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Quaternionic Darmon points on abelian varieties

Abstract. In the first part of the paper we prove formulas for the p-adic logarithm of quaternionic Darmon points on modular abelian varieties over $\mathbb{Q}$ with toric reduction at p. These formulas are amenable to explicit computations and are the first to treat Stark–Heegner type points on higher-dimensional abelian varieties. In the second part of the paper we explain how these formulas, together with a mild generalization of results of Bertolini and Darmon on Hida families of modular forms and rational points, can be used to obtain rationality results over genus fields of real quadratic fields for Darmon points on abelian varieties.

Keywords. Darmon points, modular abelian varieties, p-adic logarithm, genus fields.

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Contents

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1 - Introduction

Stark–Heegner points on elliptic curves over $\mathbb{Q}$ were defined by Darmon in [**7**] as conjectural analogues over (abelian extensions of) real quadratic fields of classical Heegner points. Since then, generalizations of these points to higher-dimensional (modular) abelian varieties have not been systematically investigated.

Building on techniques developed in [**17**], quaternionic Darmon points on p-adic tori and Jacobians of Shimura curves over $\mathbb{Q}$ were introduced in [**18**]. Recently, a rationality result for projections of Darmon points to elliptic curves (which extends to a quaternionic context the theorem obtained by Bertolini and Darmon in [**2**]) has been proved in [**21**]. Let us briefly review the arithmetic setting in which the theory of [**17**] and [**18**] takes place.

Let X_0 (respectively, X_1) denote the (compact) Shimura curve of discriminant $D > 1$ and level M (respectively, Mp) for coprime integers D, M and a prime number $p \nmid DM$ (here D is a square-free product of an *even* number of primes). Let H be the maximal torsion-free quotient of the cokernel of the degeneracy map from $H_1(X_0, \mathbb{Z})^2$ to $H_1(X_1, \mathbb{Z})$. It turns out that H is a free abelian group of rank equal to twice the dimension of the p-new quotient $J_1^{p\text{-new}}$ of the Jacobian J_1 of X_1. Let $\mathbb{H}$ be a (Hecke-stable) non-zero torsion-free quotient of H and let K be a real quadratic field in which all the primes dividing M (respectively, Dp) split (respectively, are inert). The p-adic tori on which Darmon points are naturally defined are quotients of tori of the form $T_{\mathbb{H}}(K_p) := \mathbb{H} \otimes K_p^\times$ by suitable lattices $L_{\mathbb{H}}$ contained in $T_{\mathbb{H}}(\mathbb{Q}_p) := \mathbb{H} \otimes \mathbb{Q}_p^\times$ and built, as in [**17**], via group (co)homology and p-adic integration. So far, explicit computations with Darmon points (of conductor 1) had been performed only in [**21**] in the case where $\mathbb{H}$ is associated with an elliptic curve over $\mathbb{Q}$ of conductor MDp.

While leading to rationality results for quaternionic Darmon points on elliptic curves over $\mathbb{Q}$ that provide evidence for the conjectures formulated in [**18**], the one-dimensional setting studied in [**21**] might obscure one of the key features of the theory developed in [**17**] and [**18**], that is, the possibility of defining Stark–Heegner type points of arbitrary conductor on higher-dimensional (modular) abelian varieties. It is perhaps worth remarking that constructions of this kind lie outside the scope of Darmon's original theory, since Stark–Heegner points are defined directly on elliptic curves over $\mathbb{Q}$ (when Shimura curves reduce to classical modular curves, constructions analogous to ours could be carried out using Dasgupta's p-adic uniformization of modular Jacobians, cf. [8]).

The first goal of this paper is to obtain formulas for quaternionic Darmon points on modular abelian varieties over $\mathbb{Q}$ and to explain how these formulas can be used to prove rationality results over genus fields of real quadratic fields. These results are very close to those in [**21**] (for elliptic curves) and [**10**], [28], [31], [**32**] (for motives of modular forms). However, we think that it may be interesting to have these results stated and proved independently for abelian varieties. The case of (modular) abelian varieties is considerably different from that of elliptic curves and presents some extra technical difficulties (see later in this introduction), but, at the same time, the results are neater than those for more general motives of modular forms. Moreover, along the way we take some time to review the construction of Darmon points on Jacobians of Shimura curves and some basic properties of logarithms and exponentials (in the sense of p-adic Lie groups) on the p-adic points of abelian varieties, in the hope that this will be of some use to those researchers who approach this subject for the first time. We have also added an appendix where we summarize well-known facts on the p-adic uniformization of abelian varieties over $\mathbb{Q}$ with purely toric reduction at p and we briefly explain the role they play in the context of Darmon points.

In order to describe our main results in detail, we need some further notation. With $\mathbb{H}$ as before, let $\mathcal{M}_0$ denote the abelian group of $\mathbb{H}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ with total mass 0. Following [**17**, Section 4], one can explicitly introduce a canonical cohomology class $\boldsymbol{\mu}_{\mathbb{H}} \in H^1(\Gamma, \mathcal{M}_0)$ where $\Gamma \subset \mathrm{SL}_2(\mathbb{Q}_p)$ is Ihara's group defined in (1). The abelianization of Γ is a finite group whose exponent will be denoted by t. If $\mathcal{O}_K$ is the ring of integers of K and d_K is its discriminant, let $\mathcal{O}_c := \mathbb{Z} + c\mathcal{O}_K$ be the order of K of conductor c prime to MDd_Kp. As in [**18**], multiplicative integration over $\mathbb{P}^1(\mathbb{Q}_p)$ against a representative $\mu_{\mathbb{H}}$ of $\boldsymbol{\mu}_{\mathbb{H}}$ allows us to attach a Darmon point $\mathcal{P}_{\mathbb{H},\psi} \in T_{\mathbb{H}}(K_p)/L_{\mathbb{H}}$ to every optimal embedding ψ of $\mathcal{O}_c$ into a fixed Eichler order R_0 of level M in the quaternion algebra over $\mathbb{Q}$ of discriminant D. As conjectured in [**18**, §3.2], images on modular abelian varieties of points of the form $\mathcal{P}_{\mathbb{H},\psi}$ are expected to be rational over the narrow ring class field of K of conductor c and to satisfy a suitable Shimura reciprocity law under the action of the corresponding Galois group.

Now let $\mathbb{H}_p := \mathbb{H} \otimes \mathbb{Z}_p$, let $\mathbb{X}$ be the set of primitive vectors in $\mathbb{Y} := \mathbb{Z}_p^2$ (i.e., those vectors which are not divisible by p) and write $\mathbb{D}$ for the group of $\mathbb{H}_p$-valued measures on $\mathbb{Y}$ that are supported on $\mathbb{X}$. If Γ_0 denotes the group of norm 1 elements in R_0 and $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ is the map sending (a, b) to a/b then there exists $\tilde{\boldsymbol{\mu}}_{\mathbb{H}} \in H^1(\Gamma_0, \mathbb{D})$ such that $\pi_*(\tilde{\boldsymbol{\mu}}_{\mathbb{H}}) = \mathrm{res}_{\Gamma_0}(\boldsymbol{\mu}_{\mathbb{H}})$. With self-explaining notation, we can also choose a 1-cocycle $\tilde{\mu}_{\mathbb{H}}$ representing $\tilde{\boldsymbol{\mu}}_{\mathbb{H}}$ such that $\pi_*(\tilde{\mu}_{\mathbb{H},\gamma}) = \mu_{\mathbb{H},\gamma}$ for all $\gamma \in \Gamma_0$. Let ε_c denote a generator of the group of units of $\mathcal{O}_c$ of norm 1 such that $\varepsilon_c > 1$ under a fixed embedding $K \hookrightarrow \mathbb{R}$ and set $\gamma_\psi := \psi(\varepsilon_c) \in \Gamma_0$. Moreover, denote z_ψ the fixed point of $\psi(K^\times)$ acting on $\mathbb{P}^1(K_p)$ such that $\psi(\alpha)(z_\psi, 1) = \alpha(z_\psi, 1)$ for all $\alpha \in K$. Finally, fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ be the completion of $\bar{\mathbb{Q}}_p$.

Now let A be a modular abelian variety over $\mathbb{Q}$ of dimension d and conductor $N := MDp$, which means that A is associated with a (normalized) newform f of weight 2 and level N. Write $\mathcal{O}_f$ for the ring generated over $\mathbb{Z}$ by the Fourier coefficients of f. It is known that A has purely multiplicative reduction at p, i.e., the identity component of the special fibre of the Néron model of A over $\mathbb{Z}_p$ is a torus over $\mathbb{F}_p$. Since $J_1^{p\text{-new}}$ is the maximal toric quotient of J_1 at p, it follows that A is a quotient of $J_1^{p\text{-new}}$. Moreover, if $\varepsilon \in \{\pm\}$ then the ε-eigenspace for complex conjugation acting on $\mathbb{H}_A := H_1(A(\mathbb{C}), \mathbb{Z})$, which we denote by $\mathbb{H}_A^\varepsilon$, is a free quotient of H of rank d. With notation as above, from here on we take $\mathbb{H} = \mathbb{H}_A^\varepsilon$. Set $T_A := T_{\mathbb{H}}$, $L_A := L_{\mathbb{H}}$ and $\tilde{\mu}_A := \tilde{\mu}_{\mathbb{H}}$. As a consequence of the uniformization results of [**17**], there is a Galois-equivariant isogeny

$$\varphi_A : T_A(\mathbb{C}_p)/L_A \longrightarrow A(\mathbb{C}_p)$$

defined over K_p. An explicit description of φ_A can be given in terms of the Galois-equivariant analytic isomorphism

$$(\mathbb{C}_p^\times)^d/\langle \underline{q}_1, \dots, \underline{q}_d\rangle \xrightarrow{\simeq} A(\mathbb{C}_p)$$

due to (among others) Tate, Morikawa and Mumford. Here $\langle \underline{q}_1, \dots, \underline{q}_d\rangle$ is the lattice inside $\mathbb{Q}_p^d$ generated by the Tate periods $\underline{q}_1, \dots, \underline{q}_d$ for A at p, which turns out to be commensurable to L_A (Theorem A.1). The reader can find details about these constructions in Appendix A.

Regarding ε as fixed, for every optimal embedding ψ of $\mathcal{O}_c$ into R_0 we define

$$P_{A,\psi} := \varphi_A\big(\mathcal{P}_{\mathbb{H},\psi}\big) \in A(K_p).$$

These are the Darmon points (of conductor c) on A alluded to before. To state our formula for the points $P_{A,\psi}$, recall that the theory of Lie groups gives a logarithm map

$$\log_A : A(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A(\mathbb{C}_p)) \simeq \mathbb{C}_p^d$$

on the p-adic points of A. In particular, $\log_A(P) \in \mathrm{Lie}(A(K_p))$ for all $P \in A(K_p)$. By a result of Mattuck ([**22**, Theorem 7]), if $\mathcal{K}$ is a finite extension of $\mathbb{Q}_p$ then $A(\mathcal{K})$ is compact, hence $\log_A(A(\mathcal{K}))$ is bounded. Define $\Psi_A := \log_A \circ \varphi_A$ and, by implicitly pre-composing it with the diagonal embedding and the projection onto the quotient, view Ψ_A as defined on $\bar{\mathbb{Q}}_p^\times$. Our first main result, which corresponds to Theorem 4.1 in the text, is

Theorem 1.1. $$\log_A(P_{A,\psi}) = -t \cdot \int_{\mathbb{X}} \Psi_A(x - z_\psi y)\, d\tilde{\mu}_{A,\gamma_\psi}(x, y).$$

It is relatively straightforward to show that if $\mathcal{K}$ is a complete subfield of $\mathbb{C}_p$ then both $\mathbb{H} \otimes \mathcal{K}$ and $\mathrm{Lie}(A(\mathcal{K}))$ are free $\mathcal{O}_f \otimes \mathcal{K}$-modules of rank 1 (Lemma 4.1 and Proposition 4.2). This fact, together with the boundedness of $\Psi_A|_{K_p^\times}$, ensures that the integral in Theorem 1.1 is meaningful and does indeed exist.

Let α be an endomorphism of $A_{/\mathbb{C}_p}$ and let $\alpha_* : \mathrm{Lie}(A(\mathbb{C}_p)) \to \mathrm{Lie}(A(\mathbb{C}_p))$ be the linear map induced by α. As a consequence of Theorem 1.1, we obtain

Corollary 1.1. $$\log_A\big(\alpha(P_{A,\psi})\big) = -t \cdot \alpha_*\left(\int_{\mathbb{X}} \Psi_A(x - z_\psi y)\, d\tilde{\mu}_{A,\gamma_\psi}(x, y)\right).$$

These are the first formulas for points of Stark–Heegner type on abelian varieties. When A is an elliptic curve and $c = 1$, Corollary 1.1 essentially reduces to [**21**, Corollary 3.5].

The interest for such formulas is twofold. On the one hand, since the construction of measure-valued cocycles in [**17**] is entirely explicit, Corollary 1.1 allows us to compute (at least in principle) the orbit of a Darmon point on A under the action of the endomorphism ring of A (cf. also Corollary 4.2). In this direction, it would be interesting to extend the results for elliptic curves of [**12**] to more general abelian varieties.

