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The Selberg integral and a new pair-correlation function for the zeros of the Riemann zeta-function

Abstract. The present paper is a report on joint work with Alessandro Languasco and Alberto Perelli, collected in [10], [11] and [12], on our recent investigations on the Selberg integral and its connections to Montgomery’s pair-correlation function. We introduce a more general form of the Selberg integral and connect it to a new pair-correlation function, emphasising its relations to the distribution of prime numbers in short intervals.

Keywords. Riemann zeta-function, Selberg integral, Montgomery’s pair-correlation function.

Mathematics Subject Classification (2010): 11M26, 11N05.

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Received: December 31, 2015; accepted in revised form: March 1, 2016.

This research was partially supported by the grant PRIN2010-11 *Arithmetic Algebraic Geometry and Number Theory*.

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1 - Introduction

The central problem of Analytic Number Theory is the distribution of prime numbers. How many prime numbers are there, and are they “randomly” or “evenly” distributed? The answers to these questions are known only partially, even if one assumes powerful and, as yet, unproved hypotheses like Riemann’s. Here we are interested in the distribution of prime numbers in “short intervals,” from various points of view. We give a strong quantitative version of previous results on the connection between the Selberg integral J defined in (5) and Montgomery’s pair-correlation function F defined in (9). Then we introduce a new, generalised pair-correlation function and show that it essentially is in charge of the distribution of primes in short intervals, giving a unified view of many problems.

While it is always advisable to read the original papers (like Montgomery’s [15] and the other ones quoted at appropriate places), it is extremely instructive to start with Goldston’s excellent survey [4].

2 - Distribution of prime numbers: Large scale

2.1 - The Chebyshev prime-counting functions

Let

$$\pi(x) \stackrel{\text{def}}{=} |\{p \leq x: p \text{ is prime}\}| \quad \text{and} \quad \psi(x) \stackrel{\text{def}}{=} \sum_{n \leq x} \Lambda(n),$$

where $\Lambda(n) = 0$ unless $n = p^\alpha$ for some prime p and positive integer α , and $\Lambda(p^\alpha) = \log p$. We also let

$$\text{li}(x) \stackrel{\text{def}}{=} \int_2^x \frac{dt}{\log t}.$$

It has been conjectured by Gauss at the end of the XVIII century that $\pi(x) \sim \text{li}(x)$ as $x \rightarrow +\infty$. Stated in this way, Gauss's conjecture is equivalent to Legendre's simpler statement that $\pi(x) \sim x/\log x$, but Gauss's approximation is numerically far closer to the truth. The truth of the asymptotic statement above was proved independently by Hadamard and de la Vallée Poussin in 1896 and is called the Prime Number Theorem (PNT, for short). Let

$$\Delta_\pi(x) \stackrel{\text{def}}{=} \pi(x) - \text{li}(x) \quad \text{and} \quad \Delta_\psi(x) \stackrel{\text{def}}{=} \psi(x) - x.$$

According to the strongest form of the PNT known today, there is a constant $c > 0$ such that for $x \rightarrow +\infty$

$$(1) \quad \Delta_\pi(x), \Delta_\psi(x) \ll R(x) \stackrel{\text{def}}{=} x \exp\left\{-c(\log x)^{3/5}(\log \log x)^{-1/5}\right\}.$$

The PNT shows that to a first approximation $\pi(x)$ is very close to $\text{li}(x)$ and $\psi(x)$ is very close to x . Of course, it is interesting, and in some applications critical, to know the exact degree of these approximations. It has been known since Riemann's pioneering paper [21] that Δ_π and Δ_ψ both depend on the distribution of the zeros of the zeta-function. The connection has received a strong quantitative form by Pintz [20], building on earlier work started by Ingham. Here we are mainly concerned with the case of "optimal distribution," and we will not pursue this connection further.

2.2 - The Riemann Hypothesis

In his 1859 paper referred to above, Riemann proved that the zeta-function defined by the Dirichlet series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \text{for } \sigma = \Re(s) > 1,$$

has a meromorphic continuation to the complex plane, except for the point $s = 1$ where ζ has a simple pole with residue 1. The continuation has infinitely many real zeros (the so-called “trivial zeros”) at the points $-2, -4, -6, \dots$, and also infinitely many non-real zeros in the strip $\sigma \in [0, 1]$. It is customary to denote the generic non-trivial zero of the Riemann ζ -function by $\rho := \beta + i\gamma$. These zeros are placed symmetrically with respect to the real axis and the line $\sigma = \frac{1}{2}$. Riemann wrote that it looks “likely” that all of these zeros actually lie on the line $\sigma = \frac{1}{2}$, but that he was unable to prove it. This is the original statement of the Riemann Hypothesis (RH, for short) that is still unsettled, as yet.

The Riemann Hypothesis is equivalent to either of the two statements

$$(2) \quad \pi(x) = \int_2^x \frac{dt}{\log t} + \mathcal{O}\left(x^{1/2} \log x\right) \quad \text{or} \quad \psi(x) = x + \mathcal{O}\left(x^{1/2}(\log x)^2\right).$$

Of course, this is *much stronger* than (1). Notice that by Littlewood’s Theorem [13] of 1914, we have

$$(3) \quad \Delta_\psi(x) = \psi(x) - x = \Omega_\pm\left(x^{1/2} \log \log \log x\right).$$

Here, the notation means that $\Delta_\psi(x)$ is larger than a positive multiple of $x^{1/2} \log \log \log x$ for a suitable unbounded sequence of values of x , and smaller than a negative multiple of the same function on another such sequence. In other words, the RH is very nearly optimal. It seems plausible that the correct maximal order of magnitude for $\psi(x) - x$ be $x^{1/2}(\log \log \log x)^2$. See § 15.3 of Montgomery & Vaughan [18].

