.3, namely Theorem 1.12.

Theorem 1.12. *If f is bounded and Φ-continuous on A, if the open sets are Φ-measurable, if all bounded Φ-measurable sets are of finite Φ-measure, and if the Riemann sums are bounded, then for every distinguished sequence $\{D_\nu\}$ of decompositions of A, $\overline{S}(f, \Phi, D_\nu) \to (A)\int f\,d\Phi$ and $\underline{S}(f, \Phi, D_\nu) \to (A)\int f\,d\Phi$.*

Lemma XI. If for every distinguished sequence $\{D_\nu\}$ of decompositions of A there exists the finite $\lim_\nu \overline{S}(f, \Phi, D_\nu)$, such a limit also exists for every Φ-measurable subset M of A.

The proof of this lemma is rather obvious. We next present a modified extension of HR, 13.4.32:

Theorem 1.13. *If A is separable, if f is bounded and Φ-measurable on A, if the open sets are Φ-measurable, and if all bounded Φ-measurable sets are of finite Φ-measure, then in order that for every distinguished sequence $\{D_\nu\}$ of decompositions of A there exist a finite $\lim_\nu \overline{S}(f, \Phi, D_\nu)$ [or $\lim_\nu \underline{S}(f, \Phi, D_\nu)$], it is necessary that f be Φ-continuous above (or below) on A.*

Proof. Assume f is not Φ-continuous above on A. Then there exists a bounded Φ-measurable subset M of A on which f is not Φ-continuous. In view of Lemma XI this contradicts HR, 13.4.32.

That the above condition is not sufficient may be shown readily by an example.

---

[13] By Lemma X and Lemma I, $(A)\int f\,d\Phi$ exists and is finite.

B) Φ general and $\Phi(A)$ finite.

The subject of Darboux sums is next considered under the assumptions that Φ is general (non-monotone) and $\Phi(A)$ is finite.

Definition. Let f be a function Φ-defined but not necessarily Φ-measurable on A. Let $A = \underset{i}{S} A_i$ be an arbitrary decomposition D of A into disjoint sets. Form the decomposition [see footnote (3)] $A = \underset{i}{S} A_i A' + \underset{i}{S} A_i A'' = \underset{i}{S} A_i' + \underset{i}{S} A_i''$. Let g_i' be the $\sup_{x \in A_i'} f(x)$; let g_i'' be the $\inf_{x \in A_i''} f(x)$. Let h_i' be the $\inf_{x \in A_i'} f(x)$; let h_i'' be the $\sup_{x \in A_i''} f(x)$. Form the sums $\overline{S}(f, \Phi, D) = \sum_i g_i' \Phi(A_i') + \sum_i g_i'' \Phi(A_i'')$ and $\underline{S}(f, \Phi, D) = \sum_i h_i' \Phi(A_i') + \sum_i h_i'' \Phi(A_i'')$. If these sums have meaning, call them the *Darboux sums* associated with the decomposition D, and in particular call the first one the *upper sum* and the second the *lower sum*.

With this modified extension of the definition of Darboux sums, which contains as a special case the original definition as given in HR, § 13.4, all the theorems of HR, § 13.4, on Darboux sums can be generalized rather easily. Only in HR, 13.4.31 and 13.4.32, Φ-continuity above (below) has to be replaced by Φ-continuity above (below) on A' and Φ-continuity below (above) on A''. The general method of proof of the extension of the theorems of HR, § 13.4. is to let $A = A' + A''$ and to apply the original theorems on A' and A'' separately and finally to combine the results of these applications. These proofs appear in the author's Thesis, pp. 37-42.

C) Φ general and $\Phi(A)$ infinite.

Finally the topic of Darboux sums is considered in the most general case, namely, under the assumption that $\Phi(A)$ is infinite and Φ is general. In this case the Darboux sums are defined as in the case just considered, where the decomposition $A = \underset{i}{S} A_i$ has the usual meaning. Results obtained here are analogous to those obtained in the case where Φ is monotone and $\Phi(A) = +\infty$, except that again Φ-continuity of f above (below) on A has to be replaced by Φ-continuity above (below) on A' and below (above) on A''. The details are presented in the author's Thesis, pp. 43-46.

CHAPTER II. – A discussion of the $\lim_{i\to\infty}(B_i)\int f\,d\Phi$.

In this Chapter the limit of a sequence of integrals will be considered, namely, $\lim_i (B_i)\int f\,d\Phi$, provided this limit exists and is in a certain sense independent of the decomposition of A. If A is in a general space, however, and if the A_i's and B_i's are defined as in the previous Chapter without further restriction, it will be seen from the following discussion that, although $\lim_i (B_i)\int f\,d\Phi$ is somewhat more general than $(A)\int f\,d\Phi$, yet under a certain rather natural condition (see Theorem 2.4), this limit exists if and only if $(A)\int f\,d\Phi$ exists. In any case the limit and the integral have the same value provided both exist.

Theorem 2.1. *If* $(A)\int f\,d\Phi$ *exists, then* $\lim_i (B_i)\int f\,d\Phi$ *also exists independent of the decomposition and furthermore* $(A)\int f\,d\Phi = \lim_i (B_i)\int f\,d\Phi$.