On the other hand, the expression for the p-adic logarithm of $P_{A,\psi}$ in Theorem 1.1 plays a key role in the proof of rationality theorems for genus character combinations of Darmon points on A of the type obtained in [**21**] in the case of elliptic curves. Analogues of those rationality results in this higher-dimensional setting are the subject of the second part of this paper, and now we briefly describe them.

Let A be an abelian variety of conductor $N = MDp$ attached to a (normalized) newform $f \in S_2(\Gamma_0(N))$, as above. In addition, assume that M is square-free, so that N is square-free and A is semistable. Note that the role of this semistability condition is somewhat more delicate here than it was in [**21**], as it allows us not only to apply the results of [**24**] (which we expect to hold in greater generality) but also to get information on the structure of the p-adic points of A that is a central ingredient for the proof of our rationality results.

Let G_K^+ be the narrow class group of K (i.e., by global class field theory, the Galois group over K of the narrow Hilbert class field of K). For any genus character χ of K (i.e., an unramified quadratic character of G_K^+) we define the point

$$P_\chi := \sum_{\sigma \in G_K^+} \chi(\sigma) P^{\varepsilon}_{A,\psi_\sigma} \in A(K_p),$$

where $\psi \mapsto \psi_\sigma$ denotes the Galois action on optimal embeddings of $\mathcal{O}_K$ into R_0 (that is, we take $c = 1$) and the choice of a sign $\varepsilon \in \{\pm\}$ depends on χ. Our second main result is

Theorem 1.2. *Let χ be a genus character of K associated with a pair (χ_1, χ_2) of Dirichlet characters such that $\chi_i(-MD) = -w_{MD}$ for $i = 1, 2$, where w_{MD} is the eigenvalue of the Atkin–Lehner involution W_{MD} acting on f.*

1. *There exists $n \in \mathbb{Z}$ such that $nP_\chi \in A(H_\chi)$ where H_χ is the genus field of K cut out by χ.*
2. *The point nP_χ is of infinite order if and only if $L'(f/K, \chi, 1) \neq 0$.*

This is Theorem 6.3 in the text. The reader will realize that our strategy for proving Theorem 1.2 follows [**21**] (which, in turn, was crucially inspired by [**2**]) closely. Namely, let f_∞ denote the formal q-expansion associated by Hida with f, let χ be a genus character of K corresponding to an unordered pair χ_1, χ_2 of quadratic Dirichlet characters and for $j = 1, 2$ write $L_p(f_\infty, \chi_j, k, s)$ for the Mazur–Kitagawa two-variable p-adic L-function attached to f_∞ and χ_j. As in [**21**], the cohomology class $\tilde{\boldsymbol{\mu}}_A$ can be used to introduce a p-adic L-function $L_p(f_\infty/K,\chi,k)$ attached to f_∞ and χ. In a nutshell, the strategy of [**2**] and [**21**] can be described as follows:

1. if we order the pair χ_1,χ_2 in such a way that the sign of the functional equation of the L-function $L(f,\chi_1,s)$ is -1 then a suitable factorization of $L_p(f_\infty,\chi,k)$ in terms of $L_p(f_\infty,\chi_1,k,s)$ and $L_p(f_\infty,\chi_2,k,s)$ establishes a link between the values at $k=2$ of the second derivatives of $L_p(f_\infty/K,\chi,k)$ and of the restriction of $L_p(f_\infty,\chi_1,k,s)$ to the line $s=k/2$;
2. the second derivative of $L_p(f_\infty/K,\chi,k)$ evaluated at $k=2$ is essentially computed by the integrals appearing on the right hand side of Theorem 1.1, thus providing a link with Darmon points;
3. an analogue of [**1**, Theorem 5.4] shows that the value at $k=2$ of the second derivative of $L_p(f_\infty,\chi_1,k,k/2)$ encodes the logarithm of certain linear combinations of classical Heegner points.

These three observations provide a relation between Heegner points and Darmon points and ultimately yield the rationality result for P_χ stated in Theorem 1.2.

Although, as noted above, our strategy parallels that of [**2**] and [**21**], the higher-dimensional setting of this paper brings extra complications and several of the techniques introduced in [**21**] need to be re-interpreted from a different, much more "geometric" point of view.

First of all, a generalization of the results in [**1**] to abelian varieties leads one to express the second derivative of a certain Mazur–Kitagawa L-function in terms of the square of the logarithm of a suitable (global) point on A. Here $\log_A$ takes values in $\mathrm{Lie}(A(\mathbb{C}_p))$, which is a d-dimesional $\mathbb{C}_p$-vector space with (in general) no intrinsic ring structure. Furthermore, the exponential map $\exp_A$ on $\mathrm{Lie}(A(K_p))$ is a local inverse of $\log_A$, takes values in $A(K_p)\otimes\mathbb{Q}$ (actually, in $A(K_p)$) and occurs in our recipe for $L_p(f_\infty/K,\chi,k)$. In fact, $L_p(f_\infty/K,\chi,k)$ is defined as the square of a sum of integrals of $A(K_p)\otimes\mathbb{Q}$-valued, bounded functions against $\mathbb{H}\otimes\mathbb{Z}_p$-valued measures of the form $\tilde{\mu}_{A,\gamma_\psi}$. In our context, the following problems arise:

- we need to make sense of the *square* of an element of $A(K_p)\otimes\mathbb{Q}$;
- it is not *a priori* clear that the integrals involved in the expression of $L_p(f_\infty/K,\chi,k)$ are indeed well defined.

Underlying our solution to these issues is the $\mathcal{O}_f$-module structure on the p-adic points of A and, by functoriality, on their Lie algebra. It has already been remarked that $\mathbb{H}\otimes\mathcal{K}$ and $\mathrm{Lie}(A(\mathcal{K}))$ are free $\mathcal{O}_f\otimes\mathcal{K}$-modules of rank 1 for every complete subfield $\mathcal{K}$ of $\mathbb{C}_p$. On the other hand, if $\mathcal{K}$ is a finite extension of $\mathbb{Q}_p$ then the map $\log_A$ and our fixed isomorphism $\mathrm{Lie}(A(K_p)) \simeq \mathcal{O}_f \otimes K_p$ establish an isomorphism $A(K_p) \otimes \mathbb{Q} \simeq \mathcal{O}_f \otimes K_p$ that allows us to view $A(K_p) \otimes \mathbb{Q}$ as a free $\mathcal{O}_f \otimes K_p$-module of rank 1. By fixing isomorphisms between $\mathbb{H} \otimes K_p$, $\mathrm{Lie}(A(K_p))$, $A(K_p) \otimes \mathbb{Q}$ and $\mathcal{O}_f \otimes K_p$, we can extend the arguments of [**21**] to our more general setting, thus obtaining the rationality of a suitable multiple of P_χ over the abelian extension of K predicted by the conjectures that were formulated in [**18**].

The main results that we describe below are original but, as already pointed out, a large portion of this paper has also an expository flavour. We refer the reader to [**9**] and [**10**] for related results and for generalizations to higher-dimensional Darmon cycles *à la* Rotger and Seveso ([**28**]).

2 - Measure-valued cohomology

2.1 - *Homology of Shimura curves and measures*

We recall notation and results of [**17**]. Let $D > 1$ be a square-free product of an *even* number of primes, let $M \geq 1$ be an integer prime to D and fix a prime number p not dividing MD. Let B be the (indefinite) quaternion algebra over $\mathbb{Q}$ of discriminant D. Let $R_0 \subset R_1$ be Eichler orders of B of level M and Mp, respectively. For $i = 0, 1$ write Γ_i for the group of units of norm 1 in R_i and consider the (compact) Shimura curve $X_i := \Gamma_i \backslash \mathcal{H}$, where $\mathcal{H}$ is the complex upper half-plane and the action of $B^\times$ on $\mathbb{P}^1(\mathbb{C})$ is by Möbius (i.e., fractional linear) transformations via a fixed isomorphism of $\mathbb{R}$-algebras

$$i_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{R}).$$

Let ω_p be an element in R_0 of reduced norm p that normalizes Γ_1 and denote by

$$\pi_1, \pi_2 : X_1 \longrightarrow X_0, \qquad \Gamma_1 z \stackrel{\pi_1}{\longmapsto} \Gamma_0 z, \qquad \Gamma_1 z \stackrel{\pi_2}{\longmapsto} \Gamma_0 \omega_p z$$

the two degeneracy maps. Moreover, let

$$\pi^* := \pi_1^* \oplus \pi_2^* : H_1(X_0, \mathbb{Z})^2 \longrightarrow H_1(X_1, \mathbb{Z})$$

be the map induced in singular homology by pull-back. In terms of group homology, it corresponds to the map $H_1(\Gamma_0, \mathbb{Z})^2 \to H_1(\Gamma_1, \mathbb{Z}) \simeq \Gamma_1^{\mathrm{ab}}$ induced by corestriction. Let H denote the maximal torsion-free quotient of the cokernel of π^*. If $J_1^{p\text{-new}}$ is the p-new quotient (i.e., the maximal toric quotient at p) of the Jacobian variety J_1 of X_1 and g is the dimension of $J_1^{p\text{-new}}$ then H is a free abelian group of rank $2g$. Throughout this paper fix a non-zero torsion-free quotient $\mathbb{H}$ of H.

Fix an isomorphism of $\mathbb{Q}_p$-algebras

$$i_p : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{Q}_p)$$

such that $i_p(R_0 \otimes \mathbb{Z}_p) = \mathrm{M}_2(\mathbb{Z}_p)$ and $i_p(R_1 \otimes \mathbb{Z}_p)$ is the order of $\mathrm{M}_2(\mathbb{Q}_p)$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p)$ with $c \equiv 0 \pmod p$. With $\mathbb{C}_p$ denoting the completion of an algebraic closure of $\mathbb{Q}_p$, let $B^\times$ act on $\mathbb{P}^1(\mathbb{C}_p)$ by fractional linear transformations via i_p.

Let $\mathrm{n} : B \to \mathbb{Q}$ denote the reduced norm map and define Ihara's group Γ as

$$\Gamma := \{\gamma \in R_0 \otimes \mathbb{Z}[1/p] \mid \mathrm{n}(\gamma) = 1\} \overset{i_p}{\hookrightarrow} \mathrm{SL}_2(\mathbb{Q}_p). \tag{1}$$

Write $\mathcal{M}$ for the group of $\mathbb{H}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ and $\mathcal{M}_0$ for the subgroup of $\mathcal{M}$ of those measures with total mass 0. As in [**17**, Section 4], there is a canonical left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{M}$ and $\mathcal{M}_0$. Then $B^\times$ acts on $\mathcal{M}$ and $\mathcal{M}_0$ via i_p and the embedding $B \hookrightarrow B \otimes \mathbb{Q}_p$.

2.2 - *A distinguished cohomology class*

Using harmonic cocycles on the oriented edges of the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and exploiting basic properties of radial (in the sense of [**17**, Definition 4.7]) system of representatives for the cosets in $\Gamma_1\backslash\Gamma$, one can introduce a canonical element

$$\boldsymbol{\mu}_{\mathbb{H}} \in H^1(\Gamma, \mathcal{M}_0).$$

We shall not review the construction of $\boldsymbol{\mu}_{\mathbb{H}}$ here, but rather refer the reader to [**17**, Section 4] for details. This cohomology class plays a central role in the definition of Darmon points on p-adic tori and abelian varieties.

3 - Darmon points on p-adic tori

3.1 - *Homology and p-adic integration*

Define $T_{\mathbb{H}} := \mathbb{H} \otimes_{\mathbb{Z}} \mathbb{G}_m$ where $\mathbb{G}_m$ denotes the multiplicative group (viewed as a functor on commutative $\mathbb{Q}$-algebras).

Write $\mathcal{H}_p := \mathbb{C}_p - \mathbb{Q}_p$ for Drinfeld's p-adic plane, let $D := \mathrm{Div}(\mathcal{H}_p)$ be the group of divisors on $\mathcal{H}_p$ and let $D_0 := \mathrm{Div}^0(\mathcal{H}_p)$ be the subgroup of divisors of degree 0. From the long exact sequence in homology associated with the short exact sequence induced by the degree map on divisors one can extract a boundary map $\partial : H_2(\Gamma, \mathbb{Z}) \to H_1(\Gamma, D_0)$. Now consider the multiplicative integration pairing

$$\langle\,,\rangle : D_0 \times \mathcal{M}_0 \longrightarrow T_{\mathbb{H}}(\mathbb{C}_p)$$

defined in [**17**, §5.1]. By construction, this pairing factors through $H_0(\Gamma, D_0 \otimes_{\mathbb{Z}} \mathcal{M}_0)$ and thus, by cap product, we can also define a pairing

$$H_1(\Gamma, D_0) \times H^1(\Gamma, \mathcal{M}_0) \longrightarrow T_{\mathbb{H}}(\mathbb{C}_p). \tag{2}$$

By fixing $\boldsymbol{\mu}_{\mathbb{H}} \in H^1(\Gamma, \mathcal{M}_0)$ in (2) we get a map $\mathcal{I} : H_1(\Gamma, D_0) \to T_{\mathbb{H}}(\mathbb{C}_p)$. Finally, we take

$$\Phi := \mathcal{I} \circ \partial : H_2(\Gamma, \mathbb{Z}) \longrightarrow T_{\mathbb{H}}(\mathbb{C}_p)$$

and denote by $L_{\mathbb{H}}$ the image of Φ. One can show that $L_{\mathbb{H}}$ is a Hecke-stable lattice contained in $T_{\mathbb{H}}(\mathbb{Q}_p)$ (see [**17**, Proposition 6.1]).

