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Integer polynomials with small integrals

Abstract. The smart method of Gelfond–Shnirelman–Nair allows to obtain in elementary way a lower bound for the prime counting functions $\pi(x)$ and $\psi(x)$, in terms of the integral of suitable integer polynomials. A survey on the knowledge about the method together with a new approach and some new results are presented.

Keywords. Integer polynomials, Chebyshev problem, Prime counting function.

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1 - introduction

Let $\pi(x)$ be the number of primes not exceeding x. The Prime Number Theorem (PNT), independently proved in 1896 by Hadamard and de la Vallée-Poussin, states that

$$\pi(N) \sim \frac{N}{\log N} \qquad \text{as } N \to +\infty.$$

In 1851, Chebyshev [8] made the first step towards the PNT by proving that, given $\varepsilon > 0$,

$$(c_1 - \varepsilon)\,\frac{N}{\log N} \le \pi(N) \le (c_2 + \varepsilon)\,\frac{N}{\log N}$$

where $c_1 = \log(2^{1/2}3^{1/3}5^{1/5}/30^{1/30})$, $c_2 = 6c_1/5$ and N is sufficiently large. This result was proved using an elementary approach, i.e. without the use of complex

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analysis and the Riemann zeta function. A survey of elementary methods in the study of the distribution of prime numbers may be found in Diamond [**9**].

In 1936 Gelfond and Shnirelman, see Gelfond's editorial remarks in the 1944 edition of Chebyshev's Collected Works [8, p. 287-288], proposed a new elementary and clever method for deriving a lower bound for the prime counting functions $\pi(x)$ and $\psi(x)$. In 1982 the Gelfond-Shnirelman method was rediscovered and developed by Nair, see [**12**] and [**13**]. The method of Gelfond–Shnirelman–Nair runs as follows.

Let d_N denote the least common multiple of the integers $1, 2, \ldots, N$. By using the fundamental theorem of arithmetic it is immediate to see that

$$d_N = \prod_{p \le N} p^{[\log N/\log p]}, \tag{1}$$

where p belongs to the set of prime numbers and $[\log N/\log p]$ denotes the integer part of the real number $\log N/\log p$. Then we can write

$$d_N = \exp\left(\log\left(\prod_{p \le N} p^{[\log N/\log p]}\right)\right) = \exp\left(\sum_{p \le N} [\log N/\log p] \log p\right) = \exp(\psi(N)), \tag{2}$$

where $\psi(x)$ is the summary function of the von Mangoldt Λ-function.

An elementary and smart way to proceed is to consider a polynomial with integer coefficients

$$P(x) = \sum_{n=0}^{N-1} a_n x^n$$

and let

$$I(P) = \int_0^1 P(x)\,\mathrm{d}x = \sum_{n=0}^{N-1} \frac{a_n}{n+1}.$$

Since $I(P)$ is a rational number whose denominator divides d_N, we have that $I(P)d_N$ is an integer and hence, if $I(P) \neq 0$, we get

$$d_N |I(P)| \ge 1$$

and then from (1) we obtain

$$\psi(N) \ge \log \frac{1}{|I(P)|}. \tag{3}$$

Moreover from (1) we have

$$d_N = \prod_{p \le N} p^{[\log N/\log p]} \le \prod_{p \le N} p^{\log N/\log p}$$

which gives

$$\log d_N \le \log\left(\prod_{p\le N} p^{\log N/\log p}\right) = \sum_{p\le N} \log\left(p^{\log N/\log p}\right) = \pi(N)\log N$$

and hence

$$\pi(N) \ge \frac{\log d_N}{\log N} \ge \frac{\log \dfrac{1}{|I(P)|}}{\log N}. \tag{4}$$