This theorem is a direct consequence of HR, 3.2.41. However, $\lim_i (B_i)\int f\,d\Phi$ may exist in cases where $(A)\int f\,d\Phi$ does not exist, as shown by the following example:

Example II. Let $A=[0, \infty)$. Let $\Phi=\mu_1$ [1]. Let $f(x)=+\infty$ for $0\leqq x\leqq 1$ and $f(x)=-1$ otherwise. Let $A=\underset{i}{S}A_i$ be an arbitrary decomposition of A. Then there exists a smallest index $\bar{i}$ such that the intersection $C_{\bar{i}}$ of $A_{\bar{i}}$ with the set $[0, 1]$ has the property $\Phi(C_{\bar{i}})>0$. For $i\geqq\bar{i}$, $(B_i)\int f\,d\Phi=+\infty$. Hence $\lim_i (B_i)\int f\,d\Phi=+\infty$, while $(A)\int f\,d\Phi$ does not exist.

Theorem 2.2. *In order that* $\lim_i (B_i)\int f\,d\Phi$ *exist, it is necessary that for any subset* M *of* A *such that* $\Phi(M)$ *is finite,* $(M)\int f\,d\Phi$ *exist.*

Proof. Suppose the theorem does not hold. Then there exists M with finite $\Phi(M)$ having the property that $M=M^*+M^{**}$ where M^* and M^{**} are disjoint and where $(M^*)\int f\,d\Phi=+\infty$ while $(M^{**})\int f\,d\Phi=-\infty$. Let $A_1=M$. Then however A_j $(i=2, 3, 4, \ldots)$ is chosen, $B_i\supset M^*$ and $B_i\supset M^{**}$ (for all i) and hence $(B_i)\int f\,d\Phi$ does not exist for any i. Thus $\lim_i (B_i)\int f\,d\Phi$ cannot exist.

In case $\lim_i (B_i)\int f\,d\Phi$ is finite, we have the following theorem:

Theorem 2.3. *In order that* $\lim_i (B_i)\int f\,d\Phi$ *exist independent of the decomposition and be finite it is necessary that, for every subset* M *of* A *with* $\Phi(M)$ *finite,* $(M)\int f\,d\Phi$ *be also finite.*

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[1] μ_1 is here used to denote the one-dimensional LEBESGUE measure.

Proof. Suppose the theorem does not hold. Then there exists M with finite $\Phi(M)$ such that $(M)\int f\,d\Phi = +\infty$ (or $-\infty$), since by Theorem 2.2 this integral exists. Let $A_1 = M$. Then, however A_i $(i = 2, 3, 4, \ldots)$ is chosen, $(B_i)\int f\,d\Phi = +\infty$ (or $-\infty$) and hence for such a decomposition, $\lim_i (B_i)\int f\,d\Phi = +\infty$ (or $-\infty$).

That this condition is not sufficient even for the existence of $\lim_i (B_i)\int f\,d\Phi$ is shown by the following example:

Example III. Let $A = [0, \infty)$. Let $\Phi = \mu_1$ (2). Let $f(x) = 1$ for $x \in [2n, 2n+1)$, and $f(x) = -1$ for $x \in [2n+1, 2n+2)$, for $n = 0, 1, 2, \ldots$. Let $A_i = [i-1, i)$, $(i = 1, 2, 3, \ldots)$. Then $\lim_i (B_i)\int f\,d\Phi$ does not exist.

On the other hand, if $\lim_i (B_i)\int f\,d\Phi$ is known to exist we have the following theorem:

Theorem 2.4. *If $\lim_i (B_i)\int f\,d\Phi$ exists independent of the decomposition and if a finite $\Phi(M)$ implies a finite $(M)\int f\,d\Phi$, then $(A)\int f\,d\Phi$ exists and hence* $(A)\int f\,d\Phi = \lim_i (B_i)\int f\,d\Phi$.

Proof. Suppose $(A)\int f\,d\Phi$ does not exist. Then $A = A^* + A^{**}$ where A^* and A^{**} are disjoint and $(A^*)\int f\,d\Phi = +\infty$ while $(A^{**})\int f\,d\Phi = -\infty$. By the use of Theorem 2.3 it is possible to form a decomposition $A = \underset{i}{S} A_i$ with $\Phi(A_i)$ finite, as usual, in the following manner:

Let $A_1 \subset A^* \ni (A_1)\int f\,d\Phi > 1$.

Let $A_2 \subset A^{**} \ni (A_1 + A_2)\int f\,d\Phi < -1$.

Let $A_3 \subset (A^* - A_1) \ni (A_1 + A_2 + A_3)\int f\,d\Phi > 1$.

Let $A_4 \subset (A^{**} - A_2) \ni (A_1 + \ldots + A_4)\int f\,d\Phi < -1$.

. . . .

Let $A_{2n-1} \subset (A^* - \overset{n-1}{\underset{j=1}{S}} A_{2j-1}) \ni (\overset{2n-1}{\underset{j=1}{S}} A_j)\int f\,d\Phi > 1$.

