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Large prime gaps and progressions with few primes

Abstract. We show that the existence of arithmetic progressions with few primes, with a quantitative bound on "few", implies the existence of larger gaps between primes less than x than is currently known unconditionally. In particular, we derive this conclusion if there are certain types of exceptional zeros of Dirichlet L-functions.

Keywords. Primes, prime gaps, primes in progressions, exceptional zero, exceptional character.

Mathematics Subject Classification: Primary: 11N05, 11N13; Secondary 11M20.

1 - Introduction

Estimation of the largest gap, $G(x)$, between consecutive primes less than x is a classical problem, and the best bounds on $G(x)$ are comparatively weak. The strongest unconditional lower bound on $G(x)$ is due to Ford, Green, Konyagin, Maynard and Tao [**6**], who have shown that

$$G(x) \gg \frac{\log x \, \log_2 x \, \log_4 x}{\log_3 x}, \tag{1}$$

for sufficiently large x, with $\log_k x$ the k-fold iterated natural logarithm of x, whereas the best unconditional upper bound is

$$G(x) \ll x^{0.525}, \tag{2}$$

a result due to Baker, Harman and Pintz [**1**]. Assuming the Riemann Hypothesis, Cramér [**3**] showed that

$$G(x) \ll x^{1/2} \log x.$$

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The huge distance between the lower bound (1) and upper bound (2) testifies to our ignorance about gaps between primes. Cramér [**4**] introduced a probabilistic model for primes and used it to conjecture that $\limsup_{x\to\infty} G(x)/\log^2 x \geq 1$; later, Shanks [**16**] conjectured that $G(x) \sim \log^2 x$ based on a similar model. Granville [**9**] modified Cramér's model and, based on analysis of the large gaps in the model, conjectured that $G(x) \geq (1+o(1))2e^{-\gamma}(\log x)^2$. The author, together with William Banks and Terence Tao [**2**], has created another model of primes $\leq x$, the largest gap in the model set depending on an extremal property of the interval sieve. In particular, the existence of a certain sequence of "exceptional zeros" of Dirichlet L-functions (defined below) implies that the largest gap in the model set grows faster than any constant multiple of $(\log x)^2$, and suggests that the same bound holds for $G(x)$. In this paper, we show that the existence of exceptional zeros of a certain type implies a lower bound for $G(x)$ which is larger than the right side of (1). We do not utilize probabilistic models of primes, but instead we argue directly. More generally, we derive a similar conclusion whenever there are arithmetic progressions containing few primes. We denote $\pi(x;q,b)$ the number of primes $p \leq x$ satisfying $p \equiv b \pmod q$. The prime number theorem for arithmetic progressions implies that

$$\pi(x;q,b) \sim \frac{\pi(x)}{\phi(q)}$$

for any *fixed* q, where ϕ is Euler's totient function and $\pi(x)$ denotes the number of primes $p \leq x$. It is a central problem to prove bounds on $\pi(x;q,b)$ which are uniform in q, but the best known results are only uniform for $q \leq (\log x)^{O(1)}$; see [**5**] for the classical theory.

All of the methods used to prove lower bounds on $G(x)$ utilize a simple connection between $G(x)$ and Jacobsthal's function $J(u)$, the maximum gap between integers having no prime factor $p \leq u$. A simple argument based on the prime number theorem and the Chinese Remainder Theorem implies that

$$G(x) \geq J((1/2)\log x) \tag{3}$$

if x is sufficiently large. The best bounds known today for $J(u)$ are

$$u(\log u)\frac{\log_3 u}{\log_2 u} \ll J(u) \ll u^2,$$

the lower bound proved in [**6**] and the upper bound due to Iwaniec [**11**].

T h e o r e m 1.1. *Suppose that x is large, $x > q > b > 0$ and $\pi(x;q,b) \leq \frac{\delta x}{\phi(q)}$ with $0 \leq \delta \leq 1$. Then*

$$G(e^{2u}) \geq J(u) \geq \frac{x-b}{q}.$$

where u is the smallest integer satisfying $u > 2\sqrt{x}$ and $\frac{u}{\log u} \geq \frac{10\delta x}{q}$.