From the above we can obtain a lower bound for the prime counting functions $\pi(N)$ and $\psi(N)$ from an upper bound for $|I(P)|$. The easiest way to carry on is to bound the absolute value of the integral $I(P)$

$$|I(P)| = \left|\int_0^1 P(x)\,\mathrm{d}x\right| \le \int_0^1 |P(x)|\,\mathrm{d}x \tag{5}$$

and

$$\int_0^1 |P(x)|\,\mathrm{d}x \le \max_{0\le x\le 1} |P(x)| = \|P\|_{[0,1]}, \tag{6}$$

obtaining

$$\pi(N) \ge \frac{\log\left(1/\|P\|_{[0,1]}\right)}{\log N}$$

and

$$\psi(N) \ge \log \frac{1}{\|P\|_{[0,1]}}.$$

If we could find a sequence of integer polynomials p_n, of degree n, with sufficiently small supremum norms such that

$$\lim_{n\to+\infty} \log\left(\|p_n\|_{[0,1]}^{-1/n}\right) = \lim_{n\to+\infty} -\frac{1}{n}\log\|p_n\|_{[0,1]} = 1,$$

we would obtain the best possible lower bound consistent with the PNT. This motivates the study of the quantities C_N such that

$$C_N = -\frac{1}{N-1}\log\left(\min_{\substack{P(x)\in\mathbb{Z}[x]\\ \deg(P)<N,\|P\|_{[0,1]}>0}} \|P\|_{[0,1]}\right), \tag{7}$$

the so-called integer Chebyshev problem.

2 - The classical integer Chebyshev problem

Much is known about polynomials for which the above minimum is achieved and about the quantities C_N. For small values of N it is easy to obtain the following table.

N	C_N	Extremal $p(x)$	$\|p\|_{[0,1]}$
3	0.6931	$x(1-x)$	$1/4 = 0.25$
4	0.7803	$x(1-x)(2x-1)$	$1/6\sqrt{3} = 0.0962$
5	0.6931	$x^2(1-x)^2, x(1-x)(2x-1)^2,$ $x(1-x)(5x^2-5x+1)$	$1/16 = 0.0625$
6	0.8047	$x^2(1-x)^2(2x-1)$	$1/25\sqrt{5} = 0.0178$
7	0.7803	$x^2(1-x)^2(2x-1)^2$	$1/108 = 0.0092$
8	0.7991	$x^3(1-x)^3(2x-1)$	0.0037
9	0.8010	$x^3(1-x)^3(2x-1)^2$	0.0016
10	0.8316	$x^3(1-x)^3(2x-1)(5x^2-5x+1)$	0.0005

It is quite evident a regularity in the form of the polynomials of the table, but unfortunately it is not entirely clear how to construct extremal polynomials for larger values of N. However, along the lines of [**11**], we can prove the following partial results.

Theorem 2.1. *Let $N \geq 3$ and let $p(x)$ an extremal polynomial for* (7). *Then* $x(1-x)|p(x)$.

Theorem 2.2. *Let $N \geq 6$ and let $p(x)$ an extremal polynomial for* (7). *Then* $x^2(1-x)^2|p(x)$.

Theorem 2.3. *Let $N \geq 4$, $N \neq 5$, and let $p(x)$ an extremal polynomial for* (7). *Then* $(2x-1)|p(x)$.

If we are satisfied with results for N tending to infinity, we can claim the following result of E. Aparicio [**2**].

Theorem 2.4. *Let N be sufficiently large and let $p(x)$ be an extremal polynomial for* (7). *Then*

$$(x(1-x))^{[(N-1)\lambda_1]}(2x-1)^{[(N-1)\lambda_2]}(5x^2-5x+1)^{[(N-1)\lambda_3]}|p(x),$$

with $0.1456 < \lambda_1 < 0.1495$, $0.0166 < \lambda_2 < 0.0187$ *and* $0.0037 < \lambda_3 < 0.0053$.

The best know result of this type is due to P. B. Borwein and T. Erdélyi [**7**] and is the following.