Let $A_{2n} \subset (A^{**} - \overset{n-1}{\underset{j=1}{S}} A_{2j}) \ni (\overset{2n}{\underset{j=1}{S}} A_j)\int f\,d\Phi < -1$.

. . . .

For this decomposition $\lim_i (B_i)\int f\,d\Phi$ does not exist, contrary to the hypothesis.

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(2) See footnote (1).

Theorem 2.5. *A necessary and sufficient condition that* $\lim_i (B_i)\int f\,d\Phi$ *exist independent of the decomposition and be finite is that f be Φ-summable on A.*

Necessity. Suppose the condition is not necessary. The either $(A)\int f\,d\Phi$ does not exist or $(A)\int f\,d\Phi = +\infty$ (or $-\infty$). In case it does not exist, one can form a decomposition $A = \underset{i}{S}\, A_i$ as in the proof of Theorem 2.4, forming a sequence of integrals $(B_i)\int f\,d\Phi$ possessing no limit, which is impossible. In Case $(A)\int f\,d\Phi = +\infty$ (or $-\infty$), by Theorem 2.1 we have $\lim_i (B_i)\int f\,d\Phi = +\infty$ (or $-\infty$), contrary to our assumption.

Sufficiency. This follows at once from Theorem 2.1.

Equivalent to Theorem 2.5 is the following theorem:

Theorem 2.6. *A necessary and sufficient condition that* $\lim_i (B_i)\int f\,d\Phi$ *exist and be finite is that, for all $M \subset A$ with $\Phi(M)$ finite, $(M)\int f\,d\Phi$ be bounded.*

Thus we see, as indicated previously, that $\lim_i (B_i)\int f\,d\Phi$, if finite, or more generally, under the conditions of Theorem 2.4, is not different from $(A)\int f\,d\Phi$. In order to produce in these cases a $\lim_i (B_i)\int f\,d\Phi$ even when $(A)\int f\,d\Phi$ does not exist, we modify the conditions by considering monotone sequences of a special type. For this purpose the space E is now required to be metric and the sequence of sets $\{B_i\}$ is defined by the help of concentric spheres, as follows:

Let E be a metric space, and let $a \in E$ be a fixed point. Consider the open spheres $S_{a,\varrho}$ with center at a and with radius ϱ. Let $\mathfrak{M}$ be a σ-field of subsets of E, containing the open spheres $S_{a,\varrho}$. Let Φ be a totally additive set function defined in $\mathfrak{M}$, such that all bounded Φ-measurable sets are of finite measure. Let $A \in \mathfrak{M}$, and let $\Phi(A) = +\infty$ or $-\infty$.

Let $\{\varrho_i\} \to \infty$ be a monotone increasing sequence of positive constants; let $\{S_{a,\varrho_i}\}$ be the corresponding sequence of open spheres. Let $B_i = A \cdot S_{a,\varrho_i}$ for $i = 1, 2, 3, \ldots$, and set $A_1 = B_1$ and $A_i = B_i - B_{i-1}$, $(i = 2, 3, 4, \ldots)$.

Let f be a functions Φ-measurable on A and having the property that $(A_i)\int f\,d\Phi$ exists and is finite for $i = 1, 2, 3, \ldots$. If for every sequence $\{B_i\}$ defined as above $\lim_i (B_i)\int f\,d\Phi$ exists, it will be denoted by the symbol $(A)\int^* f\,d\Phi$. Because of HR, 3.2.41 there follows at once

Theorem 2.7. *If f is Φ-integrable on A, then $(A)\int^* f\,d\Phi$ exists and is equal to $(A)\int f\,d\Phi$.*

That the existence of $(A)\int^* f\,d\Phi$ does not imply the existence of $(A)\int f\,d\Phi$ is shown by the following example:

Example IV. Let $f(x) = 1/n$ in the interval $[n-1,\ n-\frac{1}{2})$, and $f(x) = -1/n$ in the interval $[n-\frac{1}{2},\ n)$, $n = 1, 2, 3, \ldots$. Let $A = [0, \infty)$.

Let $\Phi = \mu_1$. Fort this function $\lambda^+(A) = +\infty$, $\lambda^-(A) = +\infty$ (3); from which it follows by HR, 12.7.2, that $(A)\int f \mathrm{d}\Phi$ does not exist. But for any ϱ_i ($n < \varrho_i \leqq \leqq n+1$), we have $|(B_i)\int f \mathrm{d}\Phi| < 1/(2n)$. Since $\varrho_i \to +\infty$, for $\varepsilon > 0$ there exists $\bar{i}$ such that for $i > \bar{i}$, $\varrho_i > (1/(2\varepsilon))^*$, where $(1/(2\varepsilon))^*$ denotes the first integer greater than $1/(2\varepsilon)$. Hence for $i > \bar{i}$, $|(B_i)\int f \mathrm{d}\Phi| < \frac{1}{2(1/(2\varepsilon))^*} < \frac{1}{2(1/(2\varepsilon))} = \varepsilon$. Thus $\lim_i (B_i)\int f \mathrm{d}\Phi = (A)\int^* f \mathrm{d}\Phi = 0$.

