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The isoperimetric inequality for closed surfaces. (**)

Introduction.

RADÓ [3] has shown that the *spatial isoperimetric inequality*

$$[A_L(\mathbf{S})]^3 \geqslant 36\pi[V_R(\mathbf{S})]^2$$

holds for any closed FRÉCHET surface $\mathbf{S}$, where $A_L(\mathbf{S})$ is the LEBESGUE area of $\mathbf{S}$ and $V_R(\mathbf{S})$ is an appropriately defined volume enclosed by $\mathbf{S}$.

Confining one's attention to the class, $\mathcal{S}$, of closed FRÉCHET surfaces, there are various definitions both for *area* and *enclosed volume*, and the truth of the statement $\mathbf{S} \in \mathcal{S}$ implies $[A(\mathbf{S})]^3 \geqslant 36\pi[V(\mathbf{S})]^2$ clearly depends upon the definitions of A (area) and V (enclosed volume) employed. Thus the isoperimetric inequality can be employed as a criterion to check whether a definition for area is « compatible » with a definition for enclosed volume or not. In fact, a pair (A, V) is said to be *compatible* if and only if $\mathbf{S} \in \mathcal{S}$ implies $[A(\mathbf{S})]^3 \geqslant 36\pi[V(\mathbf{S})]^2$. In this language the result of RADÓ states that (A_L, V_R) is a compatible pair; therefore, the class $\mathcal{C}$ of compatible pairs is nonvacuous.

Suppose that: 1.° (A_1, V_1) and (A_2, V_2) are distinct compatible pairs; that is, there is an $\mathbf{S} \in \mathcal{S}$ such that $A_1(\mathbf{S}) \neq A_2(\mathbf{S})$ or $V_1(\mathbf{S}) \neq V_2(\mathbf{S})$; 2.° $\mathbf{S} \in \mathcal{S}$ implies $A_1(\mathbf{S}) \geqslant A_2(\mathbf{S})$ and $V_1(\mathbf{S}) \leqslant V_2(\mathbf{S})$. Then $\mathbf{S} \in \mathcal{S}$ implies $[A_1(\mathbf{S})]^3 \geqslant [A_2(\mathbf{S})]^3 \geqslant 36\pi[V_2(\mathbf{S})]^2 \geqslant 36\pi[V_1(\mathbf{S})]^2$ and, *from the point of view of the isoperimetric inequality*, (A_2, V_2) is the « better » pair as a matter of mathematical content, since it provides a sharper inequality. This suggests a partial ordering of the class $\mathcal{C}$, writing $(A_1, V_1) < (A_2, V_2)$ if and only if conditions 1° and 2° are fulfilled. In this manner, the isoperimetric inequality can not only be employed as a criterion to check the compatibility of a definition for area and a defi-

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nition for enclosed volume, but on occasion (not always, since the order is partial) can also serve to rank two compatible pairs.

The purpose of this Note is twofold:

1° To exhibit a compatible pair (A_L, V_R^+) such that $(A_L, V_R) < (A_L, V_R^+)$.

2° To examine the consequences of the statement that the isoperimetric *equality* holds for some $\mathbf{S} \in \mathcal{S}$.

The discrepancy between V_R and V_R^+ may be very large; for example there are surfaces $\mathbf{S} \in \mathcal{S}$ for which $V_R(\mathbf{S}) = 0$ and $V_R^+(\mathbf{S}) = \infty$.

It should be mentioned that the result of RADÓ holds for other classes of closed surfaces; for example, those classes which have been called closed LEBESGUE surfaces and closed KERÉKJÁRTÓ surfaces. On the other hand, there are both analytical and topological reasons for confining one's attention to FRÉCHET surfaces (see YOUNGS [8]) and this is done here, though the conclusions to be offered are valid for the other classes mentioned above.

1. – The counting functions.

1.1. – If $\mathbf{S} \in \mathcal{S}$ then (as a matter of definition) it is represented by a mapping $f\colon X \to E_3$, where X is a 2-sphere and E_3 is Euclidean 3-space (see YOUNGS [6] and [7]). The point set S occupied by $\mathbf{S}$ is the set $f(X)$ and is independent of the representation of $\mathbf{S}$. Consequently, any component K of $E_3 - S$ is also independent of the representation of $\mathbf{S}$.