Fix $z \in K_p - \mathbb{Q}_p$ and let $\boldsymbol{d}_z \in H^2\big(\Gamma, T_{\mathbb{H}}(\mathbb{C}_p)\big)$ be the class represented by the 2-cocycle

$$d_z : \Gamma \times \Gamma \longrightarrow T_{\mathbb{H}}(K_p), \qquad (\gamma_1, \gamma_2) \longmapsto \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{s - \gamma_1^{-1}(z)}{s - z}\, d\mu_{\mathbb{H},\gamma_2}(s),$$

where the multiplicative integral on the right is defined as in [**17**, §5.1]. The class $\boldsymbol{d}_z$ does not depend on the representative $\mu_{\mathbb{H}}$ of $\boldsymbol{\mu}_{\mathbb{H}}$. Let $\bar{d}_z$ be the composition of d_z with the projection onto $T_{\mathbb{H}}(K_p)/L_{\mathbb{H}}$ and denote by $\bar{\boldsymbol{d}}_z$ the resulting class. By construction, $L_{\mathbb{H}}$ is the smallest subgroup of $T_{\mathbb{H}}(\mathbb{Q}_p)$ such that $\bar{\boldsymbol{d}}_z$ is trivial in $H^2(\Gamma, T_{\mathbb{H}}/L_{\mathbb{H}})$, so there exists $\beta_z : \Gamma \to T_{\mathbb{H}}/L_{\mathbb{H}}$ splitting $\bar{d}_z$. The map β_z is well defined only up to elements in $\mathrm{Hom}(\Gamma, T_{\mathbb{H}}/L_{\mathbb{H}})$. To deal with this ambiguity, recall that the abelianization Γ^{ab} of Γ is finite ([**18**, Proposition 2.1]). Hence if t is the exponent of Γ^{ab} then $t \cdot \beta_z$ is well defined.

3.2 - *Darmon points on p-adic tori*

Let $K = \mathbb{Q}(\sqrt{d_K})$ be a real quadratic field with discriminant d_K such that all primes dividing Dp (respectively, M) are inert (respectively, split) in K, and fix an embedding $K \hookrightarrow \mathbb{R}$. Let $c \geq 1$ be an integer prime to MDd_Kp, let $\mathcal{O}_c := \mathbb{Z} + c\mathcal{O}_K$ be the order of K of conductor c, let H_c^+ be the narrow ring class field of K of conductor c and set $G_c^+ := \mathrm{Gal}(H_c^+/K)$. The group G_c^+ is isomorphic to the narrow class group of $\mathcal{O}_c$ via the reciprocity map of global class field theory.

Write $\mathrm{Emb}(\mathcal{O}_c, R_0)$ for the set of optimal embeddings of $\mathcal{O}_c$ into R_0, i.e., the embeddings ψ of K into B such that $\mathcal{O}_c = \psi^{-1}(R_0)$. The group Γ_0 acts on $\mathrm{Emb}(\mathcal{O}_c, R_0)$ by conjugation.

Let τ denote the generator of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$. If $\psi : K \hookrightarrow B$ is an embedding of $\mathbb{Q}$-algebras then let $Q_\psi(x,y)$ be the quadratic form with coefficients in $\mathbb{Q}_p$ associated with ψ as in [**21**, §2.3]. Factor $Q_\psi(x,y)$ as $Q_\psi(x,y) = c(x - z_\psi y)(x - \bar{z}_\psi y)$ with $z_\psi, \bar{z}_\psi \in K_p - \mathbb{Q}_p$ such that $\tau(z_\psi) = \bar{z}_\psi$. Equivalently, z_ψ and $\bar{z}_\psi$ are the only fixed points for the action of $\psi(K^\times)$ on $\mathbb{P}^1(K_p)$ by fractional linear transformations via i_p.

By Dirichlet's unit theorem, the abelian group of units of norm 1 in $\mathcal{O}_c$ is free of rank 1. Let ε_c be a generator of this group such that $\varepsilon_c > 1$ with respect to the fixed embedding $K \hookrightarrow \mathbb{R}$ and set $\gamma_\psi := \psi(\varepsilon_c) \in \Gamma_0$.

Definition 3.1. The *Darmon points of conductor* c on $T_{\mathbb{H}}(K_p)/L_{\mathbb{H}}$ are the points

$$\mathcal{P}_{\mathbb{H},\psi} := t \cdot \beta_{z_\psi}(\gamma_\psi) \in T_{\mathbb{H}}(K_p)/L_{\mathbb{H}}$$

where ψ varies in $\mathrm{Emb}(\mathcal{O}_c, R_0)$.

The points $\mathcal{P}_{\mathbb{H},\psi}$ (or, rather, their images on abelian varieties) are expected to be rational over H_c^+ and to satisfy a suitable Shimura reciprocity law under the action of G_c^+: the reader is referred to [**18**, §3.2] for precise conjectures and to [**21**] for partial results in this direction.

Proposition 3.1. *The point* $\mathcal{P}_{\mathbb{H},\psi}$ *does not depend on the choice of a representative of* $\boldsymbol{\mu}_{\mathbb{H}}$. *Furthermore,* $\mathcal{P}_{\mathbb{H},\psi}$ *depends only on the* Γ_0*-conjugacy class of* ψ.

Proof. Proceed as in the proofs of [**18**, Propositions 3.4 and 3.5]. □

3.3 - *p-adic measures and Shapiro's lemma*

Let $\mathbb{H}_p := \mathbb{H} \otimes \mathbb{Z}_p$ and let $\mathbb{X} := (\mathbb{Z}_p^2)'$ be the set of primitive vectors in $\mathbb{Y} := \mathbb{Z}_p^2$, i.e., the vectors in $\mathbb{Y}$ that are not divisible by p. Let $\tilde{\mathbb{D}}$ be the group of $\mathbb{H}_p$-valued measures on $\mathbb{Y}$ and write $\mathbb{D}$ for the subgroup consisting of those measures which are supported on $\mathbb{X}$. As described in [**17**, §7.2], the group $\Sigma := \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$ acts on the left on $\tilde{\mathbb{D}}$. This action induces an action of Σ on $\mathbb{D}$ (see [**17**, Lemma 7.4]).

Denote by $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ the fibration defined by $(a,b) \mapsto a/b$. By [**17**, Theorem 7.5], the canonical map $\pi_* : H^1(\Gamma_0, \mathbb{D}) \to H^1(\Gamma_0, \mathcal{M})$ induced by π is surjective. Fix a $\tilde{\boldsymbol{\mu}}_{\mathbb{H}} \in H^1(\Gamma_0, \mathbb{D})$ such that $\pi_*(\tilde{\boldsymbol{\mu}}_{\mathbb{H}}) = \mathrm{res}_{\Gamma_0}(\boldsymbol{\mu}_{\mathbb{H}})$, where res_{Γ_0} is restriction in cohomology. As in [**17**, §7.3], we can choose a representative $\tilde{\mu}_{\mathbb{H}}$ of $\tilde{\boldsymbol{\mu}}_{\mathbb{H}}$ such that $\pi_*(\tilde{\mu}_{\mathbb{H},\gamma}) = \mu_{\mathbb{H},\gamma}$ for all $\gamma \in \Gamma_0$ (here π_* stands for the map induced by π between spaces of measures). From now on we fix such a $\tilde{\mu}_{\mathbb{H}}$.

For any compact open subset $U \subset \mathbb{Y}$ write $\mathbb{D}_U$ for the subset of the measures on $\mathbb{Y}$ which are supported on U. Thus, in particular, $\mathbb{D}_{\mathbb{X}} = \mathbb{D}$. Define $\mathbb{X}_\infty := \mathbb{Z}_p^\times \times p\mathbb{Z}_p$ and $\mathbb{X}_{\rm aff} := \mathbb{Z}_p \times \mathbb{Z}_p^\times$, so that $\mathbb{X} = \mathbb{X}_\infty \coprod \mathbb{X}_{\rm aff}$. Recall from §2.1 the element $\omega_p \in R_0$ of reduced norm p that normalizes Γ_1. Shapiro's lemma provides canonical isomorphisms

$$\mathcal{S} : H^1(\Gamma_0, \mathbb{D}) \xrightarrow{\simeq} H^1(\Gamma_1, \mathbb{D}_{\mathbb{X}_\infty}), \qquad \hat{\mathcal{S}} : H^1(\hat{\Gamma}_0, \mathbb{D}_{\omega_p\mathbb{X}}) \xrightarrow{\simeq} H^1(\Gamma_1, \mathbb{D}_{\omega_p\mathbb{X}_\infty}).$$

For a prime $\ell \nmid Mp$ let S_ℓ denote the set of elements in $R_0 \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ with non-zero norm. For a prime $\ell | Mp$ let n_ℓ be the maximal power of ℓ dividing Mp. For primes $\ell \neq p$ with $\ell | M$ fix an isomorphism of $\mathbb{Q}_\ell$-algebras

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{Q}_\ell)$$

such that $i_\ell(R_0 \otimes \mathbb{Z}_\ell)$ is the order consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_\ell)$ with $\ell^{n_\ell} | c$. For all primes $\ell | Mp$ define S_ℓ to be the inverse image via i_ℓ of the semigroup consisting of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_\ell)$ such that $\ell^{n_\ell} | c$, $a \in \mathbb{Z}_\ell^\times$ and $\det(g) \neq 0$. Now we may consider the Hecke algebra $\mathbb{T}_1$ associated with the pair (Γ_1, S_1) where $S_1 := B^\times \cap \prod_\ell S_\ell$ and the product is taken over all prime numbers ℓ (see [**17**, §2.2] for details). By definition, the Hecke operator $U_p \in \mathbb{T}_1$ is given by $U_p = \Gamma_1 g_0 \Gamma_1$ for an element $g_0 \in R_1$ of norm p. We also have the Atkin–Lehner involution $W_p = \Gamma_1 \omega_p \Gamma_1 \in \mathbb{T}_1$. Since the Hecke algebra $\mathbb{T}_1$ acts on $H^1(\Gamma_1, \mathbb{D}_{\mathbb{X}_\infty})$, we may use $\mathcal{S}$ and $\hat{\mathcal{S}}$ to define Hecke operators

$$\mathbb{U}_p := \mathcal{S}^{-1} U_p \mathcal{S} : H^1(\Gamma_0, \mathbb{D}) \to H^1(\Gamma_0, \mathbb{D}),$$
$$\mathbb{W}_p := \hat{\mathcal{S}}^{-1} W_p \mathcal{S} : H^1(\Gamma_0, \mathbb{D}) \to H^1(\hat{\Gamma}_0, \mathbb{D}_{\omega_p\mathbb{X}}).$$

Define $\hat{\tilde{\boldsymbol{\mu}}}_{\rm H} := \mathbb{W}_p \mathbb{U}_p(\tilde{\boldsymbol{\mu}}_{\rm H}) \in H^1(\hat{\Gamma}_0, \mathbb{D}_{\omega_p\mathbb{X}})$ and choose a representative $\hat{\tilde{\mu}}_{\rm H}$ of $\hat{\tilde{\boldsymbol{\mu}}}_{\rm H}$. By [**17**, eq. (45)], there are elements $m_1 \in \mathbb{D}_{\mathbb{X}_\infty}$ and $m_2 \in \mathbb{D}_{\omega_p\mathbb{X}_{\rm aff}}$ such that for all $\gamma \in \Gamma_1$ one has

$$\hat{\tilde{\mu}}_{{\rm H},\gamma} = U_p^2 \tilde{\mu}_\gamma + \gamma m_1 - m_1 \quad \text{on } \mathbb{X}_\infty\,, \qquad \hat{\tilde{\mu}}_{{\rm H},\gamma} = p\tilde{\mu}_\gamma + \gamma m_2 - m_2 \quad \text{on } p\mathbb{X}_{\rm aff}\,.$$

4 - Formulas for Darmon points on abelian varieties

4.1 - *Abelian varieties associated with newforms*

Consider a newform $f \in S_2(\Gamma_0(N))$ with $N := MDp$. Write $f(q) = \sum_{n \geq 1} a_n(f) q^n$ for the q-expansion of f, let $F := \mathbb{Q}(a_n(f) \mid n \geq 1)$ be the (totally real) number field generated by the Fourier coefficients of f and let $\mathcal{O}_F$ be the ring of integers of F. The

Fourier coefficients of f generate an order $\mathcal{O}_f$ in $\mathcal{O}_F$. Shimura's construction ([**33**, Theorem 7.14]) attaches to f an abelian variety $A = A_f$ over $\mathbb{Q}$ of dimension $d := [F : \mathbb{Q}]$. Explicitly, let $\lambda_f : \mathbb{T} \to \mathcal{O}_F$ be the algebra homomorphism such that $\lambda_f(T_\ell) = a_\ell(f)$ whenever ℓ is a prime not dividing N. Then A is the maximal abelian subvariety of J_1 killed by $\ker(\lambda_f)$; furthermore, A is $\mathbb{Q}$-simple because the cusp form f is new.