Conditions sufficient for the existence of $(A)\int^* f \mathrm{d}\Phi$ are rather easily obtained, and the author's Thesis (pp. 51-60). presents a number of such conditions, as well as a condition for the existence and finiteness of the limit $(A)\int^* fg \mathrm{d}\Phi$.

Chapter III. – Mean value theorems and inequalities.

In this Chapter the material of HR, § 14, in which Φ is generally assumed to be monotone increasing and $\Phi(A)$ finite, will be considered and, wherever possible, extended. The theory of HR, § 14.1 and § 14.2, concerning the first and second mean value theorems for integrals, is immediately extendable to the case of monotone increasing Φ with $\Phi(A) = +\infty$; while for general Φ [whether $\Phi(A)$ is finite or infinite] all this theory breaks down except HR, 14.1.1, 14.2.1 and 14.2.11, where Φ was not originally assumed to be monotone increasing. HR, § 14.3, has a purely algebraic character, while its application to Φ-integrals is discussed in HR, § 14.4; hence only HR, § 14.4, is to be generalized. We shall use again three sets of generalizing hypotheses: A) Φ monotone increasing and $\Phi(A) = +\infty$; B) Φ general and $\Phi(A)$ finite; C) Φ general and $\Phi(A)$ infinite.

A) Φ monotone increasing and $\Phi(A) = +\infty$.

The basic inequality for integrals (1), namely $(A)\int |fg| \mathrm{d}\Phi \leqq \sup|g(x)| \cdot \cdot (A)\int |f| \mathrm{d}\Phi$, where it is assumed that f is Φ-measurable, g is bounded and Φ-measurable and Φ is monotone increasing, holds in the extended case by the argument in HR. The theorem HR, 14.4.1, containing the *Hölder inequality*, may be extended as follows:

(3) Here $\lambda^+ = (A'A^+ + A''A^-)\int f \mathrm{d}\Phi$ and $\lambda^- = -[(A'A^- + A''A^+)\int f \mathrm{d}\Phi$ (see HR, p. 162).

(1) HR, § 14(4).

Theorem 3.1. *If f and g are Φ-measurable on A and if Φ is monotone increasing, then for* $p > 1$,

$$\text{(I)} \qquad (A)\int |fg|\,\mathrm{d}\Phi \leqq \{(A)\int |f|^p\,\mathrm{d}\Phi\}^{1/p}\cdot\{(A)\int |g|^{p/(p-1)}\,\mathrm{d}\Phi\}^{(p-1)/p}.$$

Proof. As in HR, 14.4.1, the theorem reduces to the case where $f \geqq 0$, $g \geqq 0$, and the functions f and g assume only finitely many finite values and no infinite values. Then two cases are distinguishable:

Case I. Both f and g are Φ-summable. In this case the proof in HR, 14.4.1, holds.

Case II. Both f and g are Φ-integrable, but at least one of them is not Φ-summable. In this case let $(A)\int f\,\mathrm{d}\Phi = +\infty$. Then also $(A)\int f^p\,\mathrm{d}\Phi = +\infty$. Again we distinguish two cases:

1) $(A)\int g^{p/(p-1)}\,\mathrm{d}\Phi = 0$. In this case $g^{p/(p-1)} =_\Phi 0$ (2), from which $g =_\Phi 0$ and hence $fg =_\Phi 0$. Thus the left side of the inequality (I) vanishes and the inequality holds (3).

2) $(A)\int g^{p/(p-1)}\,\mathrm{d}\Phi > 0$. In this case the right side of (I) is infinite, and again the inequality holds.

In case $p = 2$ one obtains from inequality (I) the extended *Schwarz inequality*, namely: $(A)\int |fg|\,\mathrm{d}\Phi \leqq \sqrt{(A)\int f^2\mathrm{d}\Phi\cdot(A)\int g^2\,\mathrm{d}\Phi}$.

In order to obtain an inequality of a more general type, a comprehensive theory was originated by E. Helly (4) and further developed by H. Hahn (5), and in HR, pp. 202-204. In this theory a « convex metric » for functions is introduced by attaching to every Φ-measurable function f on A a number $D(f)$ possessing the following properties:

1) $D(f) \geqq 0$, and in particular $D(f) = 0$ if and only if $f =_\Phi 0$.
2) $D(\varrho f) = |\varrho|\cdot D(f)$.
3) $D(f_1 + f_2) \leqq D(f_1) + D(f_2)$, provided $f_1 + f_2$ is Φ-defined.
4) $D(f) = D(|f|)$.
5) If $f = 1$ on A, and if $\Phi(A)$ is finite (6), then $D(f)$ is finite.

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(2) $f =_\Phi 0$ denotes that $f = 0$ except for a zero-set for Φ.

(3) See HR, p. 202, footnote 58.

(4) E. Hellý, Monatsh. Math. u. Phys. **31**, 60-91 (1921).

(5) H. Hahn, Jber. Deutsch. Math. Verein. **30**, 94-95 (1921) (italics); Monatsh. Math. u. Phys. **32**, 3-88 (1922).