3 - Distribution of prime numbers: Small scale

3.1 - Primes in all “short” intervals

The “additive” form of the expected main term for both π and ψ readily suggests a conjecture on the number of prime numbers contained in the interval $(x, x + y]$. We say that such an interval is *short* if $y = o(x)$.

Question 1. For $y \leq x$, is it true that

$$(4) \quad \pi(x + y) - \pi(x) \sim \int_x^{x+y} \frac{dt}{\log t} \quad \text{or} \quad \psi(x + y) - \psi(x) \sim y \quad ?$$

The asymptotic relations in (4) are true if $y/R(x) \rightarrow +\infty$. This is essentially trivial, because we can use (1) twice and the difference between the leading terms is larger than the error terms.

We may rephrase the question as: Is (4) true also for smaller y ? For *very* small y (say, $y \asymp \log x$ or smaller) it is trivially false, because most intervals will not be long enough to contain any primes at all. On the other hand, if $y = 1$ and $x + 1$ is a prime number, then the interval $(x, x + 1]$ contains “too many” primes. We assume that $y \rightarrow +\infty$ with x in order to avoid such trivialities.

The asymptotic formulae (4) are false also for $y = (\log x)^\alpha$, for any *fixed* $\alpha > 1$: this has been proved by Maier in 1985 [14], and it came as a surprise since, as we know conditionally from Selberg’s work [23], they are *usually* true; see the discussion below. Unconditionally, the relations (4) are true for $y \geq x^{7/12-\varepsilon(x)}$, provided that $\varepsilon(x) \rightarrow 0+$ as $x \rightarrow +\infty$; see Heath-Brown [9]. Of course, on the RH they are true in a wider range: by (2), for $y/(x^{1/2}(\log x)^2) \rightarrow +\infty$, but we must recall that Cramér proved that the weaker condition $y/(x^{1/2}(\log x)) \rightarrow +\infty$ suffices. Actually, his techniques also prove that the RH implies the existence of at least $x^{1/2}$ prime numbers in the interval $[x, x + y]$ provided that $y \geq Cx^{1/2} \log x$, where C is a sufficiently large constant. See Theorem 13.3 in [18].

A long-standing open problem is to bridge the gap between Maier-type results and what is known under the RH. It is plausible that we can take $y = x^\varepsilon$ for any fixed $\varepsilon > 0$, or even slightly smaller. As we remarked above, by Littlewood’s Theorem (3) we know that the error term $\Delta_\psi(x) = \psi(x) - x$ is sometimes quite large, but it is conceivable that it varies very slowly and then one can compute accurately $\psi(x + y) - \psi(x)$ for comparatively small y , although indirectly.

These topics are discussed at great length and in full detail in Chapters 13 and 15 of Montgomery & Vaughan [18]. See, in particular, § 13.1.

3.2 - Primes in “almost all” short intervals: The Selberg integral

In some applications, it is not necessary to know that either of (4) holds for *every* y , but only that it holds for most values of y . Selberg introduced a very convenient way of measuring precisely what “usually” means. Let

$$(5) \quad J(x, \theta) \stackrel{\text{def}}{=} \int_x^{2x} |\psi(t + \theta t) - \psi(t) - \theta t|^2 dt$$

denote the “variance” of the primes in short intervals, where, in the notation of the previous sections, $\theta \in [0, 1]$ is essentially y/x .

Question 2. *In which range of values for θ is it true that*

$$(6) \quad J(x, \theta) = o(x^3 \theta^2) \quad ?$$

If θ is not too small the Brun-Titchmarsh inequality in Theorem A.1 implies that $J(x, \theta) \ll x^3 \theta^2$. Furthermore, we know that (6) holds unconditionally in the range $x^{-5/6-\varepsilon(x)} \leq \theta \leq 1$, provided that $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$; see [24]. A lower bound for θ is needed, since for $\theta = 1/x$, say, $J(x, \theta)$ essentially reduces to

$$(7) \quad \sum_{n \in [x, 2x]} (\Lambda(n) - 1)^2 \sim x \log x,$$

by the Prime Number Theorem, and hence (6) is false in this range. Actually, the Brun-Titchmarsh inequality in Theorem A.1 implies, more generally, that $J(x, \theta) \ll x^3 \theta^2 (\log x)^2 (\log(2\theta x))^{-2}$ for $\theta \geq x^{-1}$, which is compatible with (7). Notice, however, that the mentioned inequality sometimes “loses” a log-factor.

If we assume the RH we have a much stronger result, which is due to Selberg in [23]:

$$(8) \quad J(x, \theta) \ll x^2 \theta (\log(2/\theta))^2 \quad \text{uniformly for } x^{-1} \leq \theta \leq 1.$$

In other words, in this range of values for θ ,

$$\psi(t + \theta t) - \psi(t) = \theta t + \mathcal{O}\left(r(x, \theta)\right) \quad \text{for “almost all” } t \in [x, 2x],$$

provided that $r(x, \theta)/((\theta x)^{1/2} \log x) \rightarrow +\infty$. More precisely, the cardinality of the set of integers $t \in [x, 2x]$ such that the asymptotic formula above fails to hold is $o(x)$ as $x \rightarrow +\infty$. When $\theta = x^{-1}(\log x)^z$ we still have that the expected asymptotic formula holds for most integers: Maier’s result implies that, in fact, there are infinitely many exceptions, although they are rather sparse.