Since $p^2 \nmid N$, part iii) of [**16**, Theorem 3] implies that $a_p(f) = \pm 1$ and A has purely multiplicative reduction at p, i.e., the identity component of the special fibre of the Néron model of A over $\mathbb{Z}_p$ is a torus over $\mathbb{F}_p$. Equivalently, the reduction of A at p is semistable of toric dimension d. Since $J_1^{p\text{-new}}$ is the maximal toric quotient of J_1 at p, it follows that A is a quotient of $J_1^{p\text{-new}}$. Finally, by construction, the ring $\mathcal{O}_f$ embeds into the ring $\mathrm{End}_{\mathbb{Q}}(A)$ of endomorphisms of A defined over $\mathbb{Q}$. In fact, $F = \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$, so that A is an abelian variety of GL_2-type ([**27**, Corollary 4.2]).

Remark 4.1. Since, by assumption, N is square-free, the abelian variety $A_{/\mathbb{Q}}$ is semistable, hence all endomorphisms of A are defined over $\mathbb{Q}$ ([**25**, Corollary 1.4, (a)]). In particular, if $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ denotes the ring of *all* endomorphisms of A then $F = \mathrm{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q}$ and $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ is commutative.

Proposition 4.1 (Mattuck). *If $\mathcal{K}$ is a finite extension of $\mathbb{Q}_p$ then $A(\mathcal{K})$ is compact.*

Proof. Let $\mathcal{K}$ be a finite extension of $\mathbb{Q}_p$. By [**22**, Theorem 7], the group $A(\mathcal{K})$ contains a subgroup of finite index that is analytically isomorphic to the product $\mathcal{O}_{\mathcal{K}}^d$ where $\mathcal{O}_{\mathcal{K}}$ is the ring of integers of $\mathcal{K}$ and, as before, d is the dimension of A. Since $\mathcal{O}_{\mathcal{K}}$ is compact, it follows that $A(\mathcal{K})$ is compact. □

Remark 4.2. Proposition 4.1 holds for arbitrary (not necessarily modular) abelian varieties. Actually, the result of Mattuck that is used in the proof of Proposition 4.1 is a special case of a structure theorem of Serre for compact p-adic manifolds ([**29**, Part II, Ch. III, Appendix 2, Theorem 2]).

Now set $\mathbb{H}_A := H_1(A(\mathbb{C}), \mathbb{Z})$, choose a sign $\varepsilon \in \{\pm\}$ and write $\mathbb{H}_A^\varepsilon$ for the ε-eigenspace for complex conjugation acting on $\mathbb{H}_A$. Thus $\mathbb{H}_A^\varepsilon$ is a free abelian group of rank d. Since A is a quotient of $J_1^{p\text{-new}}$, it follows that $\mathbb{H}_A^\varepsilon$ is a quotient of H. The ring $\mathcal{O}_f$ acts by functoriality on $\mathbb{H}_A$ and, since F is totally real, on $\mathbb{H}_A^\varepsilon$ as well. For every field $\mathcal{K}$, this endows $\mathbb{H}_A^\varepsilon \otimes \mathcal{K}$ with a canonical $\mathcal{O}_f \otimes \mathcal{K}$-module structure.

Lemma 4.1. *If $\mathcal{K}$ is a field of characteristic* 0 *then* $\mathbb{H}_A^{\varepsilon} \otimes \mathcal{K}$ *is free of rank* 1 *over* $\mathcal{O}_f \otimes \mathcal{K}$.

Proof. As remarked above, $\mathbb{H}_A^{\varepsilon}$ is an $\mathcal{O}_f$-module that is free of rank d as an abelian group, and then $\mathbb{H}_A^{\varepsilon} \otimes \mathbb{Q}$ is free of rank 1 over $\mathcal{O}_f \otimes \mathbb{Q}$ because $\mathcal{O}_f \otimes \mathbb{Q} = F$ and $[F : \mathbb{Q}] = d$. The claim for a field $\mathcal{K}$ of characteristic 0 follows immediately. □

From here on we fix an isomorphism

$$\mathbb{H}_A \otimes \mathbb{Q}_p \simeq \mathcal{O}_f \otimes \mathbb{Q}_p, \tag{3}$$

which exists by Lemma 4.1. Of course, (3) induces an isomorphism $\mathbb{H}_A \otimes \mathcal{K} \simeq \mathcal{O}_f \otimes \mathcal{K}$ for every extension $\mathcal{K}$ of $\mathbb{Q}_p$.

4.2 - *Darmon points on A*

Let $\underline{q}_1, \dots, \underline{q}_d \in (\mathbb{Q}_p^{\times})^d$ denote the Tate periods for A at p and write

$$\Phi_{\mathrm{Tate},A} : (\mathbb{C}_p^{\times})^d / \langle \underline{q}_1, \dots, \underline{q}_d \rangle \xrightarrow{\simeq} A(\mathbb{C}_p) \tag{4}$$

for Tate's analytic uniformization of A at p, which is defined over K_p (see §A.1 and §A.3).

Remark 4.3. With a slight abuse of notation, we shall sometimes write $\Phi_{\mathrm{Tate},A}$ for the map on $(\mathbb{C}_p^{\times})^d$ obtained by pre-composing (4) with the projection onto the quotient. Furthermore, we embed $\mathbb{C}_p^{\times}$ diagonally into $(\mathbb{C}_p^{\times})^d$, so that we can evaluate $\Phi_{\mathrm{Tate},A}$ at points of $\mathbb{C}_p^{\times}$.

Set $T_A^{\varepsilon} := T_{\mathbb{H}_A^{\varepsilon}}$ and recall the Hecke-stable lattice $L_A^{\varepsilon} := L_{\mathbb{H}_A^{\varepsilon}}$ inside $T_A^{\varepsilon}(\mathbb{Q}_p) \simeq (\mathbb{Q}_p^{\times})^d$. By Theorem A.1, the lattices L_A^{ε} and $\langle \underline{q}_1, \dots, \underline{q}_d \rangle$ of $(K_p^{\times})^d$ are commensurable; set $n := \left[L_A^{\varepsilon} : L_A^{\varepsilon} \cap \langle \underline{q}_1, \dots, \underline{q}_d \rangle\right]$. Raising to the n-th power and composing with $\Phi_{\mathrm{Tate},A}$ yields a Galois-equivariant isogeny

$$\varphi_A^{\varepsilon} : T_A^{\varepsilon}(\mathbb{C}_p)/L_A^{\varepsilon} \longrightarrow A(\mathbb{C}_p) \tag{5}$$

defined over K_p (see §A.3). In fact, the proof of [**17**, Theorem 1.1] shows that φ_A^{ε} can be chosen to be equivariant with respect to the actions of the relevant local Galois group (see [**17**, §7.1]). Similarly, we fix a Galois-equivariant isogeny

$$\hat{\varphi}_A^{\varepsilon} : A(\mathbb{C}_p) \longrightarrow T_A^{\varepsilon}(\mathbb{C}_p)/L_A^{\varepsilon}.$$

Finally, for every $\psi \in \mathrm{Emb}(\mathcal{O}_c, R_0)$ we define

$$P_{A,\psi}^{\varepsilon} := \varphi_A^{\varepsilon}\left(\mathcal{P}_{\mathbb{H}_A^{\varepsilon},\psi}\right) \in A(K_p).$$

These are the *Darmon points of conductor c* on A. Since the sign $\varepsilon \in \{\pm\}$ is regarded as fixed, from here until the end of this section we drop all superscripts "ε" from our notation (i.e., we write L_A for L_A^ε, φ_A for φ_A^ε, $\hat{\varphi}_A$ for $\hat{\varphi}_A^\varepsilon$, $P_{A,\psi}$ for $P_{A,\psi}^\varepsilon$ and $\mathbb{H}_A$ for $\mathbb{H}_A^\varepsilon$). By Proposition 3.1, $P_{A,\psi} = P_{A,\psi'}$ whenever ψ and ψ' are Γ_0-conjugate. In the next subsection we will give an integral expression for a suitable logarithm of the Darmon points $P_{A,\psi}$ on A.

4.3 - *Logarithms and $\mathcal{L}$-invariants*

Let $\mathbb{C}_p$ be the completion of $\bar{\mathbb{Q}}_p$ and let G be a finite-dimensional commutative Lie group over $\mathbb{C}_p$ (see [**4**, Ch. III, §1]). The Lie algebra Lie(G) of G is the tangent space of G at the identity and is a $\mathbb{C}_p$-vector space of dimension dim(G). Let G_f be the smallest open subgroup of G such that the quotient G/G_f is torsion-free. As explained in [**4**, Ch. III, §7.6], there is a canonical analytic homomorphism

$$\log : G_f \longrightarrow \mathrm{Lie}(G).$$

The map log is a local diffeomorphism and its kernel is the torsion subgroup of G_f.

Proposition 4.2. *If $\mathcal{K}$ is a complete subfield of $\mathbb{C}_p$ then* Lie($A(\mathcal{K})$) *is a free $\mathcal{O}_f \otimes \mathcal{K}$-module of rank* 1.

Proof. Denote by Lie(A) the Lie algebra of A viewed as an algebraic group over $\mathbb{Q}$, so that Lie(A) is a d-dimensional vector space over $\mathbb{Q}$. The ring $\mathcal{O}_f$ embeds into $\mathrm{End}_{\mathbb{Q}}(A)$, hence it acts by functoriality on Lie(A). This endows Lie(A) with an $\mathcal{O}_f \otimes \mathbb{Q}$-module structure, and then Lie(A) is free of rank 1 over $\mathcal{O}_f \otimes \mathbb{Q}$ because $\mathcal{O}_f \otimes \mathbb{Q} = F$ and $[F : \mathbb{Q}] = d$. By base change, we deduce that

$$\mathrm{Lie}(A_{/\mathcal{L}}) = \mathrm{Lie}(A) \otimes_{\mathbb{Q}} \mathcal{L} \simeq \mathcal{O}_f \otimes \mathcal{L} \tag{6}$$

for every field $\mathcal{L}$ of characteristic 0. Now let $\mathcal{K}$ be a complete subfield of $\mathbb{C}_p$. As remarked in [**34**, p. 2744], the Lie algebra of the $\mathcal{K}$-analytic Lie group $A(\mathcal{K})$ coincides with Lie($A_{/\mathcal{K}}$), and the proposition follows from (6). □

From now on we fix an isomorphism

$$\mathrm{Lie}(A(\mathbb{Q}_p)) \simeq \mathcal{O}_f \otimes \mathbb{Q}_p, \tag{7}$$

which in turn induces isomorphisms Lie($A(\mathcal{K})$) $\simeq \mathcal{O}_f \otimes \mathcal{K}$ for all $\mathcal{K}$ as in Proposition 4.2. In fact, we shall often implicitly view (7) as an identification.

In the special case where $G = A(\mathbb{C}_p)$, if U is an open subgroup of $A(\mathbb{C}_p)$ then $A(\mathbb{C}_p)/U$ is torsion ([**6**, Theorem 4.1]), hence $G_f = G$ and we get a surjective logarithm

$$\log_A : A(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A(\mathbb{C}_p)) \simeq \mathcal{O}_f \otimes \mathbb{C}_p \tag{8}$$

that is a $\mathbb{C}_p$-analytic homomorphism. The kernel of $\log_A$ is the torsion subgroup of $A(\mathbb{C}_p)$. As noted in [**34**, §1], if $\mathcal{K}$ is a complete subfield of $\mathbb{C}_p$ then $\log_A(A(\mathcal{K})) = \mathrm{Lie}(A(\mathcal{K}))$, hence (7) gives an isomorphism

$$\log_A : A(\mathcal{K}) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathcal{O}_f \otimes \mathcal{K} \tag{9}$$

that turns $A(\mathcal{K}) \otimes \mathbb{Q}$ into a free $\mathcal{O}_f \otimes \mathcal{K}$-module of rank 1.

Remark 4.4. It is known that $\log_A$ essentially coincides with the logarithm map introduced by Bloch and Kato in their paper [**3**] on the Tamagawa number conjecture for motives (cf. [**3** Example 3.11]).