An immediate corollary gives a lower bound on $G(x)$ assuming a lower bound on $L(q,b)$, the least prime in the progression $b \mod q$. We take $\delta = 0$ and $u = \lceil 2\sqrt{x}\,\rceil$.

Corollary 1.2. *Suppose that $L(q,b) > x$. Then $G(e^{4\sqrt{x}}) \geq \frac{x-b}{q}$.*

Theorem 1.1 is a partial converse to a theorem of Pomerance [**15**, Theorem 1], which provides a lower bound on $\max_{(b,q)=1} L(q,b)$ given a lower bound on the maximal gap between numbers coprime to m, where $(m,q) = 1$ and $m \leq q^{1-o(1)}$.

Linnik's theorem [**14**] states that $L(q,b) \ll q^L$ for some constant L; the best published result of this kind is due to Xylouris [**17**], who showed that the bound holds with $L = 5.18$ [1]. Assuming the Extended Riemann Hypothesis (ERH) for Dirichlet L-functions, we obtain a stronger bound $L(q,b) \ll_\varepsilon q^{2+\varepsilon}$ for every $\varepsilon > 0$. If, for some $c > 2$ there are infinitely many pairs (q,b) with $L(q,b) \geq q^c$ (a violation of ERH), then Corollary 1.2 implies that

$$\limsup_{X\to\infty} \frac{G(X)}{(\log X)^{2-\frac{2}{c}}} > 0.$$

It is, however, conjectured that $L(q,b) \ll \phi(q)\log^2 q$; see [**13**] for a precise version of this conjecture and for the best known lower bounds on $\max_{(b,q)=1} L(q,b)$.

We may also exceed the bound in (1) under the assumption that exceptional zeros of Dirichlet L-functions exist. Roughly speaking, an exceptional zero of $L(s,\chi)$ is a zero which is real and very close to 1. As such, their existence violates ERH for $L(s,\chi)$. Classical results (see [**5**, §14]) imply that if $c_0 > 0$ is small enough, and $q \geq 3$, then there is at most one character χ modulo q for which $L(s,\chi)$ has a zero in the region

$$\{\sigma + it \in \mathbb{C} : \sigma \geq 1 - c_0/\log(qt)\},$$

and moreover the character is real and the zero is real. We shall refer to such zeros as "exceptional zeros" with respect to c_0. Moreover, by reducing c_0 if necessary, it is known that moduli q for which an exceptional zero exists are very rare.

Siegel's theorem [**5**, Sec. 21] implies that

$$\log \frac{1}{1-\beta_q} = o(\log q) \quad (q \to \infty), \tag{4}$$

for (hypothetical) exceptional zeros β_q, although we cannot say any rate at which this occurs (the bound is *ineffective*). The exceptional zeros are also

[1] In his Ph.D. thesis [**18**], Xyloris claims a better bound $L = 5$.

known as Siegel zeros or Landau-Siegel zeros in the literature. Their existence implies a great irregularity in the distribution of primes modulo q, given by Gallagher's Prime Number Theorem [**8**]. Here we record an immediate corollary.

Proposition 1.3 (Gallagher). *For some absolute constant $B > 1$, we have the following. Suppose that χ is a real character with conductor q and $L(1-\delta,\chi) = 0$ for some $0 < \delta < 1$. Then, for all b with $\chi_q(b) = 1$ and all $x \ge q^B$, we have*

$$\pi(x;q,b) \ll \frac{\delta x}{\phi(q)}.$$

One can leverage this irregularity to prove *regularity* results about primes that are out of reach otherwise, the most spectacular application being Heath-Brown's [**10**] deduction of the twin prime conjecture from the existence of exceptional zeros (for an appropriate c_0). See Iwaniec's survey article [**12**] for background on attempts to prove the non-existence of exceptional zeros and discussion about other applications of their existence. There are also a variety of problems where one argues in different ways depending on whether or not exceptional zeros exist, a principal example being Linnik's Theorem on primes in arithmetic progressions (see, e.g., [**7**, Ch. 24]).