Notice that for $\theta = 1/x$ this strong result is essentially empty! Notice also what happens for $\theta = 1$: the average bound is slightly stronger than the pointwise one given in (2).

The proof of Selberg’s results begins observing that, without any hypothesis, we have

$$J(x, \theta) \ll \sum_{\rho_1} \sum_{\rho_2} x^{1+2\beta_1} \frac{\min(\theta^2, \gamma_1^{-2})}{(1 + |\gamma_1 - \gamma_2|)^2}.$$

If the RH holds, then $\rho = \frac{1}{2} + i\gamma$ for all zeros of the Riemann zeta-function, and therefore

$$J(x, \theta) \ll x^2 \sum_{\gamma > 0} \min(\theta^2, \gamma^{-2}) \log(\gamma),$$

by the Riemann-von Mangoldt formula A.3. Considering separately the ranges $\gamma \in (0, \theta^{-1})$ and $\gamma > \theta^{-1}$, we see that the Riemann Hypothesis implies (8). Assuming a less restrictive hypothesis on the distribution of the zeros, one obtains the result described in the next section.

3.3 - Strong bounds for J and the quasi-RH

Recall that $\rho := \beta + i\gamma$ denotes the generic non-trivial zero of the Riemann ζ -function and let

$$\Theta \stackrel{\text{def}}{=} \sup \{ \beta \leq 1 : \zeta(\beta + i\gamma) = 0 \}.$$

Essentially, $\Theta < 1$ implies that $J(x, \theta) \ll x(\theta x)^{2\Theta + \varepsilon}$ for any $\varepsilon > 0$, uniformly for $x^{\varepsilon-1} \leq \theta \leq 1$. On the RH we have $\Theta = \frac{1}{2}$ and we have the stronger bound given in (8); see e. g., Saffari & Vaughan [22], Lemma 6. We remark that we do not even know whether $\Theta < 1$ or not, as yet.

It is interesting that it is also possible to invert the arrow, as in Pintz's results quoted above. We can deduce information concerning the horizontal distribution of the zeros of the Riemann zeta-function from hypothetical strong upper bounds for J .

Remark. Assume that $J(x, 1) \ll x^\delta$ for some $\delta \in [2, 3]$. Then $\Theta \leq \frac{1}{2}(\delta - 1)$.

Notice that *no* uniformity is needed. This is a result in the author's paper [25]. The proof is little more than Cauchy's inequality.

3.4 - Application to an approximation problem

The results described in the previous section are obviously interesting in themselves. Here we quickly quote an application to an approximation problem which is relevant for the circle method. We have two exponential sums, one containing explicitly the prime numbers, and we know that they are close to each other in the neighbourhood of 0. The Selberg integral gives, essentially, the L^2 -norm of the difference. Let

$$S(\alpha) \stackrel{\text{def}}{=} \sum_{n \leq x} \Lambda(n) e^{2\pi i n \alpha} \quad \text{and} \quad T(\alpha) \stackrel{\text{def}}{=} \sum_{n \leq x} e^{2\pi i n \alpha}.$$

It is well known that T gives a good approximation to S for α small since, in fact, $S(0) = \psi(x)$ and $T(0) = [x]$. In Diophantine problems like the one described by Brüdern, Cook & Perelli in [1], the circle is replaced by the whole real line which is dissected into three pieces: the main term arises as an integral over a neighbourhood

of 0 and the width of the neighbourhood is reflected in the quality of the final result. According to Lemma 1 in [1] we have

$$\int_{-1/(\theta x)}^{1/(\theta x)} |S(\alpha) - T(\alpha)|^2 d\alpha \ll \frac{1}{(\theta x)^2} (J(x, \theta) + x) + \theta x (\log x)^2.$$

Here we see the important role of uniformity in θ : the length of the integration interval depends on it. Ideally, we would like to have $\theta = 1/(2x)$ so that there is no “minor arc,” which, by definition, is the set of α 's where we have comparatively poor information on the exponential sum S .

4 - Montgomery's pair-correlation function

We now assume RH until the end. Following Montgomery, we let

$$(9) \quad F(x, T) \stackrel{\text{def}}{=} \sum_{\gamma_1, \gamma_2 \in [0, T]} \frac{4x^{i(\gamma_1 - \gamma_2)}}{4 + (\gamma_1 - \gamma_2)^2} = \sum_{\gamma_1, \gamma_2 \in [0, T]} x^{i(\gamma_1 - \gamma_2)} \omega(\gamma_1 - \gamma_2),$$

where $\omega(x) = 4/(4 + x^2)$. The weight ω arises naturally as the Fourier transform of $e^{-2|x|}$, apart from a normalisation factor. Montgomery [15] proved that

$$F(x, T) \sim \frac{T}{2\pi} \log x \quad \text{as } T \rightarrow +\infty$$

uniformly for $T^\epsilon \leq x \leq T$, and gave the following conjecture.

Conjecture 1 [Montgomery]. *For $T \rightarrow +\infty$ we have*

$$F(x, T) \sim \frac{T}{2\pi} \log T \quad \text{uniformly for } T \leq x \leq T^A.$$

In other words, only the “diagonal” terms where $\gamma_1 = \gamma_2$ give a contribution, in view of the Riemann-von Mangoldt formula in Theorem A.3 below, whereas the other terms, more or less, cancel out.