1.2. – If o is selected as the origin in E_3, then any point $z \in (E_3 - o)$ determines a vector from o to z. With this convention and $z \in (E_3 - S)$ define

$$(1) \qquad {}_z\varphi(x) = (f(x) - z) \,/\, |\, f(x) - z\,|, \qquad x \in X.$$

This formula provides a mapping

$${}_z\varphi\colon \; X \to Y$$

where Y is the unit sphere with center o.

1.3. – The 2-dimension ČECH cohomology groups $H^2(X)$ and $H^2(Y)$ – with the additive group of integers taken as the coefficient group – are cyclic infinite. Suppose $\mathbf{x}$ and $\mathbf{y}$ are generators of $H^2(X)$ and $H^2(Y)$ respectively, and consider the homomorphism

$${}_z\varphi^*\colon \; H^2(Y) \to H^2(X)$$

2. – The enclosed volumes.

2.1. – Consider $\mathbf{S} \in \mathcal{S}$ as in 1.1 and the concepts developed in the preceding section. Define

$$V_R(\mathbf{S}) = \int \gamma(\mathbf{S}; z) ,$$

$$V_R^+(\mathbf{S}) = \int \Gamma(\mathbf{S}; z)$$

where 3-dimension LEBESGUE integration is understood over E_3 with positive infinite values permitted.

The non-negative number $V_R(\mathbf{S})$ is the RADÓ volume enclosed by $\mathbf{S}$ (see RADÓ [3]) and may be interpreted as

$$\sum [\text{meas}\,(K)]\gamma(\mathbf{S}; K) ,$$

the summation being over all components K of $S - E_3$. With this interpretation the motivation is obvious and natural.

2.2. – In consequence of theorem 1 one has

THEOREM 2. If $\mathbf{S} \in \mathcal{S}$ then $V_R^+(\mathbf{S}) \geqslant V_R(\mathbf{S})$.

It is a simple matter to construct an $\mathbf{S} \in \mathcal{S}$ such that $V_R^+(\mathbf{S}) > V_R(\mathbf{S})$. Indeed, examples exist with $V_R^+(\mathbf{S}) = (1/36\pi)^{1/2}[A_L(\mathbf{S})]^{3/2}$ and $V_R(\mathbf{S}) = 0$, showing that the isoperimetric *inequality* may be trivially satisfied for V_R while the isoperimetric *equality* holds for V_R^+.

To show that $(A_L, V_R) < (A_L, V_R^+)$ it remains but to prove that (A_L, V_R^+) is a compatible pair. (See Introduction.) The results required for the proof are now tabulated. If no reference is given the theorems follow easily from the previous development.

2.3. – If $\mathbf{S} \in \mathcal{S}$ and $\mathfrak{X}$ is cyclic (see 1.6) then $V_R(\mathbf{S}) = V_R^+(\mathbf{S})$.

2.4. – *Ciclic additivity of* V_R^+. If $\mathbf{S} \in \mathcal{S}$ then

$$V_R^+(\mathbf{S}) = \sum V_R^+(\mathbf{S}_{\mathfrak{E}}) , \quad \mathfrak{E} \in \mathcal{E} . \quad \text{Moreover}$$

$$V_R^+(\mathbf{S}) = \sum V_R(\mathbf{S}_{\mathfrak{E}}) , \quad \mathfrak{E} \in \mathcal{E} \quad \text{by 2.3.}$$

(It may be of interest to note that V_R is not cyclicly additive.)

2.5. – If $\mathbf{S} \in \mathcal{S}$, then $A_L(\mathbf{S}) = \sum A_L(\mathbf{S}_{\mathfrak{E}})$, $\mathfrak{E} \in \mathcal{E}$. (See YOUNGS [7]).

2.6. – It is now possible to prove the compatibility of the pair (A_L, V_R^+).

THEOREM 3. If $\mathbf{S} \in \mathcal{S}$, then $[A_L(\mathbf{S})]^3 \geqslant 36\pi[V_R(\mathbf{S})]^2$.