Theorem 2.5. *Let k be a positive integer,*

$$\begin{aligned}
&p_0(x) = x,\\
&p_1(x) = 1-x,\\
&p_2(x) = 2x-1,\\
&p_3(x) = 5x^2-5x+1,\\
&p_4(x) = 13x^3-19x^2+8x-1,\\
&p_5(x) = 13x^3-20x^2+9x-1,\\
&p_6(x) = 29x^4-58x^3+40x^2-11x+1,\\
&p_7(x) = 31x^4-61x^3+41x^2-11x+1,\\
&p_8(x) = 31x^4-63x^3+44x^2-12x+1,\\
&p_9(x) = 941x^8-3764x^7+6349x^6-5873x^5+3243x^4-1089x^3+216x^2-23x+1
\end{aligned}$$

and

$$P_{210}(x) = p_0^{67}\, p_1^{24}\, p_2^{9}\, p_3\, p_4\, p_5\, p_6^{3}\, p_7\, p_8\, p_9.$$

Then $(P_{210}(x))^k$ divides every extremal polynomial $p(x)$ for (7)*, provided N is sufficiently large.*

It was proved by Shnirelman, see [**11**], that the sequence C_N converges to a limit C. Borwein and Erdélyi [**7**] showed that $C \in (0.85866, 0.86577)$ and the lower bound was improved by Flammang [**10**] to 0.85912. The best known result to date, due to Pritsker [**14**], is that $C \in (0.85991, 0.86441)$. See also [**1**], [**5**], [**6**] and [**16**].

Therefore, following this lines, we can get a lower bound in the form

$$\pi(N) \geq C\ \frac{N}{\log N},$$

only for a constant C less than 0.87, which is quite far from what is expected by the PNT.

3 - A new approach to the method of Gelfond–Shnirelman

In order to avoid the trouble above we can deal with the problem in a different way. From the definition of $I(P)$ we have that

$$|I(P)| = \left|\int_0^1 P(x)\,\mathrm{d}x\right| = \left|\sum_{n=0}^{N-1} \frac{a_n}{n+1}\right| = \frac{1}{d_N}\left|\sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n\right|.$$

Since $d_N/(n+1)$ and a_n are integers for every $n = 0, 1, \ldots, N-1$, we have that

$$\left|\sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n\right|$$

is an integer and then the small positive value of $|I(P)|$ is $1/d_N$ and it is reached if

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n = \pm 1.$$

Without loss of generality we can deal with the linear diophantine equation

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} x_n = 1,$$

with integer coefficients $d_N/(n+1)$. Since the integer coefficients d_N, $d_N/2$, …, d_N/N are relatively prime, we have that for all N there exists a polynomial of degree $< N$ such that $I(P) = 1/d_N$. This leads to define the following sets of polynomials.

Definition 3.1. Let $N \geq 2$. We define

$$Z_N = \{P(x) \in \mathbb{Z}[x],\ \deg(P) < N\},$$

$$R_N = \{P(x) \in \mathbb{Z}[x],\ \deg(P) < N, I(P) = 0\}$$

and

$$S_N = \{P(x) \in \mathbb{Z}[x],\ \deg(P) < N, I(P) = 1/d_N\},$$

where d_N denotes the least common multiple of the integers $1, 2, \ldots, N$.

It is simple to verify that, for every N, Z_N is a free $\mathbb{Z}$-module and R_N is a submodule of Z_N and then it is also free. S_N is the affine space of the integer polynomials with positive and minimal integral on $[0, 1]$.

4 - Properties of the sets R_N

We start giving a theorem about the structure of the modules R_N.