4.1 - The link between F and J

A number of authors studied the connection between hypothetical asymptotic formulae for J and F , with increasing precision. The following table lists the

major contributions.

- Goldston & Montgomery (1987) [6]
- Montgomery & Soundararajan (2002) [16]
- Chan (2003) [2]
- Languasco, Perelli & Z. (2012) [10].

Apart from a possible lack of precision, the argument leading to (8) sketched at the end of §3.2 shows that J depends crucially on the distribution of zeros of ζ . Injecting the information contained in Montgomery's Conjecture 1, one gets the one-term asymptotic formula

$$(10) \quad J(x, \theta) \sim \frac{3}{2} x^2 \theta \log(1/\theta),$$

and conversely, superseding (8). More precisely, if the asymptotic relation in Conjecture 1 holds uniformly for $T \in [x^{B_1} L^{-3}, x^{B_2} L^3]$ then (10) holds uniformly for $\theta \in [x^{-B_2}, x^{-B_1}]$, where $L = \log x$ and $0 < B_1 \leq B_2 \leq 1$ are fixed. A similar assumption on the range for (10) allows to recover Conjecture 1 in the corresponding range. This is Theorem 2 of Goldston & Montgomery [6]. From now on, we drop all reference to uniformity ranges, for simplicity. More generally, the authors mentioned above set out to pursue the following

Goal. Compare the size of the "error terms" R_J and R_F in expansions like

$$\begin{aligned} J(x, \theta) &= c_1 x^2 \theta \log(1/\theta) + c_2 x^2 \theta + R_J(x, \theta) \\ F(x, T) &= c_3 T \log T + c_4 T + R_F(x, T) \end{aligned}$$

in suitable ranges of uniformity.

In both cases the main parameter is x . The expected values of the constants c_j are known. We may therefore restate our goal in a more precise form. Let R_J and R_F be defined implicitly by

$$J(x, \theta) = \frac{3}{2} x^2 \theta (\log(1/\theta) + 1 - \gamma - \log(2\pi)) + R_J(x, \theta)$$

and

$$F(x, T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + R_F(x, T).$$

We expect that R_J is small if and only if R_F is, provided that the assumed bound holds in a wide enough range of values. In fact, assuming RH Montgomery & Soundararajan proved that, essentially, $R_J(x, \theta) = o(x^2 \theta)$ if and only if the imaginary parts of the

zeros of the Riemann zeta-function are “well distributed” in the sense that $R_F(x, T) = o(T)$. In other words, the diagonal contribution $\gamma_1 = \gamma_2$ dominates, as expected. Actually, it seems necessary to assume slightly more than just $R_F(x, T) = o(T)$ in order to get $R_J(x, \theta) = o(x^2\theta)$: see the detailed comment to equation (1.8) in [10]. We stress again the importance of the uniformity ranges in the hypothesis.

Following Chan’s results in [2], Languasco, Perelli and the author studied relations between hypothetical bounds of the type

$$R_J(x, \theta) = \mathcal{O}\left(x^2\theta^{1+\alpha}\right) \quad \text{and} \quad R_F(x, T) = \mathcal{O}(T^{1-\beta}).$$

We state a weakened and simplified form of our results: for full details see [10]. Essentially, for $\alpha, \beta > 0$, we have

$$(11) \quad R_J(x, \theta) \ll x^2\theta^{1+\alpha} \implies R_F(x, T) \ll T^{1-\alpha/(\alpha+3)}$$

$$(12) \quad R_F(x, T) \ll T^{1-\beta} \implies R_J(x, \theta) \ll x^2\theta^{1+\beta/2}.$$

Here we are totally neglecting log-powers and uniformity in the various parameters, for simplicity of statement, since our results are as general as they are cumbersome. According to the heuristics given by Montgomery & Soundararajan in [16], a plausible range for α and β is the interval $\left(0, \frac{1}{2}\right)$.

For the proof we need various forms of abelian-tauberian results, extending the technique introduced by Goldston & Montgomery in [6]. Essentially, we have a hypothetical average of a certain function, say J , and we need to transform it into a different average of the same function, before we can use standard Fourier-transform techniques. This leads to some loss of uniformity. This also shows why we use J instead of the perhaps more “natural”

$$\tilde{J}(x, y) \stackrel{\text{def}}{=} \int_x^{2x} |\psi(t+y) - \psi(t) - y|^2 dt,$$

because J is more directly connected to sums over zeros, via the explicit formula for ψ . In fact, using J we see that each summand in the finite sum over zeros factors as $t^\rho c(\rho, \theta)$ as in (13) below, and this leads to simpler handling.

However, there is a standard way to relate $J(x, \theta)$ and $\tilde{J}(x, \theta x)$ and the two asymptotic formulae differ only in the “secondary main term,” that is, in the value of the constant c_2 above. For full details, see Saffari & Vaughan [22].