(6) The condition « and if $\Phi(A)$ is finite » is not expressly stated in HR, pp. 202-204, since there only sets A with finite $\Phi(A)$ are considered.

In order to extend this theory to the case of infinite $\Phi(A)$, a new independent property 5*) is added.

5*) Let f_1 be Φ-defined on a Φ-measurable subset M of A, with convex metric $D_M(f_1)$. Let $M_1 \subset M_2 (\in \mathfrak{A})$, where $\overline{\Phi}(M_2 - M_1) > 0$. Let $f_2 =_\Phi f_1$ on M_1, $f_2 =_\Phi 0$ on $(M_2 - M_1)$. Then $D_{M_2}(f_2) = D_{M_1}(f_1)$.

That the property 5*) is independent of properties 1)–5) may be shown by the following example:

Example V. Let $D_M(f) = \Phi(M)\cdot(M)\int|f|\,d\Phi$, where M is any Φ-measurable subset of A. It is clear that $D_M(f)$ possesses properties 1)–5). That it does not possess property 5*) may be shown as follows:

Let M_1 be a Φ-measurable subset of A having finite measure. Let f be Φ-summable on M_1 and have the additional property that $D_{M_1}(f) > 0$. Let $M_1 \subset M_2$ and let $\overline{\Phi}(M_2 - M_1) > 0$. Let $f' =_\Phi f$ on M; let $f' =_\Phi 0$ on $(M_2 - M_1)$. Then $D_{M_2}(f') = \Phi(M_2)\cdot(M_2)\int|f'|\,d\Phi = \Phi(M_2)\cdot(M_1)\int|f|\,d\Phi > \Phi(M_1)\cdot(M_1\int|f|\,d\Phi = D_{M_1}(f)$. Hence 5*) does not hold.

As in HR, § 14(4.2), let Δg be defined as follows for every Φ-measurable function g on A:

$$\Delta(g) = \sup_{D(f)=1} (A)\int|fg|\,d\Phi\,,$$

where f runs through all Φ-measurable functions f on A with $D(f) = 1$, and let $\Delta(g)$ be called the *metric polar to* $D(f)$. The properties of $\Delta(g)$ are analogous to properties 1)–4) of $D(f)$. Only the fact that $\Delta g = 0$ implies $g =_\Phi 0$ [corresponding to a part of 1)] requires a special proof:

Let $g \neq_\Phi 0$ on $B \subseteq A$, where $\Phi(B) > 0$. There exists $\varepsilon > 0$ and $C \subseteq B$, with $0 < \Phi(C) < +\infty$, such that $|g| > \varepsilon$ on C. Let $f = 1$ on C, $f = 0$ otherwise. Then $0 < D_A(f) < +\infty$, by 1), 5), and 5*). Let $f^* = f/D_A(f)$. Then $D_A(f^*) = 1$, by 2), and $(A)\int|f^*g|\,d\Phi = (C)\int|f^*g|\,d\Phi > \varepsilon\cdot\Phi(C)/D_A(f) > 0$. Therefore $\Delta(g) > 0$.

The extension of HR, 14.4.2, holds by the argument in HR provided that in the statement of this theorem property 5*) is added. Thus if $D(f)$ satisfies the conditions 1)–5) and 5*), the extension of the inequality HR, § 14 (4.21), holds, namely, $(A)\int|fg|\,d\Phi \leqq D(f)\cdot\Delta(g)$. The extension of the remaining theorems of HR, § 14.4, all hold as in HR, except that in the case of HR, 14.4.3, it is assumed that property 5*) is possessed by $D(f)$.

The examples discussed in HR, § 14.4, are all valid under the assumption that $\Phi(A) = +\infty$, namely:

Example 1. $D(f) = (A)\int|f|\,d\Phi$. If p designates the Φ-supremum of $|g|$, then $\Delta(g) = p$. As in HR, $(A)\int|fg|\,d\Phi \leqq p\cdot D(f)$, and hence $\Delta(g) \leqq p$. To prove the opposite inequality one may let $\{p_\nu\}$ be a monotone increasing sequence of numbers such that $p_\nu \to p$, set $A_\nu^* = A[|g| > p_\nu] = \underset{i}{S} A_\nu^* A_i = \underset{i}{S} A_{\nu,i}$, and then consider the two cases: 1) there exists $\bar{\nu}$ such that for $\nu > \bar{\nu}$, $\Phi(A_\nu^*)$ is finite, and 2) for all ν, $\Phi(A_\nu^*)$ is infinite. (See author's Thesis, p. 66.)

Example 2. $D(f) = \Phi$-supremum of $|f|$. Then $\Delta(g) = (A)\int |g| \mathrm{d}\Phi$. The argument is as in HR.