Let $\mathcal{L}_p^D$ be the $\mathcal{L}$-invariant introduced in [**17**, Definition 3.2]. Since $\mathbb{H}_A$ is Hecke-stable, we can consider the endomorphism $\mathcal{L}_A$ of $\mathbb{H}_A$ induced by $\mathcal{L}_p^D$. As a consequence of [**17**, Theorem 7.1], we know that the map

$$\mathrm{id}_{\mathbb{H}_A} \otimes \log_p - \mathcal{L}_A \otimes \mathrm{ord}_p : \mathbb{H}_A \otimes \mathbb{C}_p^\times \longrightarrow \mathbb{H}_A \otimes \mathbb{C}_p \simeq \mathcal{O}_f \otimes \mathbb{C}_p\,, \tag{10}$$

where $\log_p$ is the branch of the p-adic logarithm such that $\log_p(p) = 0$ and ord_p is the p-adic valuation, vanishes identically on L_A. Hence (10) induces a $\mathbb{C}_p$-analytic homomorphism

$$\log_{\mathcal{L}_A} : A(\mathbb{C}_p) \xrightarrow{\hat{\varphi}_A} T_A(\mathbb{C}_p)/L_A \longrightarrow \mathcal{O}_f \otimes \mathbb{C}_p$$

that we may call the *$\mathcal{L}$-invariant logarithm* (or, better, the *$\mathcal{L}$-invariant logarithm with sign ε*) of A. In fact, since the logarithm $\log_A$ on $A(\mathbb{C}_p)$ is uniquely determined by the property that its tangent map is the identity ([**4**, Ch. III, §7.6]), a computation shows that there is $\lambda \in (\mathcal{O}_f \otimes \mathbb{C}_p)^\times$ such that $\log_{\mathcal{L}_A} = \lambda \cdot \log_A$. We shall not use this formula in what follows; rather, we shall implicitly view it as a justification for extending the arguments of [**17**] and [**21**] in the way that we describe below.

Composing $\log_A$ with the map φ_A introduced in (5) gives an analytic homomorphism

$$\Psi_A := \log_A \circ \varphi_A : T_A(\mathbb{C}_p)/L_A \longrightarrow \mathcal{O}_f \otimes \mathbb{C}_p$$

that is $\mathcal{O}_f \otimes K_p$-valued on $T_A(K_p)/L_A$. By pre-composing it with the projection onto the quotient (respectively, with the diagonal embedding and the projection onto the quotient) we may also view Ψ_A as defined on $T_A(\mathbb{C}_p)$ (respectively, on $\mathbb{C}_p^\times$).

4.4 - *A formula for Darmon points on A*

Our goal is to describe an integral formula for Darmon points on A (or, rather, for their p-adic logarithm). With notation as in §3.3, we set $\tilde{\mu}_A := \tilde{\mu}_{\mathbb{H}_A}$. Now let $\psi \in \mathrm{Emb}(\mathcal{O}_c, R_0)$ and recall the Darmon point $P_{A,\psi} \in A(K_p)$ defined in §4.2.

The first main result of this article is

Theorem 4.1. $\log_A(P_{A,\psi}) = -t \cdot \int_{\mathbb{X}} \Psi_A(x - z_\psi y) d\tilde{\mu}_{A,\gamma_\psi}(x,y).$

As in §3.1, the integer t is the exponent of Γ^{ab}. Note that the integral on the right does indeed make sense, as the function $\Psi_A|_{K_p^\times}$ is bounded thanks to the compactness of $A(K_p)$ ensured by Proposition 4.1, the measure $\tilde{\mu}_{A,\gamma_\psi}$ takes values in $\mathbb{H}_A \otimes \mathbb{Z}_p \subset \mathbb{H}_A \otimes K_p$, the function $\Psi_A|_{K_p^\times}$ takes values in $\mathcal{O}_f \otimes K_p$ and $\mathbb{H}_A \otimes K_p \simeq \mathcal{O}_f \otimes K_p$ by (3).

Proof. First of all, by definition of the point $P_{A,\psi}$, there is an equality

$$\log_A(P_{A,\psi}) = \Psi_A(\mathcal{P}_{\mathbb{H}_A,\psi}).$$

Moreover, $x - z_\psi y \in \mathcal{O}_{K_p}^\times$ when $(x,y) \in \mathbb{X}$; here $\mathcal{O}_{K_p}$ is the ring of integers of K_p. Define the 1-cochain ρ on Γ_0 with values in $T_A(K_p)$ by the formula $\rho_\gamma := -\int_{\mathbb{X}} \Psi_A(x - z_\psi y) d\tilde{\mu}_{A,\gamma}$. Keeping the notation of §3.3, define also the 1-cochain $\hat{\rho}$ on $\hat{\Gamma}_0$ with values in $T_A(K_p)$ by

$$\hat{\rho}_\gamma := -\int_{\mathbb{X}} \Psi_A(x - z_\psi y) d\hat{\tilde{\mu}}_{A,\gamma} + \int_{\mathbb{X}_\infty} \Psi_A(x - z_\psi y) d(\gamma m_1 - m_1) + \int_{p\mathbb{X}_{\mathrm{aff}}} \Psi_A(x - z_\psi y) d(\gamma m_2 - m_2).$$

Following *verbatim* the proofs of [**17**, Propositions 7.10 and 7.13] after replacing the branch $\log_p$ of the p-adic logarithm with Ψ_A, we see that ρ and $\hat{\rho}$ split the restrictions of $\Psi_A(d)$ to Γ_0 and $\hat{\Gamma}_0$, respectively. Thus $\Psi_A(\boldsymbol{d}_z) = \Delta(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}})$ where Δ is the connecting map in the Mayer–Vietoris exact sequence and $\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}$ is the class of $\rho - \hat{\rho}$ in $H^1(\Gamma_1, T_A(K_p))$. We know that $\Psi_A(\bar{d}_z)$ is split by $\Psi_A(\beta_z)$, so $\delta(\Psi_A(\beta_z)) = \Delta(\rho - \hat{\rho})$ where δ is the connecting map on cochains. Since $\Gamma = \Gamma_0 *_{\Gamma_1} \hat{\Gamma}_0$ and t annihilates Γ^{ab}, we conclude by applying the group-theoretic result of [**21**, Lemma 3.4]. □

At this point it is important to observe that the validity of the formula in Theorem 4.1 is independent of the isomorphisms fixed in (7) and (3). In fact, any two isomorphisms of the form (7) or (3) differ by multiplication by an element of $(\mathcal{O}_f \otimes K_p)^\times$.

Remark 4.5. It would be desirable to perform our computations directly via a pairing of the form

$$\mathrm{Lie}(A) \otimes_{\mathrm{End}_{\mathbb{Q}}(A)} \mathbb{H}_A \longrightarrow \mathrm{Lie}(A),$$

without needing to fix isomorphisms $\mathrm{Lie}(A(K_p)) \simeq \mathcal{O}_f \otimes K_p$ and $\mathbb{H}_A \otimes K_p \simeq \mathcal{O}_f \otimes K_p$.

Now let α be an endomorphism of $A_{/\mathbb{C}_p}$ and let $\alpha_* : \mathrm{Lie}(A(\mathbb{C}_p)) \to \mathrm{Lie}(A(\mathbb{C}_p))$ be the linear map induced by α. By [**34**, §1], the maps $\log_A$ and α commute, in the sense that

$$\log_A \circ \alpha = \alpha_* \circ \log_A. \tag{11}$$

As a consequence of Theorem 4.1, we obtain

Corollary 4.1. $\log_A\big(\alpha(P_{A,\psi})\big) = -t \cdot \alpha_*\left(\int_{\mathbb{X}} \Psi_A(x - z_\psi y) d\tilde{\mu}_{A,\gamma_\psi}(x,y)\right).$

Proof. Apply α_* to both sides of the formula in Theorem 4.1 and use equality (11). □

While the validity of Corollary 4.1 is unconditional, now we describe a refinement that depends (at least in part) on a technical assumption. By [**4**, Ch. III, §7, Proposition 10], there exists an open subgroup V of $A(\mathbb{C}_p)$ such that $\log_A(V)$ is open in $\mathrm{Lie}(A(\mathbb{C}_p))$ and $\log_A|_V$ is the right inverse of the restriction of the exponential map

$$\exp_A : \mathrm{Lie}(A(\mathbb{C}_p)) \longrightarrow A(\mathbb{C}_p)$$

to the open subgroup $\log_A(V)$. Furthermore, $\exp_A\big(\mathrm{Lie}(A(\mathcal{K}))\big) \subset A(\mathcal{K})$ for every complete subfield $\mathcal{K}$ of $\mathbb{C}_p$.

Corollary 4.2. *Let $V \subset A(\mathbb{C}_p)$ and α be as above.*

1. *If $\alpha(P_{A,\psi}) \in V$ then*

$$\alpha(P_{A,\psi}) = \exp_A\left(-t \cdot \alpha_*\left(\int_{\mathbb{X}} \Psi_A(x - z_\psi y) d\tilde{\mu}_{A,\gamma_\psi}(x,y)\right)\right).$$

2. *There exists an integer $m \geq 1$ such that*

$$mP_{A,\psi} = \exp_A\left(-mt \cdot \int_{\mathbb{X}} \Psi_A(x - z_\psi y) d\tilde{\mu}_{A,\gamma_\psi}(x,y)\right)$$

for every $\psi \in \mathrm{Emb}(\mathcal{O}_c, R_0)$.

Proof. To prove part (1) apply $\exp_A$ to both sides of the equality in Proposition 4.1. As for part (2), observe that $V' := A(K_p) \cap V$ is an open subgroup of $A(K_p)$. On the other hand, by Proposition 4.1, the group $A(K_p)$ is compact, so V' has finite index, say m, in $A(K_p)$. But $P_{A,\psi}$ lies in $A(K_p)$, hence $mP_{A,\psi}$ belongs to $V' \subset V$. The desired formula follows from part (1) by taking as α the multiplication-by-m map on $A(\mathbb{C}_p)$. $\square$

In other words, part (1) of Corollary 4.2 says that if $\alpha(P_{A,\psi})$ is "sufficiently close" to the identity of A then we can exhibit a formula for $\alpha(P_{A,\psi})$ itself.

Remark 4.6. It is worth stressing that the integer m appearing in part (2) of Corollary 4.2 depends exclusively on the Lie structure of the group $A(\mathbb{C}_p)$ of p-adic points of A, and not on the Darmon point $P_{A,\psi}$. It would be very interesting to calculate m, so as to be able to compute an explicit multiple of $P_{A,\psi}$ for any $\psi \in \mathrm{Emb}(\mathcal{O}_c, R_0)$.

Setting $\boldsymbol{Q} := \langle \underline{q}_1, \dots, \underline{q}_d \rangle$, it is convenient to introduce the homomorphism

$$\log_{\boldsymbol{Q}} : (\mathbb{C}_p^\times)^d/\boldsymbol{Q} \xrightarrow{\Phi_{\mathrm{Tate},A}} A(\mathbb{C}_p) \xrightarrow{\log_A} \mathrm{Lie}(A(\mathbb{C}_p)) \simeq \mathcal{O}_F \otimes \mathbb{C}_p, \tag{12}$$

where $\log_A$ is the logarithm on the p-adic points of A introduced in (8) and the isomorphism on the right is a consequence of (7). Recall from §4.2 that $n := \left[L_A : L_A \cap \langle \underline{q}_1, \dots, \underline{q}_d \rangle\right]$. One has the formula

$$\log_A\left(\varphi_A^{\pm}(x)\right) = n \log_{\boldsymbol{Q}}(x) \qquad \text{for all } x \in T_A^{\pm}(\bar{\mathbb{Q}}_p)/L_A^{\pm}. \tag{13}$$

We will also denote by $\log_{\boldsymbol{Q}}$ the map on $(\mathbb{C}_p^\times)^d$ obtained by pre-composing $\log_{\boldsymbol{Q}}$ with the canonical projection. Finally, we evaluate the function $\log_{\boldsymbol{Q}}$ at points of $\bar{\mathbb{Q}}_p^\times$ by embedding $\bar{\mathbb{Q}}_p^\times$ diagonally into $(\mathbb{C}_p^\times)^d$ and projecting onto $(\mathbb{C}_p^\times)^d/\boldsymbol{Q}$.

Corollary 4.3. $\log_A(P_{A,\psi}^{\pm}) = -nt \cdot \int_{\mathbb{X}} \log_{\boldsymbol{Q}}(x - z_\psi y) d\tilde{\mu}_{A,\gamma_\psi}^{\pm}(x, y).$

Proof. Immediate from Theorem 4.1 and formula (13). $\square$

In the rest of the paper, set

$$m := \left[A(K_p) : V'\right] \tag{14}$$

where $V' = A(K_p) \cap V$ is the subgroup of $A(K_p)$ defined in the proof of Corollary 4.2.

In order to interpret the statement of the following lemma, one should keep Remark 4.3 in mind.

Lemma 4.2. *If* $x \in (K_p^\times)^d$ *then* $x^m \in \Phi_{\mathrm{Tate},A}^{-1}(V)$.

Proof. Since the map $\Phi_{\mathrm{Tate},A}$ is Galois-equivariant and defined over K_p, if $x \in (K_p^\times)^d$ then $\Phi_{\mathrm{Tate},A}(x) \in A(K_p)$. On the other hand, one has

$$\Phi_{\mathrm{Tate},A}(x^m) = m\Phi_{\mathrm{Tate},A}(x) \in mA(K_p) \subset V',$$

and we are done because V' is a subset of V. □

5 - Hida families and rational points on abelian varieties

The goal of this section is to extend the results of [**1**] to the case of a newform of weight 2 whose Fourier coefficients are not necessarily rational integers. As most of the arguments of [**1**] carry over *mutatis mutandis* to our more general setting (as was already anticipated in [**1**, Remark 5]), we will content ourselves with highlighting the main novelties occurring here.