Theorem 4.1. *A basis B_N of the module R_N can be constructed by adding to a basis B_{N-1} of the module R_{N-1} a suitable polynomial $q(x) \in R_N$. More precisely*

1. *if N is a prime: $q(x) = 1 - Nx^{N-1}$;*
2. *if N is a power of a prime: $q(x) = x^{n-1} - px^{N-1}$, where $N = p^k$, $n = p^{k-1}$ and $k \geq 2$;*
3. *otherwise: $q(x) = a_1 x^{n_1 - 1} + a_2 x^{n_2 - 1} - x^{N-1}$, where p_1 and p_2 are distinct primes dividing N, a_1 and a_2 are such that $a_1 p_1 + a_2 p_2 = 1$, $n_1 = N/p_1$ and $n_2 = N/p_2$.*

Proof. Let N be prime and $p(x) \in R_N$. Then we can write

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{N-1} x^{N-1}$$

with

$$a_0 d_N + a_1 \frac{d_N}{2} + a_2 \frac{d_N}{3} + \ldots + a_{N-2} \frac{d_N}{N-1} + a_{N-1} \frac{d_N}{N} = 0.$$

Since N is a prime number we have that $d_N = N d_{N-1}$ and then

$$N | d_N, N | \frac{d_N}{2}, N | \frac{d_N}{3}, \ldots N | \frac{d_N}{N-1}$$

and N does not divide d_N/N. From this it follows that a_{N-1}/N is an integer.

Now we define

$$r(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{N-2} x^{N-2} + \frac{a_{N-1}}{N},$$

which implies

$$p(x) = r(x) + a_{N-1} x^{N-1} - \frac{a_{N-1}}{N} = r(x) - \frac{a_{N-1}}{N} \left(1 - N x^{N-1}\right).$$

Then the first assertion of the theorem is proved, since $r(x) \in R_{N-1}$.

To prove the second assertion we let $N = p^k$. In this case $d_N = p\, d_{N-1}$ and more precisely

$$d_N = \prod_{q \leq N} q^{[\log q / \log N]} = p^k \prod_{q \leq N, q \neq p} q^{[\log q / \log N]} = N m,$$

where $(m,p)=1$ and q runs over primes. From this it follows that

$$p|d_N, p|\frac{d_N}{2}, p|\frac{d_N}{3}, \ldots p|\frac{d_N}{N-1}$$

and p does not divide d_N/N, hence a_{N-1}/p is an integer.

Now we define $n=p^{k-1}$ and

$$r(x)=a_0+a_1x+a_2x^2+\ldots+a_{N-2}x^{N-2}+\frac{a_{N-1}}{p}x^{n-1},$$

which implies

$$p(x)=r(x)+a_{N-1}x^{N-1}-\frac{a_{N-1}}{p}x^{n-1}=r(x)-\frac{a_{N-1}}{p}\big(x^{n-1}-px^{N-1}\big).$$

Then also the second assertion of the theorem is proved, since $r(x)\in R_{N-1}$.

To prove the last assertion we observe that if N is neither prime nor a power of a prime then there exist two primes $p_1\neq p_2$ both dividing N. Let a_1 and a_2 integers such that $a_1p_1+a_2p_2=1$, we define $n_1=N/p_1$, $n_2=N/p_2$ and

$$r(x)=a_0+a_1x+a_2x^2+\ldots+a_{N-2}x^{N-2}+a_1a_{N-1}x^{n_1-1}+a_2a_{N-1}x^{n_2-1}.$$

We conclude that

$$p(x)=r(x)-a_{N-1}\big(a_1x^{n_1-1}+a_2x^{n_2-1}-x^{N-1}\big).$$

and then the proof of the theorem is complete, since $r(x)\in R_{N-1}$. □

Using Theorem 4.1 we can fully describe the sets R_N. By the definition we have

$$\begin{aligned}R_2&=\{p(x)\in\mathbb{Z}[x], p(x)=a_0+a_1x, 2a_0+a_1=0\}\\&=\{p(x)\in\mathbb{Z}[x], p(x)=a_0(1-2x), a_0\in\mathbb{Z}\}.\end{aligned}$$