We select a few, significant, steps of the proof. The first one is the approximate equality

$$F(x, T) = \frac{1}{\pi} \int_0^T \left| \sum_{\gamma} \frac{x^{i\gamma}}{1 + (t - \gamma)^2} \right|^2 dt + \mathcal{O}\left((\log T)^3\right).$$

Let $c(\rho, \theta) := ((1 + \theta)^\rho - 1)/\rho$. The first application of a tauberian result yields the asymptotic formula for

$$\int_{\mathbb{R}} \left| \sum_{|\gamma| \leq Z} c(\rho, \theta) \frac{x^{i\gamma}}{1 + (t - \gamma)^2} \right|^2 dt,$$

where the interval $[0, T]$ has been replaced by the whole real line, only the zeros with “small” imaginary parts are included and there is the coefficient of the “right” shape, namely c . Another application coupled with the Plancherel formula gives the asymptotic formula for

$$\int_x^{2x} \left| \sum_{|\gamma| \leq Z} c(\rho, \theta) t^\rho \right|^2 dt,$$

where Z is at our disposal. This is essentially J , since

$$(13) \quad \psi(t + \theta t) - \psi(t) - \theta t = - \sum_{|\gamma| \leq T} c(\rho, \theta) t^\rho + \mathcal{O}\left(\frac{t}{T} (\log tT)^2\right),$$

by the Explicit formula A.2, where T is a free parameter. Recalling that $\beta < 1$ for every zero of the Riemann ζ -function, we have

$$c(\rho, \theta) = \int_1^{1+\theta} t^{\rho-1} dt \ll \min\left(\theta, \frac{1}{|\gamma|}\right).$$

This means that we have to choose, essentially, $T = \theta^{-1}$ and a suitable value for Z . Among other things, here we improve a crucial technical lemma (Lemma 1 of [10]) that allows us to reach the square root in (12) instead of the fourth root as in Chan [2].

4.2 - A different version of Montgomery’s pair-correlation conjecture

A slightly weaker form of Montgomery’s Conjecture 1 describes the distribution of the “gaps” between zeros of the Riemann zeta-function. For the detailed relationship between the two Conjectures, see [4], Theorem 4.

Conjecture 2 [Montgomery]. *For fixed α and β with $0 < \alpha < \beta$ we have*

$$\sum_{\substack{\gamma_1, \gamma_2 \in [0, T] \\ ((\gamma_1 - \gamma_2) \log T) / 2\pi \in [\alpha, \beta]}} 1 \sim \frac{1}{2\pi} T \log T \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du,$$

as $T \rightarrow +\infty$.

The function inside the integral on the far right is known as the pair-correlation function for the zeros of the Riemann ζ -function. In view of the Riemann-von Mangoldt formula A.3, the average spacing between consecutive zeros of the Riemann zeta-function with imaginary parts in $[0, T]$ is $2\pi/\log T$.

5 - A general pair-correlation function

5.1 - A unifying approach

Let $\tau \in [0, 1]$ and define

$$F(x, T, \tau) \stackrel{\text{def}}{=} \sum_{\gamma_1, \gamma_2 \in [-T, T]} \frac{4x^{i(\gamma_1 - \gamma_2)}}{4 + \tau^2(\gamma_1 - \gamma_2)^2}.$$

It is clear that $F(x, T, 1)$ is (essentially) the same as $F(x, T)$. This function is already present, albeit in a slightly different guise, in Heath-Brown & Goldston [5]; they do not directly pursue its relations with primes in short intervals, but use it simply as a technical device.

Moreover, $F(x^{1/\tau}, T, \tau)$ is the pair-correlation function for $Z_\tau(s) = \zeta(s/\tau)$, where Z_τ is (almost) an element of the Selberg Class of degree and conductor

$$d = \frac{1}{\tau} \quad \text{and} \quad q = \left(\frac{1}{\tau}\right)^{1/\tau}$$

respectively. This means that we can use well-known heuristics for the Selberg class to make plausible guesses concerning $F(x, T, \tau)$. Of course, Z_τ is an element of the Selberg Class only for $\tau = 1$. See Murty & Perelli [19] for the discussion of the pair-correlation function in the Selberg Class.

5.2 - Properties of the general pair-correlation function

Notice that $F(x, T, 0) = |\Sigma(x, T)|^2$ where

$$(14) \quad \Sigma(x, T) \stackrel{\text{def}}{=} \sum_{|\gamma| \leq T} x^{i\gamma}$$

is the exponential sum that appears in Landau's explicit formula A.4. There is a strong conjecture of Gonek [7] concerning the behaviour of Σ : see Conjecture 3. We remark that $F(x, T, \tau)$ is difficult to estimate for τ very small ($\leq 1/T$); in fact the trivial bound

$$F(x, T, \tau) \ll \min(T, \tau^{-1})T \log^2 T$$

becomes very large. When $\tau = 1$, the trivial bound for $F(x, T)$ is only slightly larger than the expected truth, by just a log factor.

5.3 - *The generalised Selberg Integral*

From now on we assume that $\tau > 0$ in order to avoid trivial statements. Let

$$J(x, \tau, \theta) \stackrel{\text{def}}{=} \int_x^{x(1+\tau)} |\psi(t + \theta t) - \psi(t) - \theta t|^2 dt.$$

Here we are dealing with “short intervals” in two different ways. The obvious conjecture is $J(x, \tau, \theta) \ll x^{2+\varepsilon}\tau\theta$.

Assumption [Hypothesis $H(\eta)$]. *We assume that the RH holds and that*

$$F(x, T, \tau) \ll Tx^\varepsilon \quad \text{uniformly for} \quad \begin{cases} x^\eta \leq T \leq x \\ x^\eta/T \leq \tau \leq 1 \end{cases}$$

for some fixed $\eta > 0$ and every $\varepsilon > 0$.

This assumption is “justified” by Gonek’s Conjecture 3 for τ small, and by an obvious generalisation of Montgomery’s for τ large. In [11], Languasco, Perelli and the author give a variant of $H(\eta)$ valid for $\eta = 0$, which we omit for brevity. The following results are Theorem 1 and 2, respectively, of [11].