Example 3. $D(f) = \{(A)\int |f|^p \mathrm{d}\Phi\}^{1/p}$ with $p > 1$. Then $\Delta(g) = \{(A)\int |g|^{p/(p-1)} \mathrm{d}\Phi\}^{(p-1)/p}$. In order to prove that $D(f)$ in this example possesses property 3), one must show that the *Minkowski inequality* holds, namely:

$$\text{(II)} \qquad \{(A)\int |f_1 + f_2|^p \mathrm{d}\Phi\}^{1/p} \leqq \{(A)\int |f_1|^p \mathrm{d}\Phi\}^{1/p} + \{(A)\int |f_2|^p \mathrm{d}\Phi\}^{1/p}.$$

If the right member of the inequality (II) is finite, all products $a_i \cdot (\Phi(A_i B_j))$ and $b_j \cdot (\Phi(A_i B_j))$ (see HR, p. 205) are finite, and the inequality holds as in HR, while if the right side of the inequality is (positively) infinite, the relation holds trivially. That $\Delta(g)$ has the above form can be proved as in HR in case $(A)\int |g|^{p/(p-1)} \mathrm{d}\Phi < +\infty$. If this integral is infinite, then it is easy to give a direct proof for that form of $\Delta(g)$.

B) Φ general and $\Phi(A)$ finite.

HR, § 14(4) and 14.4.1, cannot be generalized in this case since for monotone decreasing Φ it is possible to reverse the inequalities involved. The remaining theorems of HR, § 14.4, can easily be generalized by the use of the following modification of the definition of $\Delta(g)$. The polar metric $\Delta(g)$ was defined for monotone increasing Φ as follows: $\Delta(g) = \sup_{D(f)=1} (A)\int |fg| \mathrm{d}\Phi$. Here, for a general Φ, we replace this definition by: $\Delta(g) = \sup_{D(f)=1} (A)\int |fg| \mathrm{d}\overline{\Phi}$.

Again $\Delta(g)$ possesses properties analogous to 1), 2), 3) and 4). Analogs to the previously mentioned Examples 1-3 can be constructed, by generally replacing Φ by $\overline{\Phi}$ in those examples.

The *Minkowski inequality* (II) breaks down in case of general Φ, since in the case of monotone decreasing Φ, if p is an odd integer, the inequality is reversed.

C) Φ general and $\Phi(A)$ infinite.

The results of the preceding case (i.e., Φ general and $\Phi(A)$ finite) hold here, except that again property 5*) must be added to the properties possessed by $D(f)$; in particular, this must be done in the case of HR, 14(4.21) and 14.4.3. Furthermore $\Delta(g)$ is again defined by: $\Delta(g) = \sup_{D(f)=1}(A)\int|fg|\,d\overline{\Phi}$.

Chapter IV. – Theorems on convergence.

1. – Mean convergence.

Definition. Let the functions f and f_ν $(\nu=1, 2, 3, \ldots)$ be Φ-measurable on A, and let p be a positive number. If, for $\nu=1, 2, 3, \ldots$, the function $|f_\nu|^p$ is Φ-summable on A and if $(A)\int|f_\nu - f|^p\,d\overline{\Phi} \to 0$, then the sequence $\{f_\nu\}$ is said to be *convergent in the mean of order p to f*.

This definition of mean convergence (HR, § 15.1), when applied to the case of infinite $\Phi(A)$, enables us to extend without difficulty most of the theory of HR, § 15.1. In particular, the extensions of HR, 15.1.1, 15.1.11, 15.1.2, 15.1.21 (Necessity), and 15.1.3 - 15.1.5, hold by the arguments in HR. The theorem HR, 15.1.23, can also be extended directly, but requires a new proof. Thus we have:

Theorem 4.1. *If the f_ν are Φ-measurable, if $\{f_\nu\}$ is asymptotically convergent to f* (1), *and if there is a g such that $|g|^p$ is Φ-summable and $|f_\nu| \leqq_\Phi |g|$ for all ν, then $\{f_\nu\}$ is convergent in the mean of order p to f.*

Proof. Let $A = \underset{i}{\mathfrak{S}} A_i$ where the A_i's are disjoint and $\Phi(A_i)$ is finite for $i=1, 2, 3, \ldots$. Since $|g|^p$ is Φ-summable on A it is also $\overline{\Phi}$-summable on A. Hence for $\eta > 0$ there exists $\bar{i}$ such that for $i_2 > i_1 > \bar{i}$ we have $\sum_{i=i_1+1}^{i_2}(A_i)\int|g|^p\,d\overline{\Phi} = (\underset{i=i_1+1}{\overset{i_2}{\mathfrak{S}}} A_i)\int|g|^p\,d\overline{\Phi} < \eta$. Therefore if $R_{\bar{i}}$ denotes the set $A - \underset{i=1}{\overset{\bar{i}}{\mathfrak{S}}} A_i$, we have $(R_{\bar{i}})\int|g|^p\,d\overline{\Phi} \leqq \eta$. Since $|f_\nu| \leqq_\Phi |g|$ and $\{f_\nu\}$ is asymptotically convergent to f, then $(R_i)\int|f_\nu|^p\,d\overline{\Phi} \leqq \eta$ by the extension of HR, 12.1.7; and by HR, 10.3.33, 10.3.64, and the extension of HR, 12.1.7, we have $(R_{\bar{i}})\int|f|^p\,d\overline{\Phi} \leqq \eta$. By the extension of HR, 15.1.1, and by HR, 10.3.64, for all ν,

$$(R_{\bar{i}})\int|f - f_\nu|^p\,d\overline{\Phi} \leqq 2^p\cdot(R_{\bar{i}})\int|f|^p\,d\overline{\Phi} + 2^p\cdot(R_{\bar{i}})\int|f_\nu|^p\,d\overline{\Phi} \leqq 2^{p+1}\eta.$$