Then a basis B_2 of the set R_2 is

$$B_2=\{1-2x\}.$$

Using several times Theorem 4.1 we can get a basis B_N of the set R_N for many values of N:

$$\begin{aligned}
B_3&=\{1-2x,1-3x^2\},\\
B_4&=\{1-2x,1-3x^2,-x+2x^3\},\\
B_5&=\{1-2x,1-3x^2,-x+2x^3,1-5x^4\},\\
B_6&=\{1-2x,1-3x^2,-x+2x^3,1-5x^4,x(1-x-x^4)\},\\
B_7&=\{1-2x,1-3x^2,-x+2x^3,1-5x^4,x(1-x-x^4),1-7x^6\},\\
B_8&=\{1-2x,1-3x^2,-x+2x^3,1-5x^4,x(1-x-x^4),1-7x^6,x^3-2x^7\}\ldots
\end{aligned}$$

5 - Properties of the sets S_N

It is much more complicated to describe the sets S_N. Since S_N are affine spaces, we can write

$$S_N = \{\overline{p}(x) + r(x) : r(x) \in R_N\},$$

where $\overline{p}(x)$ is a fixed polynomial of S_N. For small values of N it is simple to find such a suitable polynomial:

$$\begin{array}{ll} N = 3 & \overline{p}(x) = x(1-x) \\ N = 4 & \overline{p}(x) = x^2(1-x) \\ N = 5 & \overline{p}(x) = x^2(1-x)(1-2x) \\ N = 6 & \overline{p}(x) = x^3(1-x)^2 \\ N = 7 & \overline{p}(x) = x^2(1-x)(x-2x^3) \\ N = 8 & \overline{p}(x) = x^3(1-x)^2(1-3x^2) \end{array}$$

Unfortunately it is very difficult to find out such a polynomial for a generic value of N. However we may provide some theorems about their roots and their factorization.

Theorem 5.1. *Let $N \geq 2$. Then there exists an integer polynomial $P(x) \in S_N$ with $x = 0$ (or $x = 1$) as a root of degree $\sim N/2$.*

Theorem 5.2. *Let N be an even natural number. Then there exists an integer polynomial $P(x) \in S_N$ with $N - 1$ roots on $(0, 1)$ and $N - 1$ changes of sign.*

On the other hand we can prove that in the set S_N there are also integer polynomials with at most one root and one change of sign.

Theorem 5.3. *Let $N \geq 2$. Then there exists an integer polynomial $P(x) \in S_N$ with at most one root on $(0, 1)$ and at most one change of sign on $(0, 1)$.*

For the proof of the above theorems see [3]. At last we can investigate the factorization of polynomials in S_N, similarly as shown in Section 2 for the classical integer Chebyshev problem.

Theorem 5.4. *For every $N \geq 3$ there exists a polynomial $p(x) \in S_N$ such that $p(0) = p(1) = 0$, namely $p(x) = x(1 - x)q(x)$ with $q(x) \in \mathbb{Z}[x]$.*

Proof. We can directly prove that the theorem is true for $3 \leq N \leq 7$. Then we let $N \geq 8$ and $p(x) \in S_N$, that is

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{N-1} x^{N-1}$$

and

$$a_0 d_N + a_1 \frac{d_N}{2} + a_2 \frac{d_N}{3} + a_3 \frac{d_N}{4} + \ldots + a_{N-2} \frac{d_N}{N-1} + a_{N-1} \frac{d_N}{N} = 1. \tag{8}$$

The Diophantine equation

$$a_3 \frac{d_N}{4} + a_4 \frac{d_N}{5} + \ldots + a_{N-2} \frac{d_N}{N-1} + a_{N-1} \frac{d_N}{N} = 1$$

has an integer solution $(a_3, a_4, \ldots, a_{N-1})$, since for $N \geq 8$ we have

$$\left(\frac{d_N}{4}, \frac{d_N}{5}, \ldots, \frac{d_N}{N-1}, \frac{d_N}{N} \right) = 1.$$

Setting $a_0 = 0$, $a_1 = 2(a_3 + a_4 + \ldots + a_{N-1})$ and $a_2 = -3(a_3 + a_4 + \ldots + a_{N-1})$ we have that $(a_0, a_1, a_2, \ldots, a_{N-1})$ is a solution of (8) and verifies $p(0) = a_0 = 0$ and

$$p(1) = a_0 + a_1 + a_2 + \ldots + a_{N-1} = 0,$$

which concludes the proof of the theorem. □

At the cost of some complications we can prove a similar result also including the factor $(2x - 1)$.