Theorem 5.1. *If assumption $H(\eta)$ holds for some $\eta \in (0, 1)$, then*

$$J(x, \tau, \theta) \ll x^{2+\varepsilon}\tau\theta$$

uniformly for $x^{-1} \leq \theta \leq x^{-\eta}$ and $\theta x^\eta \leq \tau \leq 1$.

As an immediate corollary, we have that

$$\psi(x + y) - \psi(x) = y + \mathcal{O}\left(y^{1/2}x^\varepsilon\right)$$

for “almost all” $x \in [x, x(1 + \tau)]$ and $y \in [1, x^{1-\eta}]$.

Theorem 5.2. *Let $\varepsilon > 0$ be small. If assumption $H(\eta)$ holds for some $\eta \in (0, 1/2 - 5\varepsilon)$, then*

$$\psi(x + y) - \psi(x) = y + \begin{cases} \mathcal{O}\left(y^{2/3}x^{\eta/3+\varepsilon}\right) & \text{for } x^{\eta+5\varepsilon} \leq y \leq x^{1/2}, \\ \mathcal{O}\left(y^{1/3}x^{1/6+\eta/3+\varepsilon}\right) & \text{for } x^{1/2} \leq y \leq x^{1-\eta}. \end{cases}$$

We quote (2.3) of [11] to show the explicit connection between the “short” Selberg integral and the generalised pair-correlation function that we use in the proof of Theorem 5.1. In fact we have

$$\int_x^{x+y} \left| \sum_{|\gamma| \leq U} \frac{(t+h)^\rho - t^\rho}{\rho} \right|^2 dt \ll \frac{h^2 y}{x} \max_{x \leq t \leq x+h} F\left(t, U, \frac{y}{t}\right).$$

The proof of Theorem 5.2 uses the “inertia” property of ψ : $\psi(x+y) - \psi(x)$ does not change by more than $\asymp \log x$ as x increases by 1. Hence, if

$$\psi(x+y) - \psi(x) - y$$

is abnormally large in absolute value for some $x = x_0$, it is also large for “many” values of x around x_0 , which is impossible by the previous Theorem 5.1. We also adapt an argument in Gonek [7]. In the proof of Theorem 5.1 we need suitable, uniform generalisations of the results in the original paper by Heath-Brown [8].

5.4 - More consequences

One of the more interesting corollaries of our work in [11] is the fact that we obtain explicit estimates for $\Delta_\psi(x) = \psi(x) - x$. More precisely, we show that there is a direct, quantitative connection between the size of Δ_ψ and

1. the hypothetical uniformity in τ ,
2. the hypothetical estimate for $F(x, T, \tau)$.

We pick an example: see Theorem 3 and Remark 3 in [11] for a complete discussion of this important topic. Assume the RH and that $F(x, T, \tau) \ll T \log T$ uniformly for $U \leq T \leq x^{1/2}$. Then, if the choice

$$\tau = \min\left(1, \frac{\log^4 U}{\log^3 x}\right),$$

is admissible, it yields

$$\Delta_\psi(x) \ll x^{1/2} \log^2 U.$$

6 - Asymptotic formulae

6.1 - The asymptotic formula for $F(x, T, \tau)$

Let

$$S(x, \tau) \stackrel{\text{def}}{=} \sum_{n \geq 1} \frac{A^2(n)}{n} a^2(n, x, \tau),$$

where

$$a(n, x, \tau) \stackrel{\text{def}}{=} \begin{cases} (n/x)^{1/\tau} & \text{if } n \leq x \\ (x/n)^{1/\tau} & \text{if } n > x. \end{cases}$$

Notice that $S(x, \tau) \ll \tau \log x$ by the Brun-Titchmarsh inequality, if τ is not too small. In [12], Languasco, Perelli and the author gave the following asymptotic result.

Theorem 6.1. *As $x \rightarrow +\infty$ we have*

$$F(x, T, \tau) \sim \frac{T}{\pi} \frac{S(x, \tau)}{\tau} + \frac{T \log^2 T}{\pi \tau x^{2/\tau}} + \text{smaller order terms}$$

uniformly for $\tau \geq 1/T$, provided that $TS(x, \tau) = \infty$ ($\max(x, (\log T)^3/\tau$).

Of course, this reduces to Montgomery’s result for $\tau = 1$. We remark that Theorem 6.1 shows the same phenomenon of yielding an asymptotic formula only at “extreme ranges.”

We use a σ -uniform version of the main Lemma in Montgomery’s original paper [15]. He used his crucial representation for the function

$$(15) \quad (2\sigma - 1) \sum_{\gamma} \frac{x^{i\gamma}}{(\sigma - 1/2)^2 + (t - \gamma)^2}$$

for $\sigma = \frac{3}{2}$, while we have to take $\sigma = \frac{1}{2} + \tau^{-1}$. Since we are interested in very *small* values of τ , our estimates have to be uniform with respect to σ , and this leads to a fair amount of complication in detail. We also have to take care of other uniformity aspects.