5.1 - *Hida's formal q-expansion*

Embed $\mathbb{Z}$ into the weight space

$$\mathcal{X} := \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times (1 + p\mathbb{Z}_p) \tag{15}$$

by sending $k \in \mathbb{Z}$ to the map $\left(x \mapsto x^{k-2}\right)$. Hida theory associates with f a neighbourhood $\mathbb{U}_f$ of 2 in $\mathcal{X}$ and a formal q-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n(\kappa) q^n \tag{16}$$

where $a_1 = 1$ and a_n is a rigid analytic function on $\mathbb{U}_f$ such that $\sum_{n=1}^{\infty} a_n(k) q^n$ for an even $k \in \mathbb{U}_f \cap \mathbb{Z}_{\geq 2}$ is the q-expansion of a p-stabilized (in the sense of [**11**, Definition 2.5]) weight k eigenform f_k on $\Gamma_0(N)$ and $f_2 = f$. In fact, we choose $\mathbb{U}_f$ small enough so that the results of [**21**, §3.3], which are based on a suitable "control theorem" ([**20**], [**21**, Theorem 3.1]), hold. We explicitly remark that none of the computations with Hida families and L-functions of cusp forms performed in [**21**] (in particular, those of a cohomological nature contained in [**21**, §4.2] and the elaboration on Popa's work [**24**] described in [**21**, §4.3]) uses the assumption made in *loc. cit.* (and in [**1**]) that the Fourier coefficients of f are rational. This means that these results apply word for word to our present setting, so we shall not duplicate the arguments here but rather refer (especially in Section 6 below) to the relevant sections of [**21**] for details.

5.2 - *Mazur–Kitagawa p-adic L-functions*

As in [**1**, Definition 1.11], to a primitive quadratic Dirichlet character χ and the formal q-expansion f_∞ in (16) one can attach the Mazur–Kitagawa two-variable p-adic L-function $L_p(f_\infty, \chi, k, s)$ in which the pair (k, s) varies in $\mathbb{U}_f \times \mathcal{X}$. This function is defined in terms of the measure-valued modular symbols that were studied by Greenberg and Stevens in [**11**, Section 6] and satisfies an interpolation property with respect to the special values of the classical L-functions $L(f_k, \chi, s)$ that is described in [**1**, Theorem 1.12].

5.3 - *Heegner points*

Suppose that we can write N in the form $N = pN^+N^-$ where p is, as before, a prime, N^- is the square-free product of an *odd* number of primes and the integers p, N^+ and N^- are pairwise coprime. Denote by X the Shimura curve over $\mathbb{Q}$ of discriminant N^-p and level N^+ and let J denote its Jacobian variety. Fix a (surjective) modular parametrization

$$\pi_A : J \longrightarrow A$$

defined over $\mathbb{Q}$, whose existence is ensured by the Jacquet–Langlands correspondence and Faltings's isogeny theorem. Now let K' be an imaginary quadratic field that is admissible in the sense of [**1**, Definition 3.1], let $H_{K'}$ denote its Hilbert class field and write $G_{K'} := \mathrm{Gal}(H_{K'}/K')$ for the corresponding Galois group over K'. Finally, let χ be a quadratic character of $G_{K'}$ and let $H_\chi = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be the biquadratic (respectively, quadratic) field cut out by χ if χ is non-trivial (respectively, trivial). Let χ_1 and χ_2 denote the quadratic Dirichlet characters associated with χ; order them in such a way that $\chi_1(-N) = w_N$ and $\chi_2(-N) = -w_N$ where $-w_N$ is the sign in the functional equation for $L(f, s)$.

Proceeding as in [**1**, §4.3], one can introduce a Heegner divisor $\tilde{P}_\chi$ of degree 0 on X and define $P_\chi := \pi_A(\tilde{P}_\chi) \in A$. It turns out that P_χ belongs to $A(H_\chi)^\chi$ (respectively, $A(K') \otimes \mathbb{Q}$) if $\chi \neq 1$ (respectively, $\chi = 1$).

Theorem 5.1. *The point P_χ is non-torsion if and only if $L'(f/K', \chi, 1) \neq 0$; in this case, the group $A(H_\chi)^\chi$ has rank one. Furthermore, the image of P_χ in $A(H_\chi) \otimes \mathbb{Q}$ belongs to $\big(A(\mathbb{Q}(\sqrt{d_1})) \otimes \mathbb{Q}\big)^{\chi_1}$.*

Proof. The first assertion is a consequence of Zhang's formula of Gross–Zagier type for Shimura curve parametrizations ([**36**, Theorem 1.2.1]), while the statement about the rank follows from the methods of Kolyvagin, as extended by

Kolyvagin and Logachëv to the case of modular abelian varieties ([**14**], [**15**]; cf. also [**35**, Theorem A]). Finally, the last claim can be proved as [**1**, Theorem 4.7]. □

Let $L_p(f_\infty/K', \chi, k)$ be the analogue of the p-adic L-function defined in [**1**, Definition 3.5]. As in [**1**, §4.4], one has the formula

$$\frac{d^2}{dk^2} L_p(f_\infty/K', \chi, k)_{k=2} = \begin{cases} 2\log_A(P_\chi)^2 & \text{if } \chi_1(p) = a_p(f), \\ 0 & \text{if } \chi_1(p) = -a_p(f), \end{cases} \tag{17}$$

which is the counterpart of [**1**, Corollary 4.10] and is used in the proof of Theorem 5.2. Thanks to isomorphism (3), the square on the right hand side of (17) is to be understood as a product in $\mathcal{O}_f \otimes \mathbb{Q}_p$.

5.4 - *Factorization of p-adic L-functions*

We are interested in the following factorization result, which can be proved by means of the arguments described in [**1**, §5.1].

Proposition 5.1. *For all $k \in \mathbb{U}_f$ there is a factorization*

$$L_p(f_\infty/K', \chi, k) = \eta(k) L_p(f_\infty, \chi_1, k, k/2) L_p(f_\infty, \chi_2, k, k/2),$$

where η is a p-adic analytic function on $\mathbb{U}_f$ such that $\eta(2) \in F^\times$.

Remark 5.1. The Jacquet–Langlands (JL, for short) correspondence establishes Hecke-equivariant isomorphisms between classical and quaternionic modular forms (see, e.g., [**1**, Theorem 2.4]). In fact, the JL correspondence between modular forms is a consequence of a canonical isomorphism between the corresponding Hecke algebras (see, e.g., [**13**, Section 2.4] and [**19**, §6.3]). Let ϕ_2 be a quaternionic form associated with f via the JL correspondence. As in [**1**, p. 395], one can use the Bruhat–Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$ to define the Petersson norm $\langle \phi_2, \phi_2 \rangle \in F^\times$. With notation as in Proposition 5.1 one has $\eta(2) = \langle \phi_2, \phi_2 \rangle$.

5.5 - *Hida families and rational points*

The main result of this section is the following theorem, which extends [**1**, Theorem 5.4] to the case of modular abelian varieties.

Theorem 5.2. *Suppose that there are at least two distinct prime numbers of semistable reduction for A. Let χ_1 be a Dirichlet character of conductor prime to N* *such that* $\chi_1(-N) = w_N$ *and* $\chi_1(p) = a_p(f)$*. Then*

1. *the p-adic L-function* $L_p(f_\infty, \chi_1, k, k/2)$ *vanishes to order at least* 2 *at* $k = 2$*;*
2. *there exist a global point* $\mathbf{P}_{\chi_1} \in \big(A(\mathbb{Q}(\sqrt{d_1})) \otimes \mathbb{Q}\big)^{\chi_1}$ *and a number* $s \in F^\times$ *such that*
$$\frac{d^2}{dk^2} L_p(f_\infty, \chi_1, k, k/2)_{k=2} = s \log_A(\mathbf{P}_{\chi_1})^2;$$
3. *the point* $\mathbf{P}_{\chi_1}$ *is of infinite order if and only if* $L'(f/\mathbb{Q}, \chi_1, 1) \neq 0$*;*
4. *the image of* s *in* $F^\times/(F^\times)^2$ *is equal to that of* $L^*(f, \psi, 1)$*, where* ψ *is any quadratic Dirichlet character satisfying*
   (a) $\psi(\ell) = \chi_1(\ell)$ *for all primes* ℓ *dividing* $N' = N/p$*;*
   (b) $\psi(p) = -\chi_1(p)$*;*
   (c) $L(f, \psi, 1) \neq 0$.

Proof. The main ingredients needed here are provided by Theorem 5.1, formulas (17) and Proposition 5.1 and then one can argue as in the proof of [**1**, Theorem 5.4]. □

6 - Rationality results for Darmon points over genus fields

In this final section we extend the rationality result of [**21**] to the case of higher-dimensional (modular) abelian varieties. In order to avoid indulging in arguments that have already been expounded elsewhere, we freely refer to definitions and computations in [**21**] whenever they carry over to our more general setting without significant change; the reader is advised to keep copies of [**1**] and [**21**] close at hand.

6.1 - *p-adic L-functions over real quadratic fields*

Let $\mathbb{U}_f$ be the p-adic neighbourhood of 2 in $\mathcal{X}$ introduced in [**21**, Proposition 3.2]. Let $\langle x \rangle$ denote the principal unit attached to $x \in \mathbb{Q}_p^\times$, which is defined as the unique element of $1 + p\mathbb{Z}_p$ such that $x = p^{\mathrm{ord}_p(x)}\zeta_x\langle x \rangle$ where ζ_x is a $(p-1)$-th root of unity. Let $\psi : K \hookrightarrow B$ be an optimal embedding of $\mathcal{O}_K$ into R_0 and let Q_ψ be the quadratic form defined in §3.2. Let $m \geq 1$ be the integer defined in (14), then for all $k \in \mathbb{U}_f$ and all pairs $(x, y) \in \mathbb{X}$ set

$$Q_\psi^{(k)}(x, y) := \exp_A\left(\frac{k-2}{2}\log_{\boldsymbol{Q}}\big(\langle Q_\psi(x, y)\rangle^m\big)\right) \in A(K_p) \otimes \mathbb{Q} \simeq \mathcal{O}_f \otimes K_p, \tag{18}$$

where $\log_{\boldsymbol{Q}}$ is the map defined in (12). Observe that $Q_\psi^{(k)}(x, y)$ lies in $A(K_p) \otimes \mathbb{Q}$

(and not just in $A(\mathbb{C}_p)\otimes\mathbb{Q}$) because the argument of $\exp_A$ belongs to Lie($A(K_p)$). In fact, $Q_\psi^{(k)}$ is naturally $A(K_p)$-valued: here we are implicitly composing the exponential $\exp_A$ with the natural (localization) map $A(K_p)\to A(K_p)\otimes\mathbb{Q}$. This shows, by [**22**, Theorem 7], that $Q_\psi^{(k)}$ is p-adically bounded.

Remark 6.1. Recall from (15) that the weight space $\mathcal{X}$ is the disjoint union of $p-1$ copies of the disc in $\mathbb{Q}_p$ of centre 1 and radius 1 and that $\mathbb{U}_f$ is an open neighbourhood of 2 in $\mathcal{X}$. In light of this, for any fixed pair $(x,y)\in\mathbb{X}$ the map sending $k\in\mathbb{U}_f$ to $Q_\psi^{(k)}(x,y)\in A(K_p)$ may be interpreted, from a Lie-theoretic point of view, as a suitable restriction of a one-parameter subgroup of the p-adic Lie group $A(K_p)$.

In the following definition, the measure $\tilde{\mu}^\pm_{A,\gamma_\psi}$ is as in §4.3.

Definition 6.1. The *partial square root p-adic L-function* attached to f_∞, K and ψ is

$$\mathcal{L}_p^\pm(f_\infty,\psi,k):=\int_{\mathbb{X}} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{A,\gamma_\psi}(x,y)$$

where k varies in $\mathbb{U}_f$.

When $k\in\mathbb{U}_f$ is kept fixed, in Definition 6.1 we are integrating the function $Q_\psi^{(k)}$ defined on $\mathbb{X}$ with values in $A(K_p)\otimes\mathbb{Q}$ against the $\mathbb{H}_A\otimes\mathbb{Z}_p$-valued measure $\tilde{\mu}^\pm_{A,\gamma_\psi}$. This does indeed make sense, as we fixed in (3) (respectively, in (9)) an isomorphism $\mathbb{H}_A\otimes\mathbb{Q}_p\simeq\mathcal{O}_f\otimes\mathbb{Q}_p$ (respectively, $A(K_p)\otimes\mathbb{Q}\simeq\mathcal{O}_f\otimes K_p$). Geometrically, what we are doing here is to exploit the natural action of $\mathrm{End}_{\mathbb{Q}}(A)$ on the homology $\mathbb{H}_A$. In other words, Definition 6.1 gives a function

$$\mathcal{L}_p^\pm(f_\infty,\psi,-):\mathbb{U}_f\longrightarrow\mathcal{O}_f\otimes K_p.$$

As in [**21**, Proposition 4.16], it can be checked that $\mathcal{L}_p^\pm(f_\infty,\psi,k)$ is independent of the choice of a representative $\tilde{\mu}^\pm_A$ of $\tilde{\boldsymbol{\mu}}^\pm_A$ and only depends on the Γ_0-conjugacy class of ψ. Analogous considerations apply to the L-functions of Definition 6.2 below.