Proof. Suppose $\mathbf{S}_{\mathfrak{C}}$, $\mathfrak{C} \in \mathcal{C}$ is the cyclic decomposition of $\mathbf{S}$ (see 1.8).

$$\begin{aligned} A_L(\mathbf{S}) &= \sum A_L(\mathbf{S}_{\mathfrak{C}}), \quad \mathfrak{C} \in \mathcal{C}, && \text{by 2.5},\\ &\geqslant (36\pi)^{1/3} \sum [V_R(\mathbf{S}_{\mathfrak{C}})]^{2/3}, && \text{by RADÓ's theorem},\\ &\geqslant (36\pi)^{1/3}[\sum V_R(\mathbf{S}_{\mathfrak{C}})]^{2/3} &&\\ &\geqslant (36\pi)^{1/3}[V_R^+(\mathbf{S})]^{2/3} && \text{by 2.4}. \end{aligned}$$

3. – The isoperimetric equality.

3.1. – If the admissible class of closed surfaces is restricted to an elementary range, then it is well known that the isoperimetric equality holds, if and only if, the surface is a *geometric 2-sphere.* For closed FRÉCHET surfaces this is false as shown by TONELLI [4]. However certain necessary conditions are available.

3.2. – THEOREM 4. If $\mathbf{S} \in \mathcal{S}$ while $0 < A_L(\mathbf{S}) < \infty$ and $[A_L(\mathbf{S})]^3 = 36\pi[V_R^+(\mathbf{S})]^2$, then the cyclic decomposition of $\mathbf{S}$ consists of a single closed FRÉCHET surface.

Proof. The hypothesis implies that the chain of *inequalities* employed in the proof of theorem 3 becomes a chain of *equalities*. Hence $0 < \sum [V_R(\mathbf{S}_{\mathfrak{C}})]^{2/3} = [\sum V_R(\mathbf{S}_{\mathfrak{C}})]^{2/3} < \infty$ and there is precisely one surfaces $\mathbf{S}^*$ in the cyclic decomposition of $\mathbf{S}$ such that $V_R(\mathbf{S}^*) > 0$.

It follows that $A_L(\mathbf{S}_{\mathfrak{C}}) > 0$ if and only if $\mathbf{S}_{\mathfrak{C}} = \mathbf{S}^*$. But $A_L(\mathbf{S}_{\mathfrak{C}}) > 0$ for $\mathfrak{C} \in \mathcal{C}$, see RADÓ [1]. Hence $\mathcal{C}$ has exactly one element and the theorem is proved.

Corollary. Given the hypothesis of the theorem, the middle space $\mathfrak{X}$ of $\mathbf{S}$ has a unique true cyclic element, and this element is a 2-sphere.

It is only necessary to recall that any true cyclic element of a space which is the monotone image of a 2-sphere is itself a 2-sphere. (See WHYBURN [5].)

It thus appears that the 2-sphere which one might expect, in analogy with the classical result, occurs in the middle space and not necessarily in E_3.

It may be (to use the terminology of the proof of theorem 4) that the set S^* occupied by the closed FRÉCHET surface $\mathbf{S}^*$ is a geometric 2-sphere in E_3 and $\mathbf{S}^*$ is representable by a topological mapping. The answer to this conjecture appears remote.

As a concluding remark it should be mentioned that theorem 4 and the corollary are true *a fortiori* if V_R^+ is replaced by V_R.

Bibliography.

[1]. RADÒ, T.: *On continuous path-surfaces of zero area.* Annals of Math. **44**, 173-191 (1943).

[2]. — —: *On continuous mappings of* PEANO *spaces.* Trans. Am. Math. Soc. **58**, 420-454 (1945).

[3]. — —: *The isoperimetric inequality and the* LEBESGUE *definition of surface area.* Trans. Am. Math. Soc. **61**, 530-555 (1947).

[4]. TONELLI, L.: *Sulla proprietà di minimo della sfera.* Rend. Circ. Mat. Palermo **39**, 1-30 (1915).

[5]. WHYBURN, G. T.: *Analytic Topology.* Am. Math. Soc. Coll. Pub., vol. **28**, 1942.

[6]. YOUNGS, J. W. T.: *Curves and surfaces.* Am. Math. Monthly **51**, 1-11 (1944).

[7]. — —: *The topology of* FRÉCHET *surfaces.* Annals of Math. **45**, 753-785 (1944).

[8]. — —: LEBESGUE, FRÉCHET *and* KERÉKJÁRTÓ *varieties.* Am. J. Math. **70**, 481-496 (1948).

[9]. — —: *Remarks on cyclic additivity.* Bull. Amer. Math. Soc. **55**, 427-432 (1949).

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