The first “main term” for $F(x, T, \tau)$ arises as follows: our version of Montgomery’s representation for (15) is the sum of several terms, the most important being

$$R_1(x, t, \tau) \stackrel{\text{def}}{=} -x^{-1/2} \left(\sum_{n \leq x} A(n) \left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n > x} A(n) \left(\frac{x}{n}\right)^{\sigma+it} \right),$$

where, here and until the end of the section, $\sigma = \frac{1}{2} + \tau^{-1}$. We square out and integrate over $[-T, T]$ using Corollary 3 of Montgomery & Vaughan [17] for the L^2 -average of Dirichlet series over intervals. Thus, neglecting error terms, we have

$$\int_{-T}^T |R_1(x, t, \tau)|^2 dt \sim 2T \sum_{n \geq 1} \frac{A(n)^2}{n} a(n, x, \tau)^2 = 2TS(x, T).$$

On the other hand, the integral of the square of (15) over the same interval is

$$\sim 4\tau^2 \int_{-\mathbb{R}} \left| \sum_{\gamma \in [-T, T]} \frac{x^{i\gamma}}{1 + \tau^2(t - \gamma)^2} \right|^2 dt,$$

by the Riemann-von Mangoldt formula of Theorem A.3. Squaring out, integrating term by term and computing the relevant residue, we conclude that this is

$$\sim 2\pi\tau F(x, T, \tau).$$

The residue turns out to be a constant multiple of $\tau^{-1}\omega(\tau(\gamma_1 - \gamma_2))$.

The second “main term” in Theorem 6.1 arises computing the contribution of another term in the development of (15), namely

$$R_2(x, t, \tau) \stackrel{\text{def}}{=} \frac{1}{4}x^{1/2-\sigma+it}(\log(\sigma^2 + t^2) + \log((1-\sigma)^2 + t^2)).$$

Of course, this second main term is negligible unless x is quite small.

6.2 - The asymptotic formula for $S(x, \tau)$

As remarked above, if $\tau \geq x^{\varepsilon-1}$ then $S(x, \tau) \ll \tau \log x$. Moreover, if $y \leq x$ and

$$\psi(x+y) - \psi(x) \sim y \quad \text{uniformly for } y \geq x^{\beta+\varepsilon}$$

then

$$S(x, \tau) \sim \tau \log x \quad \text{uniformly for } \tau \geq x^{\beta+\varepsilon-1}.$$

However, S is erratic for $\tau \leq 1/x$. Essentially, it reduces to the single term given by the prime power closest to x . If x itself is a prime number, then $S(x, 1/x) \asymp (\log x)^2/x$, but if the prime power nearest to x is, say, $x \pm c \log x$ for some $c > 0$, then $S(x, 1/x) \asymp (\log x)^2/x^{1+c}$, which is much smaller. Of course, gaps between consecutive primes of size $\gg \log x$ occur infinitely often, by the PNT.

6.3 - The asymptotic formula for $J(x, \tau, \theta)$

Finally, in [12], Languasco, Perelli and the author also gave the following asymptotic result.

Theorem 6.2. *Assume the “Generalised Montgomery Conjecture.” Then*

$$J(x, \tau, \theta) \sim \left(1 + \frac{\tau}{2}\right) \tau \theta x^2 \log(1/\theta)$$

uniformly for $1/x \leq \theta \leq x^{-\varepsilon}$ and $\theta^{1/2-\varepsilon} \leq \tau \leq 1$.

Of course, the first factor here is relevant only if $\tau \gg 1$, when Theorem 6.2 is a consequence of earlier results. We left it for ease of comparison. The proof requires a suitable, stronger version of the technique introduced by Goldston & Montgomery in [6], with particular care for the τ -uniformity aspect.

A - The tools

Here we briefly summarise the main tools that we need.

Theorem A.1 [Brun-Titchmarsh inequality]. *For $x > 0$ and $y \geq 2$ we have*

$$\pi(x+y) - \pi(x) \leq \frac{2y}{\log y} \left(1 + \mathcal{O}((\log y)^{-1}) \right).$$

This is the special case $q = 1$ of Theorem 3.9 of Montgomery & Vaughan [18]. The error term may be omitted. The two results that follow are classical: they are Theorem 12.5 and Corollary 14.3 respectively of [18].

Theorem A.2 [Explicit formula]. *If $x \geq 2$ is not an integer and $T \geq 2$, we have*

$$\begin{aligned} \psi(x) = x - \sum_{\substack{\zeta(\beta+iy)=0 \\ \beta \in (0,1) \\ |y| \leq T}} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log(1-x^{-2}) \\ + \mathcal{O}\left(\frac{x}{T}(\log xT)^2 + (\log x) \min\left(1, \frac{x}{T\langle x \rangle}\right)\right), \end{aligned}$$

where $\langle x \rangle$ denotes the distance from x to the nearest prime power different from x itself. If x is an integer, the term $-\frac{1}{2}A(x)$ is to be added to the left hand side.

This is a very convenient representation of $\Delta_\psi(x) = \psi(x) - x$, since it contains the “free” parameter T (free in the sense that it does not appear in the left-hand side). This means that its value can be chosen in an optimal way in different applications. The upper bound in (2) is an almost immediate deduction of the Riemann Hypothesis, using also the Riemann-von Mangoldt formula below.

Theorem A.3 [Riemann-von Mangoldt formula]. *Let $N(T)$ denote the number of zeros of the Riemann zeta-function lying in the rectangle in the complex plane with sides $[0, 1]$ along the real axis and $[0, T]$ along the complex one. Then*

$$N(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + \mathcal{O}(\log T).$$

Theorem A.4 [Landau’s formula]. *We have*

$$\Sigma(x, T) = -\frac{A(n_x)}{2\pi} \frac{e^{iT \log(x/n_x)} - 1}{i \log(x/n_x)} + \mathcal{O}\left(x(\log(xT))^2 + \frac{\log T}{\log x}\right),$$

where n_x is the prime-power nearest to x and Σ is the exponential sum over zeros defined in (14). If $x = n_x$, then the main term is $-T\Lambda(x)/2\pi$.