Remark 6.2. By Lemma 4.2, the power $\langle Q_\psi(x,y)\rangle^m$ lies in $\Phi^{-1}_{\mathrm{Tate},A}(V)$ for all $(x,y)\in\mathbb{X}$. Therefore if $k\geq 2$ is an integer in $\mathbb{U}_f$ then $Q_\psi^{(k)}(x,y)$ is the $m(k-2)/2$-fold self product of $Q_\psi(x,y)$. This justifies the restriction of $\mathbb{U}_f$ to the residue class of 2 modulo $p-1$ in $\mathcal{X}$, and also explains the appearance of the exponent m in (18).

Let $H_K^+=H_1^+$ denote the narrow Hilbert class field of K and write $G_K^+:=$

$\mathrm{Gal}(H_K^+/K)$ for its Galois group over K. Let $\mathrm{Pic}^+(\mathcal{O}_K)$ denote the narrow class group of K, so that $\mathrm{Pic}^+(\mathcal{O}_K) \simeq G_K^+$ via the reciprocity map of global class field theory. As in [**21**, §2.5], fix a set $\{\psi_\sigma\}_{\sigma\in G_K^+}$ of representatives of the Γ_0-conjugacy classes of oriented optimal embeddings of $\mathcal{O}_K$ into R_0 (for the notion of *oriented embedding* see, e.g., [**21**, Definition 2.4]).

Definition 6.2. Let χ be a quadratic character of G_K^+ of sign $-\varepsilon$ and let k vary in $\mathbb{U}_f$.

1. The *square root p-adic L-function* attached to f_∞ and χ is

$$\mathcal{L}_p(f_\infty/K,\chi,k) := \sum_{\sigma\in G_K^+} \chi(\sigma)\mathcal{L}_p^\varepsilon(f_\infty,\psi_\sigma,k).$$

2. The *p-adic L-function* attached to f_∞ and χ is

$$L_p(f_\infty/K,\chi,k) := \mathcal{L}_p(f_\infty/K,\chi,k)^2.$$

Of course, the square in part (2) of Definition 6.2 is to be interpreted as a square in $\mathcal{O}_f \otimes K_p$. Note that

$$\mathcal{L}_p^\pm(f_\infty/K,\psi,2) = \int_{\mathbb{X}} d\tilde{\mu}^\pm_{A,\gamma_\psi} = \int_{\mathbb{P}^1(\mathbb{Q}_p)} d\pi_*\left(\mu^\pm_{A,\gamma_\psi}\right) = 0$$

for all $\psi \in \mathrm{Emb}(\mathcal{O}_K, R_0)$, hence

$$\mathcal{L}_p(f_\infty/K,\chi,2) = \sum_{\sigma\in G_K^+} \chi(\sigma)\mathcal{L}_p^\varepsilon(f_\infty,\psi_\sigma,k) = 0. \tag{19}$$

As in [**21**], we shall also have use for p-adic L-functions attached to real quadratic fields in which p splits (such functions play a crucial role, for instance, in the proof of Theorem 6.2 below). In this case the definitions are somewhat more involved, and we refer the reader to [**21**, §4.5] for all details and results that are needed here. In particular, the interpolation formulas given in [**21**, Theorem 4.19] and [**21**, Theorem 4.25] apply without change.

6.2 - *Review of genus characters*

A *genus character* of K is an unramified quadratic character of G_K^+. Let $\sigma_K \in G_K^+$ denote the image of the class in $\mathrm{Pic}^+(\mathcal{O}_K)$ of the ideal $(\sqrt{d_K})$ via the reciprocity map; the *sign* of a genus character χ is $\chi(\sigma_K) \in \{\pm 1\}$. A genus character χ of K cuts out the genus field H_χ of K given by

$$H_\chi = \mathbb{Q}\big(\sqrt{d_1}, \sqrt{d_2}\big)$$

where $d_K = d_1 d_2$. Since χ is quadratic, the extension $H_\chi/\mathbb{Q}$ is either biquadratic or quadratic, the latter case occurring precisely when χ is trivial (and then, of course, $H_\chi = K$). Let χ_1, χ_2 and ε_K be the characters associated with the quadratic fields $\mathbb{Q}(\sqrt{d_1})$, $\mathbb{Q}(\sqrt{d_2})$ and K, respectively, so that $\varepsilon_K = \chi_1 \cdot \chi_2$. It is well known that the genus characters of K are in bijection with the factorizations of d into a product of relatively prime discriminants d_1 and d_2 or, equivalently, with the unordered pairs (χ_1, χ_2) of primitive quadratic Dirichlet characters of coprime conductors satisfying $\varepsilon_K = \chi_1 \cdot \chi_2$ (the trivial character corresponds to the factorization $d = 1 \cdot d$). For more details see, e.g., [**5**, Ch. 14, §G].

6.3 - *Derivatives of p-adic L-functions and Darmon points*

The rationality results we are interested in will concern the Darmon points $P^{\pm}_{A,\psi}$ of conductor $c = 1$ introduced in §4.2. For every $\psi \in \mathrm{Emb}(\mathcal{O}_K, R_0)$ set $J^{\pm}_{\psi} := \log_A(P^{\pm}_{A,\psi}) \in \mathrm{Lie}(A(K_p)) \simeq \mathcal{O}_F \otimes K_p$, then for a genus character χ of K of sign $-\varepsilon \in \{\pm\}$ define

$$J_\chi := \sum_{\sigma \in G_K^+} \chi(\sigma) J^{\varepsilon}_{\psi_\sigma} = \log_A(P_\chi)$$

with $P_\chi := \sum_{\sigma \in G_K^+} \chi(\sigma) P^{\varepsilon}_{A,\psi_\sigma} \in A(K_p)$. Recall the integers n and t in Corollary 4.3.

Theorem 6.1. *With notation as before, there is an equality*

$$\frac{d}{dk} \mathcal{L}_p(f_\infty/K, \chi, k)_{k=2} = \frac{w_{MD}\chi_1(-MD) - 1}{2nt} J_\chi .$$

Proof. Mimic the proof of [**21**, Theorem 4.31]. □

Corollary 6.1. *There is an equality*

$$\frac{d^2}{dk^2} L_p(f_\infty/K, \chi, k)_{k=2} = \begin{cases} J_\chi^2/(nt)^2 & \text{if } \chi_1(-MD) = -w_{MD}\,; \\ 0 & \text{if } \chi_1(-MD) = w_{MD}\,. \end{cases}$$

Proof. The second derivative of $L_p(f_\infty/K, \chi, k)$ evaluated at $k = 2$ is equal to the sum

$$2\left(\frac{d}{dk}\mathcal{L}_p(f_\infty/K, \chi, k)_{k=2}\right)^2 + 2\mathcal{L}_p(f_\infty/K, \chi, 2)\, \frac{d^2}{dk^2}\mathcal{L}_p(f_\infty/K, \chi, k)_{k=2} .$$

But $\mathcal{L}_p(f_\infty/K, \chi, 2) = 0$ by (19), and the result follows from Theorem 6.1. □

6.4 - *A factorization formula for p-adic L-functions*

For $j = 1, 2$ let $L_p(f_\infty, \chi_j, k, s)$ be the Mazur–Kitagawa p-adic L-function associated with f_∞ and χ_j; we refer to [**1**, §1.4] for its definition and main properties.

The following factorization result is the counterpart of [**21**, Theorem 4.33].

Theorem 6.2. *There exist a neighbourhood $\mathbb{U} \subset \mathbb{U}_f$ of 2 and a p-adic analytic function η on $\mathbb{U}$ such that*

1. $\eta(k) \neq 0$ *for all* $k \in \mathbb{U}$ *and* $\eta(2) \in (F^\times)^2$;
2. *for all* $k \in \mathbb{U}$ *there is a factorization*

$$L_p(f_\infty/K, \chi, k) = \eta(k) L_p(f_\infty, \chi_1, k, k/2) L_p(f_\infty, \chi_2, k, k/2).$$

Proof. Proceed as in the proof of [**21**, Theorem 4.33], keeping in mind that, with notation as in *loc. cit.*, $\langle \phi_2, \phi_2 \rangle \in F^\times$ (cf. Remark 5.1) and, by construction, $L_p(f_\infty/K'_{i,j}, \chi'_{i,j}, 2)$ is a square in $F^\times$ (see [**1**, §3.2]). □

6.5 - *The rationality result*

The rationality result we want to prove is the following

Theorem 6.3. *Let χ be a genus character of K corresponding to a pair (χ_1, χ_2) such that $\chi_i(-MD) = -w_{MD}$ for $i = 1, 2$.*

1. *There exists a point* $\mathbf{P}_\chi \in A(H_\chi)^\chi$ *and a number* $c \in F^\times$ *such that*

$$J_\chi = c \log_A(\mathbf{P}_\chi).$$

2. *The point* $\mathbf{P}_\chi$ *is of infinite order if and only if* $L'(f/K, \chi, 1) \neq 0$.
3. *A suitable integral multiple of* P_χ *belongs to the natural image of* $A(H_\chi)^\chi$ *in* $A(K_p)$. *In particular,* P_χ *coincides with the image of a global point in* $A(K_p) \otimes \mathbb{Q}$.

This result extends [**21**, Theorem 5.1] to the case of higher-dimensional abelian varieties. As will be clear, we follow the proofs of [**2**, Theorem 4.3] and of [**21**, Theorem 5.1] closely.

Proof. Order χ_1, χ_2 in such a way that $\mathrm{sign}(A, \chi_1) = -1$. Then $\chi_1(p) = -w_p = a_p$ and Theorem 5.2 ensures that there are a global point $\mathbf{P}_{\chi_1} \in A\big(\mathbb{Q}(\sqrt{d_1})\big)^{\chi_1}$, which is torsion if and only if $L'(f/\mathbb{Q}, \chi_1, 1) = 0$, and a number $s \in F^\times$ such that

$$\frac{d^2}{dk^2} L_p(f_\infty, \chi_1, k, k/2)_{k=2} = s \log_A(\mathbf{P}_{\chi_1})^2 \tag{20}$$

and

$$s \equiv L^*(f, \psi, 1) \pmod{(F^\times)^2} \tag{21}$$

for *any* primitive Dirichlet character ψ for which $L(f, \psi, 1) \neq 0$, $\psi(p) = -\chi(p)$ and $\psi(\ell) = \chi(\ell)$ for all primes $\ell | MD$. Here the *algebraic part* $L^*(f, \psi, 1) \in F$ of the special value $L(f, \psi, 1)$ is defined as in [**2**, eq. (24)].

From Theorem 6.2 and the fact that, again by Theorem 5.2, $L_p(f_\infty, \chi_1, k, k/2)$ vanishes of order at least 2 at $k = 2$, it follows that

$$\frac{d^2}{dk^2} L_p(f_\infty/K, \chi, k)_{k=2} = \eta(2) \frac{d^2}{dk^2} L_p(f_\infty, \chi_1, k, k/2)_{k=2} L_p(f_\infty, \chi_2, 2, 1). \tag{22}$$

First suppose that $L'(f/K, \chi, 1) \neq 0$. In this case

$$L_p(f_\infty, \chi_2, 2, 1) = 2L^*(f, \chi_2, 1) \in F^\times.$$

Set $u := L^*(f, \chi_2, 1)$. By (21), there exists $r \in F^\times$ such that $r^2 = s/u$. Define $\mathbf{P}_\chi := \mathbf{P}_{\chi_1}$ in this case. Setting $v := ntru \in F^\times$, a combination of (20), (22) and Corollary 6.1 yields

$$J_\chi^2 = \eta(2)(nt)^2 su \log_A(\mathbf{P}_{\chi_1})^2 = \left(v\sqrt{\eta(2)}\right)^2 \log_A(\mathbf{P}_\chi)^2,$$

from which we obtain the desired result for $c := \pm\, v\sqrt{\eta(2)} \in F^\times$ (recall that, by Theorem 6.2, $\eta(2)$ is a square in $F^\times$).

Now suppose that $L'(f/K, \chi, 1) = 0$. In this case

$$L_p(f_\infty, \chi_2, 2, 1) = 2L^*(f, \chi_2, 1) = 0,$$

and the result is a consequence of (22) and Corollary 6.1 upon setting $\mathbf{P}_\chi := 0$. This completes the proof of parts (1) and (2).