Choose $\varepsilon > 0$. Let $2^{p+1}\eta = \varepsilon/2$. Since the hypotheses of the theorem hold on A, they hold on the Φ-measurable subset $B_{\bar{i}} = \underset{i=1}{\overset{\bar{i}}{\mathfrak{S}}} A_i$. Thus by HR, 15.1.23, there exists $\bar{\nu}$ such that $(B_{\bar{i}})\int|f - f_\nu|^p\,d\overline{\Phi} < \varepsilon/2$ for $\nu > \bar{\nu}$, and

(1) For the definition of asymptotic convergence see HR, p. 126.

$(A)\int|f-f_\nu|^p\,\mathrm{d}\overline{\Phi}=(B_{\bar{i}})\int|f-f_\nu|^p\,\mathrm{d}\overline{\Phi}+(R_{\bar{i}})\int|f-f_\nu|^p\,\mathrm{d}\overline{\Phi}<\varepsilon$ for $\nu>\bar{\nu}$. Thus the sequence $\{f_\nu\}$ is convergent in the mean of order p to f.

Theorems HR, 15.1.21, (Sufficiency); HR, 15.1.22, 15.1.6 and 15.1.61, are not extendable. Counter-examples are easy to construct. (See author's Thesis, pp. 79-82.) It is to be noted that the original proofs of these theorems make use of the Φ-summability of a Φ-measurable and Φ-bounded function on a set of finite measure. Such a function is not necessarily Φ-summable in case $\Phi(A)$ is infinite.

2. – Integrable and completely integrable sequences.

Definition. Let $\{f_\nu\}$ be a sequence of Φ-integrable functions which is asymptotically convergent to f on A. The sequence $\{f_\nu\}$ is called *Φ-integrable* if f is also Φ-integrable and if $(A)\int f_\nu\,\mathrm{d}\Phi\to(A)\int f\,\mathrm{d}\Phi$. It is also said that $\{f_\nu\}$ *converges in a Φ-integrable manner to f* (cf. HR, § 15.2, p. 213).

The extension of HR, 15.2.1, holds by the argument in HR.

In the discussion of the extension of the remainder of HR, § 15.2, it is natural first to use without change the definition of completely Φ-integrable sequences of functions, thus producing a direct extension of this portion of the theory.

A) Direct extension.

Definition. Let $\{f_\nu\}$ be a sequence of Φ-integrable functions which is asymptotically convergent to f on A. The sequence $\{f_\nu\}$ is called *completely Φ-integrable* if f is also Φ-integrable and if for every Φ-measurable subset M of A the relation $(M)\int f_\nu\,\mathrm{d}\Phi\to(M)\int f\,\mathrm{d}\Phi$ holds. In this case $\{f_\nu\}$ is said to *converge in a completely Φ-integrable manner to f* (2).

With this definition, extensions of HR, 15.2.2 (Sufficiency), 15.2.221 (Necessity), 15.2.222 (Necessity), part of HR, 15.2.223 (3), and HR, 15.2.31, 15.2.311, 15.2.5 and 15.2.51, all hold by the arguments in HR; while HR, 15.2.22 (Necessity) and 15.2.3, can be extended, but require new proofs (4). On the other hand, HR, 15.2.21, 15.2.22 (Sufficiency), 15.2.222 (Sufficiency), 15.2.23, 15.2.232 and 15.2.4, cannot be extended (5). Furthermore, two ques-

(2) See HR, p. 412.

(3) The proof that $|f|$ and almost all $|f_\nu|$ are Φ-summable holds in the extended case by the argument in HR; the convergence of $\{|f_\nu|\}$ to $|f|$ is not known.

(4) See author's Thesis, pp. 84-86.

(5) For counter-examples, see author's Thesis, pp. 84-86.

tions remain unsettled, namely, the extendibility of HR, 15.2.2 (Necessity), and the convergence of $\{|f_\nu|\}$ to $|f|$ in HR, 15.2.223. The obstacle that appears is that, if no restriction is made as to the choice of the Φ-measurable subsets M of A, sequences of functions can be constructed which are difficult to classify as to complete Φ-integrability.

Hence we introduce a modified (but also quite natural) extension of the definition of complete Φ-integrability. This modification has the advantage that it enables all questions of the extendibility of the theory to be answered completely and that (in contrast to the above direct extension) most theorems of HR, § 15.2, now permit extensions.

B) **Modified extension.**

Definition. Let $\{f_\nu\}$ be a sequence of Φ-integrable functions which is asymptotically convergent to f on A. The sequence $\{f_\nu\}$ is called *completely Φ-integrable in a modified manner* if f is also Φ-integrable and if for every Φ-measurable subset M of A with finite Φ-measure, $(M)\int f_\nu d\Phi \to (M)\int f d\Phi$ (6).