This result can be found in Ford & Zaharescu [3] and it is unconditional.

Conjecture 3 [Gonek]. For $x, T \geq 2$ we have

$$\Sigma(x, T) \ll Tx^{-1/2+\varepsilon} + T^{1/2}x^\varepsilon.$$

Acknowledgments. I warmly thank my coauthors Alessandro Languasco and Alberto Perelli, and the organisers of the “Third Italian Number Theory Meeting.” Thanks are also due to the anonymous Referee for pointing out some inaccuracies.

References

- [1] J. BRÜDERN, R. J. COOK and A. PERELLI, *The values of binary linear forms at prime arguments*, Sieve methods, exponential sums and their application in number theory (Cardiff, 1995), G. R. H. Greaves et al., ed., Cambridge Univ. Press, Cambridge 1997, 87-100.
- [2] T. H. CHAN, *More precise pair correlation of zeros and primes in short intervals*, J. London Math. Soc. (2) **68** (2003), no. 3, 579-598.
- [3] K. FORD and A. ZAHARESCU, *On the distribution of imaginary parts of zeros of the Riemann zeta function*, J. Reine Angew. Math. **579** (2005), 145-158.
- [4] D. A. GOLDSTON, *Notes on pair correlation of zeros and prime numbers*, in “Recent perspectives in random matrix theory and number theory”, F. Mezzadri and N. C. Snaith, eds., London Math. Soc. Lecture Note Ser., **322**, Cambridge University Press, Cambridge 2005, 79-110.
- [5] D. A. GOLDSTON and D. R. HEATH-BROWN, *A note on the differences between consecutive primes*, Math. Ann. **266** (1984), no. 3, 317-320.
- [6] D. A. GOLDSTON and H. L. MONTGOMERY, *Pair correlation of zeros and primes in short intervals*, Analytic number theory and Diophantine problems (Stillwater, OK, 1984), A. C. Adolphson et al., ed., Progr. Math., **70**, Birkhäuser Boston, Boston, MA 1987, 183-203.
- [7] S. M. GONEK, *An explicit formula of Landau and its applications to the theory of the zeta-function*, Contemp. Math., **143**, Amer. Math. Soc., Providence, RI 1993, 395-413.
- [8] D. R. HEATH-BROWN, *Gaps between primes, and the pair correlation of zeros of the zeta-function*, Acta Arith. **41** (1982), 85-99.
- [9] D. R. HEATH-BROWN, *The number of primes in a short interval*, J. Reine Angew. Math. **389** (1988), 22-63.

- [10] A. LANGUASCO, A. PERELLI and A. ZACCAGNINI, *Explicit relations between pair correlation of zeros and primes in short intervals*, J. Math. Anal. Appl. **394** (2012), 761-771.
- [11] A. LANGUASCO, A. PERELLI and A. ZACCAGNINI, *An extension of the pair-correlation conjecture and applications*, Math. Res. Lett., **23** (2016), no. 1, 201-220.
- [12] A. LANGUASCO, A. PERELLI and A. ZACCAGNINI, *An extended pair-correlation conjecture and primes in short intervals*, Trans. Amer. Math. Soc., to appear.
- [13] J. E. LITTLEWOOD, *Sur la distribution des nombres premiers*, C. R. Math. Acad. Sci. Paris **158** (1914), 1869-1872.
- [14] H. MAIER, *Primes in short intervals*, Michigan Math. J. **32** (1985), 221-225.
- [15] H. L. MONTGOMERY, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc., Providence, R.I. 1973, 181-193.
- [16] H. L. MONTGOMERY and K. SOUNDARARAJAN, *Beyond pair correlation*, Paul Erdős and his mathematics, I (Budapest, 1999), Bolyai Soc. Math. Stud., **11**, János Bolyai Math. Soc., Budapest 2002, 507-514.
- [17] H. L. MONTGOMERY and R. C. VAUGHAN, *Hilbert's inequality*, J. London Math. Soc. (2) **8** (1974), 73-82.
- [18] H. L. MONTGOMERY and R. C. VAUGHAN, *Multiplicative number theory. I. Classical theory*, Cambridge Univ. Press, Cambridge 2007.
- [19] M. R. MURTY and A. PERELLI, *The pair correlation of zeros of functions in the Selberg class*, Internat. Math. Res. Notices 1999, no. 10, 531-545.
- [20] J. PINTZ, *On the remainder term of the prime number formula and the zeros of the Riemann's zeta-function*, Number Theory (Noordwijkerhout 1983), Lecture Notes in Math., **1068**, Springer, Berlin 1984, 186-197.
- [21] G. F. B. RIEMANN, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Königl. Preuss. Akad. Wiss. Berlin (1859), 671-680, in "Gesammelte Mathematische Werke" (ed. H. Weber), reprint, Dover Publications, New York 1953.
- [22] B. SAFFARI and R. C. VAUGHAN, *On the fractional parts of x/n and related sequences. II*, Ann. Inst. Fourier **27** (1977), 1-30.
- [23] A. SELBERG, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** (1943), 87-105.
- [24] A. ZACCAGNINI, *Primes in almost all short intervals*, Acta Arith. **84** (1998), no. 3, 225-244.
- [25] A. ZACCAGNINI, *A conditional density theorem for the zeros of the Riemann zeta-function*, Acta Arith. **93** (2000), no. 3, 293-301.

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