Theorem 5.5. *Let $N \geq 4$.*

1. *If N is not a power of 2, then there exists a polynomial $p(x) \in S_N$ such that $p(0) = p(1) = p(1/2) = 0$, namely such that $p(x) = x(1 - x)(2x - 1)q(x)$ with $q(x) \in \mathbb{Z}[x]$;*
2. *If N is a power of 2, then there does not exist a polynomial $p(x) \in S_N$ such that $(2x - 1) | p(x)$.*

Proof. We can directly prove that the theorem is true for $4 \leq N \leq 24$. Let $N \geq 25$ and $p(x) = (2x - 1)(b_0 + b_1 x + b_2 x^2 + \ldots + a_{N-2} x^{N-2})$. The condition

$$\int_0^1 p(x) \, \mathrm{d}x = \frac{1}{d_N}$$

is equivalent to

$$\sum_{k=1}^{N-2} \frac{d_N\, k}{(k+1)(k+2)} b_k = 1. \tag{9}$$

If N is a power of 2, then all the coefficients

$$\frac{d_N\, k}{(k+1)(k+2)}$$

are even and thus the equation (9) has no solutions, therefore there does not exist a polynomial $p(x) \in S_N$ such that $(2x-1)|p(x)$.

If N is not a power of 2, then we are able to prove that the coefficients

$$\frac{d_N\, k}{(k+1)(k+2)}$$

are relatively prime. In order to prove the coprimality, we suppose on the contrary that there exists a prime p dividing

$$\frac{d_N\, k}{(k+1)(k+2)}$$

for every $k = 1, 2, \dots, N-2$. Let $H = p^j$, with $j = \max\{i : p^i \le N\}$ and observe that p does not divide d_N/H. Then at least one of the two coefficients

$$\frac{d_N\,(H-1)}{H(H+1)} \quad \text{and} \quad \frac{d_N\,(H-2)}{(H-1)H}$$

is not divisible by p, a contradiction. By the coprimality of the coefficients of the Diophantine equation (9) it follows that there exists $p(x) \in S_N$ such that $(2x-1)|p(x)$.

To have also the factors x and $(1-x)$ it is sufficient to note that the integer H defined above is greater than 7, since $N \ge 25$, and then there exists a solution $(b_4, b_5, \dots, b_{N-2})$ of the Diophantine equation

$$\sum_{k=4}^{N-2} \frac{d_N\, k}{(k+1)(k+2)}\, b_k = 1. \tag{10}$$

We conclude the proof as above by setting $b_0 = b_1 = 0$, $b_2 = 9(b_4 + b_5 + \ldots + b_{N-2})$ and $b_3 = -10(b_4 + b_5 + \ldots + b_{N-2})$. □

Following the same lines we can prove also this last theorem.

Theorem 5.6. *Let $N \geq 4$ and let $0 < m < n$ be natural numbers such that $(n, m) = 1$.*

1. *If N is not a power of a prime, then there exists $p(x) \in S_N$ such that $(nx - m) | p(x)$;*
2. *If N is a power of a prime p, then there exists $p(x) \in S_N$ such that $(nx - m) | p(x)$ if and only if $(p, n) = 1$.*

6 - Open problem

In the standard method of Gelfond–Shnirelman–Nair we bound the absolute value of the integral

$$|I(P)| = \left| \int_0^1 P(x)\,\mathrm{d}x \right| \leq \int_0^1 |P(x)|\,\mathrm{d}x \tag{11}$$

and then

$$\int_0^1 |P(x)|\,\mathrm{d}x \leq \max_{0 \leq x \leq 1} |P(x)| = \|P\|_{[0,1]}, \tag{12}$$

to obtain

$$\pi(N) \geq \frac{\log\left(1/\|P\|_{[0,1]}\right)}{\log N}.$$

As observed in the introduction, following this line we can get a lower bound in the form

$$\pi(N) \geq C\ \frac{N}{\log N},$$

only for constant C much smaller than 1. It is not clear if this is only a consequence of the bound by the supremum norm on the interval $[0, 1]$ in (12) or if the inequality (11) is also involved.