Equivalent to this definition is the following:

Definition. Let the function f be Φ-integrable on A. The sequence $\{f_\nu\}$ is said to converge in a *modified completely Φ-integrable manner* to f on A if it is completely Φ-integrable (7) to f on every Φ-measurable subset M of A with finite Φ-measure.

With this modification of the notion of complete Φ-integrability, the extension of the essential part of the theory of HR, § 15.2, is valid. Certain minor changes, however, must be made in the extensions of HR, 15.2.223, 15.2.23, 15.2.232 and 15.2.4 (8). Only HR, 15.2.2 (Necessity) (9), cannot be extended, as seen by the following counter-example, which shows that a sequence $\{f_\nu\}$ of Φ-integrable functions which is asymptotically convergent to

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(6) Cf. HR, § 15.2, p. 214. Note that in the case of an infinite $\Phi(A)$, the property of complete Φ-integrability includes this modified property; but that this modified property becomes also identical with complete Φ-integrability in case $\Phi(A)$ is finite.

(7) Complete Φ-integrability on M could here be replaced by Φ-integrability on M without changing the meaning of the definition.

(8) In HR, 15.2.223, replace «$|f|$ and almost all $|f_\nu|$ are also Φ-summable on A» by «$|f|$ is also Φ-summable on A»; in HR, 15.2.23 and 15.2.232, omit «f and all f_ν are Φ-summable»; and in HR, 15.2.4, replace «subset M of A» by «subset M of A with finite $\Phi(M)$».

(9) Professor ROSENTHAL states that the following changes are to be made in the proof of HR, 15.2.2 (Necessity): Insert in the third line from the bottom of page 215 (HR),

a Φ-summable function f may be completely Φ-integrable in a modified manner on A even if $(A)\int|f_\nu - f|\,\mathrm{d}\overline{\Phi} \to 0$ does not hold.

Example VI. Let $A = [0, \infty)$ and $\Phi = \mu_1$. For $\nu = 1, 2, 3, \ldots$ set $f_\nu = 1/\nu$ on $[2k, 2k+1)$, $f_\nu = -1/\nu$, on $[2k+1, 2k+2)$ for all integers k with $0 \leqq k < \nu$, and $f_\nu = 0$ on $[2\nu, \infty)$. Let $f = 0$ on A. In this example for every Φ-measurable set $M \subset A$ with finite $\Phi(M)$, we have, for all ν, $|(M)\int f_\nu \,\mathrm{d}\Phi| \leqq (M)\int |f_\nu|\,\mathrm{d}\Phi \leqq$ $\leqq 1/\nu \cdot \Phi(M)$ and $(M)\int f\,\mathrm{d}\Phi = 0$. Hence $(M)\int f_\nu\,\mathrm{d}\Phi \to (M)\int f\,\mathrm{d}\Phi$, i.e., $\{f_\nu\}$ converges to f in a *modified completely Φ-integrable manner*. But for all ν, $(A)\int|f_\nu - f|\,\mathrm{d}\Phi = 2$.

The general technique used in constructing proofs of the modified extensions of the theorems of HR, §, 15.2, is the following: If certain hypotheses of a theorem hold on A, they hold also on all Φ-measurable subsets M of A with finite $\Phi(M)$. Hence the conclusion also holds on M. If this conclusion includes complete Φ-integrability, then the condition of modified complete Φ-integrability on A is automatically satisfied.

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after the definition of B: « If the indices $i_\varkappa$ are chosen according to this method, then not only are 1) and 2) satisfied, but also

1a) $|(B_{i_1} + B_{i_2} + \ldots + B_{i_\varkappa}) \cdot (B - B_{i_{\varkappa+1}}) \int f_{\nu_{i_{\varkappa+1}}} \mathrm{d}\Phi| < \varepsilon/8$, and

2a) $|(B_{i_{\varkappa+1}} + B_{i_{\varkappa+2}} + \ldots) \cdot \{B - [B_{i_1} + \ldots + B_{i_\varkappa}]\} \int f_{\nu_{i_\varkappa}} \mathrm{d}\Phi| < \varepsilon/8$. »

Replace the first sentence of page 216 by the following: « Moreover, $|(B)\int f_{\nu_{i_\varkappa}} \mathrm{d}\Phi| \geqq$ $\geqq |(B_{i_\varkappa})\int f_{\nu_{i_\varkappa}} \mathrm{d}\Phi| - |(B_{i_1} + \ldots + B_{i_{\varkappa-1}}) \cdot (B - B_{i_\varkappa}) \int f_{\nu_{i_\varkappa}} \mathrm{d}\Phi| - |(B_{i_{\varkappa+1}} + B_{i_{\varkappa+2}} + \ldots) \cdot \{B - [B_{i_1} +$ $+ \ldots + B_{i_\varkappa}]\} \int f_{\nu_{i_\varkappa}} \mathrm{d}\Phi|$ and hence by (2) and by 1a) and 2a), $|(B)\int f_{\nu_{i_\varkappa}} \mathrm{d}\Phi| > \varepsilon/4$. »