Apply Proposition 1.3 with $x = q^B$. Recalling 4, we see that the quantity u in Theorem 1.1 satisfies

$$u \asymp \frac{\delta x \log x}{q}$$

and consequently that $\log u \asymp \log q$. We conclude that

Theorem 1.4. *Suppose that χ is a real character with conductor q and that $L(1-\delta,\chi) = 0$ for some $0 < \delta < 1$. Then*

$$G(e^{2u}) \gg \frac{u}{\delta \log u}, \tag{5}$$

for some u satisfying $\log u \asymp \log q$.

For example, if k is fixed and there exist infinitely many exceptional zeros $\delta = \delta_q$ satisfying $\delta_q \le (\log q)^{-k}$, we see that there is an unbounded set of X for which

$$G(X) \gg_k (\log X)(\log_2 X)^{k-1}$$

this improves upon (1) for $k \ge 2$. Similarly, if there is an infinite set of q satisfying $\delta = \delta_q = q^{-\varepsilon(q)}$, where $\varepsilon(q) \to 0$ very slowly, then for an unbounded set of X,

$$G(X) > X^{1+\delta(X)}$$

with $\delta(X) \to 0$ very slowly.

2 - Proof of Theorem 1.1

Let u be as in the theorem, and let

$$y = \frac{x-b}{q}. \tag{6}$$

Too show that $J(u) \geq y$, it suffices to find residue classes $a_p \mod p$, one for each prime $p \leq u$, which together cover $[0, y]$. For each prime $p \leq u/2$ with $p \nmid q$, define a_p by

$$qa_p + b \equiv 0 \pmod{p}.$$

Recall that $(b, q) = 1$. In this way, if $0 \leq n \leq y$ and $n \not\equiv a_p \pmod{p}$ for all such p, then $m = qn + b$ has no prime factor $\leq u/2$. Also, $x = qy + b < (u/2)^2$ by hypothesis, and thus m is prime. Let

$$\mathcal{N} = \{0 \leq n \leq y : n \not\equiv a_p \pmod{p}, \forall p \leq u/2 \text{ with } p \nmid q\}.$$

It follows from the hypothesis of the theorem that

$$|\mathcal{N}| \leq \pi(qy + b; q, b) = \pi(x; q, b) \leq \frac{\delta x}{\phi(q)}.$$

Next, we choose residue classes a_p for primes $p|q$ with $p \leq u/2$ using a greedy algorithm, successively selecting for each p a residue class $a_p \mod p$ which covers at least a proportion $1/p$ of the elements remaining uncovered. As $u > 2\sqrt{x} > 2\sqrt{q}$, there is at most one prime $p|q$ satisfying $p > u/2$. Letting $\mathcal{N}'$ denote the set of $n \in [0, y]$ not covered by $\{a_p \mod p : p \leq u/2\}$, we have

$$|\mathcal{N}'| \leq |\mathcal{N}| \prod_{p|q, p \leq u/2} \left(1 - \frac{1}{p}\right) \leq 2|\mathcal{N}| \frac{\phi(q)}{q} \leq \frac{2\delta x}{q}.$$

By hypothesis,

$$|\mathcal{N}'| \leq \frac{u}{5 \log u},$$

which, by the prime number theorem, is less than the number of primes in $(u/2, u]$ for u large enough (as $u > \sqrt{x}$, this happens if x is large enough). Thus, we may associate each number $n \in \mathcal{N}'$ with a distinct prime in $p_n \in (u/2, u)$. Choosing $a_{p_n} \equiv n \pmod{p_n}$ for each $n \in \mathcal{N}'$ then ensures that $\{a_p \mod p : p \leq u\}$ covers all of $[0, y]$, as desired. □

Remark. We have made no use in the proof of estimates for numbers lacking large prime factors, a common feature in unconditional lower bounds on $G(x)$. There does not seem to be any advantage to this in our argument.

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