If the set S_N contains polynomials of constant sign in $(0, 1)$ for all N, or at least for infinitely many values of N, the limit of the method would be only due to the inequality (12).

It is simple to verify that for very small values of N these positive polynomials exist. For S_3, $\deg(P) = 2$ and $d_3 = 6$, we have the positive polynomial $P(x) = x(1 - x)$ and for S_4, $\deg(P) = 3$ and $d_3 = 12$, we have the positive polynomial $P(x) = x^2(1-x)$. For S_N with larger values of N is not simple to determine what happens, and this leads to the following question.

Conjecture. *For every $N \geq 3$, or at least for infinitely many values of N, there exists an integer polynomial $p(x) \in S_N$ such that $p(x) \geq 0$ in the interval $[0,1]$.*

An answer to the posed question in the maximum generality does not appear simple but it is possible to obtain some results about small values of N. A straightforward way to obtain a negative conclusion about the existence of integer polynomials of S_N non-negative in $[0,1]$ is to consider $0 \leq x_1 < x_2 < x_3 \ldots < x_n \leq 1$ and a generic polynomial $p(x) \in S_N$ in the form

$$p(x) = \sum_{k=0}^{N-1} a_k x^k.$$

Since $p(x) \in S_N$, we have

$$\int_0^1 p(x)\,\mathrm{d}x = \frac{1}{d_N},$$

that is

$$\sum_{k=0}^{N-1} \frac{d_N}{k+1}\, a_k = 1$$

and consider the following linear Diophantine system composed of an equality and n inequalities

$$\begin{cases} \displaystyle\sum_{k=0}^{N-1} \frac{d_N}{k+1}\, a_k = 1 \\ p(x_1) \geq 0 \\ p(x_2) \geq 0 \\ \ldots \\ p(x_n) \geq 0. \end{cases} \tag{13}$$

If we are able to prove that, for a fixed value of N, the above linear system has no integer solutions $a_1, a_2 \ldots a_{N-1}$, we obtain that there does not exist an integer polynomial $p(x) \in S_N$ such that $p(x) \geq 0$ in the interval $[0,1]$.

By the branch and cut algorithm, used in many mathematical software systems, we can verify in deterministic way that for $N = 5$ and $x_k = k/4$, with $k = 0, 1, \ldots, 4$, the system (13) has no integer solutions, although it has infinitely many real solutions, which implies that there are no integer polynomials $p(x) \in S_5$ such that $p(x) \geq 0$ in the interval $[0, 1]$. Hence we disproved the strong form of the conjecture.

For $N = 6$ there exists the polynomial $p(x) = x^3(1 - x)^2 \in S_6$, non-negative for all values of $x \in [0, 1]$. Then the case $N = 5$ might appear as an exceptional case. Instead we can verify that for many values of N there does not exist a polynomial in S_N non-negative in $[0, 1]$. More precisely we can verify that there does not exist an integer polynomial $p(x) \in S_N$ such that $p(x) \geq 0$ in the interval $[0, 1]$ for all $7 \leq N \leq 20$, with the only exclusion of the case $N = 10$, for which we have the polynomial $p(x) = x^3(1 - x)^4(2x - 1)^2$.

Thus not only the conjecture about the positivity of the polynomials is not true for all values of N but, with the exclusion of small N, seems to be satisfied only for a minority of the values of N. It still remains open the question whether the conjecture is true at least for infinitely many values of N.

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