Finally, write $c = a/b$ with $a \in \mathcal{O}_f$ and $b \in \mathbb{Z}$. Since $J_\chi = \log_A(P_\chi)$, part (1) and the commutativity relation (11) give the equality

$$\log_A(bP_\chi) = \log_A(a\mathbf{P}_\chi). \tag{23}$$

As remarked in §4.1, the elements of $\mathcal{O}_f$ act on A as endomorphisms defined over $\mathbb{Q}$, hence $a\mathbf{P}_\chi \in A(H_\chi)$. Therefore, since the kernel of $\log_A$ is the torsion subgroup of $A(\mathbb{C}_p)$, part (3) follows from (23). □

A - The p-adic uniformization of abelian varieties

We review the p-adic uniformization theory of abelian varieties A over $\mathbb{Q}$ with purely multiplicative reduction at p and explicitly describe the isogeny φ_A^ε appearing in §4.2.

A.1 - *Review of Tate–Morikawa–Mumford theory*

Let $A_{/\mathbb{Q}}$ be an abelian variety of dimension d with purely multiplicative reduction at p. This means that the identity component of the special fibre of the Néron model of A over $\mathbb{Z}_p$ is a torus over $\mathbb{F}_p$; if this torus is *split* over $\mathbb{F}_p$ (which we do not assume) then we say that A has purely *split* multiplicative reduction at p. Such an abelian variety is a higher-dimensional analogue of Tate's p-adic elliptic curve (see, e.g., [**30**, §A.1.1]). Thanks to results due to (among others) Tate, Morikawa and Mumford, which now we briefly describe, this condition guarantees that A admits an analytic uniformization locally at p. Details can be found in [**26**, Section III], while the rigid analytic point of view is nicely exposed in [**23**, Section 4].

Set $G_{\mathbb{Q}_p} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. There are free abelian groups $\mathcal{M}$ and $\mathcal{N}$ of rank d and an admissible (in the sense of [**26**, Section III, §2]) homomorphism $\alpha : \mathcal{M} \to \mathrm{Hom}(\mathcal{N}, \mathbb{Q}_p^\times)$ that fit into a short exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{N}, \bar{\mathbb{Q}}_p^\times) \xrightarrow{\theta} A(\bar{\mathbb{Q}}_p) \longrightarrow 0$$

of $G_{\mathbb{Q}_p}$-modules (with $\mathcal{M}$ and $\mathcal{N}$ being regarded as trivial Galois modules). In other words, there is a Galois-equivariant analytic uniformization

$$\theta : \mathrm{Hom}(\mathcal{N}, \bar{\mathbb{Q}}_p^\times)/\mathcal{M} \xrightarrow{\simeq} A(\bar{\mathbb{Q}}_p)$$

that expresses the geometric points of $A_{/\mathbb{Q}_p}$ as a quotient of a d-dimensional p-adic torus by a sublattice of full rank. Reversing the roles of $\mathcal{M}$ and $\mathcal{N}$, one obtains an analogous parametrization for the dual abelian variety of A.

Choose $\mathbb{Z}$-bases $\{x_1, \dots, x_d\}$ and $\{y_1, \dots, y_d\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, and for every $j = 1, \dots, d$ define $\underline{q}_j := \big(\alpha(x_j)(y_1), \dots, \alpha(x_j)(y_d)\big) \in (\mathbb{Q}_p^\times)^d$. It follows that θ induces, by continuity, an analytic isomorphism

$$\Phi_{\mathrm{Tate},A} : (\mathbb{C}_p^\times)^d \big/ \big\langle \underline{q}_1, \dots, \underline{q}_d \big\rangle \xrightarrow{\simeq} A(\mathbb{C}_p)$$

of $G_{\mathbb{Q}_p}$-modules. The vectors $\underline{q}_1, \dots, \underline{q}_d$ are the d-dimensional analogue of the period q that appears in the theory of Tate's elliptic curves (see, e.g., [**30**, §A.1.1]); they will henceforth be referred to as the *Tate periods* for A at p.

Remark A.1 If the reduction of A at p is split then the map $\Phi_{\mathrm{Tate},A}$ is defined over $\mathbb{Q}_p$. On the other hand, if the reduction of A at p is purely multiplicative but not necessarily split then a result of Mumford and Raynaud shows the existence of an unramified extension H of $\mathbb{Q}_p$ such that $\Phi_{\mathrm{Tate},A}$ is defined over H (cf. [**26**, Theorem 3.2.2]).

A.2 - *Modularity and commensurable lattices*

Assume that the d-dimensional abelian variety $A_{/\mathbb{Q}}$ has conductor MDp and is modular. As remarked in the introduction and in §4.1, it is known that A has purely multiplicative reduction at p and is a quotient of $J_1^{p\text{-new}}$.

Recall the d-dimensional p-adic torus T^ε_A and the Hecke-stable lattice L^ε_A inside $T^\varepsilon_A(\mathbb{Q}_p)$ defined in §4.1. Two lattices Λ_1 and Λ_2 in $(K_p^\times)^d$ are *commensurable* if $\Lambda_1 \cap \Lambda_2$ has finite index both in Λ_1 and in Λ_2.

The following result, asserting that L^ε_A is commensurable to the lattice of Tate periods for A at p, extends [**17**, Theorem 7.16] from elliptic curves to abelian varieties of higher dimension.

Theorem A.1. *The lattices L^ε_A and $\langle \underline{q}_1, \dots, \underline{q}_d \rangle$ are commensurable in $(K_p^\times)^d$.*

Proof. By multiplicity one, A is, up to isogeny, the unique quotient of $J_1^{p\text{-new}}$ on which the action of the Hecke operators T_ℓ for primes $\ell \nmid N$ factors through λ_f. Similarly, $T^\varepsilon_A(\mathbb{C}_p)$ is the unique quotient of $T_H(\mathbb{C}_p)$ on which the Hecke action factors through λ_f and complex conjugation acts as $\varepsilon 1$. Then [**17**, Theorem 1.1] implies that $T^\varepsilon_A/L^\varepsilon_A$ and A are isogenous over K_p, which amounts to saying that L^ε_A and $\langle \underline{q}_1, \dots, \underline{q}_d \rangle$ are commensurable in $(K_p^\times)^d$. □

A.3 - *The isogeny φ^ε_A*

For simplicity, we set $\mathcal{J} := J_1^{p\text{-new}}$. As pointed out before, $\mathcal{J}$ is the maximal quotient of J_1 having purely multiplicative reduction at p. Since the p-adic uniformization of $\mathcal{J}$ can be explicitly described in terms of the abelian group of degree 0 divisors on the supersingular points of X_0 in characteristic p (see, e.g., [**17**, §3.5]; cf. also [8, pp. 448–449], which treats the case of Jacobians of modular curves), it turns out that the map $\Phi_{\mathrm{Tate},\mathcal{J}}$ is defined over K_p. Keeping in mind that A is a quotient of $J_1^{p\text{-new}}$, it follows that the p-adic uniformization $\Phi_{\mathrm{Tate},A}$ of A is defined over K_p as well.

By Theorem A.1, $L^\varepsilon_A \cap \langle \underline{q}_1, \dots, \underline{q}_d \rangle$ has finite index in L^ε_A; set $n := \left[L^\varepsilon_A : L^\varepsilon_A \cap \langle \underline{q}_1, \dots, \underline{q}_d \rangle\right]$. Raising to the n-th power gives a Galois-equivariant isogeny of p-adic tori

$$T^\varepsilon_A(\mathbb{C}_p)/L^\varepsilon_A \longrightarrow (\mathbb{C}_p^\times)^d / \langle \underline{q}_1, \dots, \underline{q}_d \rangle \tag{24}$$

over $\mathbb{Q}_p$, and then composing (24) with $\Phi_{\mathrm{Tate},A}$ yields a Galois-equivariant isogeny

$$\varphi^\varepsilon_A : T^\varepsilon_A(\mathbb{C}_p)/L^\varepsilon_A \longrightarrow A(\mathbb{C}_p)$$

defined over K_p. This gives an explicit construction of the isogeny introduced in (5).

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References

[1] M. Bertolini and H. Darmon, *Hida families and rational points on elliptic curves*, Invent. Math. **168** (2007), no. 2, 371-431.

[2] M. Bertolini and H. Darmon, *The rationality of Stark–Heegner points over genus fields of real quadratic fields*, Ann. of Math. (2) **170** (2009), no. 1, 343-370.

[3] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, in "The Grothendieck Festschrift", vol. I, P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Yu. Manin and K. A. Ribet (eds.), Progr. Math., **86**, Birkhäuser, Boston, MA 1990, 333-400.

[4] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie – Chapitres 2 et 3*, Reprint of the 1972 original, Springer, Berlin 2006.

[5] H. Cohn, *A classical invitation to algebraic numbers and class fields*, Universitext, Springer-Verlag, New York-Heidelberg 1978.

[6] R. F. Coleman, *Reciprocity laws on curves*, Compositio Math. **72** (1989), no. 2, 205-235.

[7] H. Darmon, *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications*, Ann. of Math. (2) **154** (2001), no. 3, 589-639.

[8] S. Dasgupta, *Stark–Heegner points on modular Jacobians*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 3, 427-469.

[9] M. Greenberg and M. A. Seveso, *p-adic families of modular forms and p-adic Abel–Jacobi maps*, Special volume in honor of Glenn Stevens' 60th birthday, Ann. Math. Qué. **40** (2016), no. 2, 397-434.

[10] M. Greenberg, M. A. Seveso and S. Shahabi, *Modular p-adic L-functions attached to real quadratic fields and arithmetic applications*, J. Reine Angew. Math., to appear, DOI:10.1515/crelle-2014-0088.

[11] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math **111** (1993), no. 2, 407-447.

[12] X. Guitart and M. Masdeu, *Overconvergent cohomology and quaternionic Darmon points*, J. Lond. Math. Soc. (2) **90** (2014), no. 2, 495-524.

[13] H. Hida, *Hilbert modular forms and Iwasawa theory*, Oxford Mathematical Monographs, Oxford University Press, Oxford 2006.

[14] V. A. Kolyvagin and D. Yu. Logachëv, *Finiteness of the Shafarevich–Tate group and the group of rational points for some modular abelian varieties*, Leningrad Math. J. **1** (1990), no. 5, 1229-1253.

[15] V. A. Kolyvagin and D. Yu. Logachëv, *Finiteness of* Ш *over totally real fields*, Math. USSR–Izv. **39** (1992), no. 1, 829-853.

[16] W.-C. W. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), no. 4, 285-315.

[17] M. Longo, V. Rotger and S. Vigni, *On rigid analytic uniformizations of Jacobians of Shimura curves*, Amer. J. Math. **134** (2012), no. 5, 1197-1246.

[18] M. Longo, V. Rotger and S. Vigni, *Special values of L-functions and the arithmetic of Darmon points*, J. Reine Angew. Math. **684** (2013), 199-244.

[19] M. Longo and S. Vigni, *Quaternion algebras, Heegner points and the arithmetic of Hida families*, Manuscripta Math. **135** (2011), no. 3-4, 273-328.

[20] M. Longo and S. Vigni, *A note on control theorems for quaternionic Hida families of modular forms*, Int. J. Number Theory 8 (2012), no. 6, 1425-1462.

[21] M. Longo and S. Vigni, *The rationality of quaternionic Darmon points over genus fields of real quadratic fields*, Int. Math. Res. Not. IMRN **2014**, no. 13, 3632-3691.

[22] A. Mattuck, *Abelian varieties over p-adic ground fields*, Ann. of Math. (2) **62** (1955), no. 1, 92-119.

[23] M. Papikian, *Rigid-analytic geometry and the uniformization of abelian varieties*, in "Snowbird lectures in algebraic geometry", R. Vakil (ed.), Contemp. Math., **388**, Amer. Math. Soc., Providence, RI 2005, 145-160.

[24] A. A. Popa, *Central values of Rankin L-series over real quadratic fields*, Compos. Math. **142** (2006), no. 4, 811-866.

[25] K. A. Ribet, *Endomorphisms of semi-stable abelian varieties over number fields*, Ann. Math. (2) **101** (1975), no. 3, 555-562.

[26] K. A. Ribet, *Galois action on division points of Abelian varieties with real multiplications*, Amer. J. Math. **98** (1976), no. 3, 751-804.

[27] K. A. Ribet, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. **253** (1980), no. 1, 43-62.

[28] V. Rotger and M. A. Seveso, *$\mathcal{L}$-invariants and Darmon cycles attached to modular forms*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 6, 1955-1999.

[29] J.-P. Serre, *Lie algebras and Lie groups*, 2nd edition, Lecture Notes in Math., **1500**, Springer-Verlag, Berlin 1992.

[30] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, revised reprint of the 1968 original, Research Notes in Mathematics, **7**, A K Peters, Wellesley, MA 1998.

[31] M. A. Seveso, *p-adic L-functions and the rationality of Darmon cycles*, Canad. J. Math. **64** (2012), no. 5, 1122-1181.

[32] M. A. Seveso, *The Teitelbaum conjecture in the indefinite setting*, Amer. J. Math. **135** (2013), no. 6, 1525-1557.

[33] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, Princeton, NJ 1971.

[34] Yu. G. Zarhin, *p-adic abelian integrals and commutative Lie groups*, J. Math. Sci. **81** (1996), no. 3, 2744-2750.

[35] S.-W. Zhang, *Heights of Heegner points on Shimura curves*, Ann. of Math. (2) **153** (2001), no. 1, 27-147.

[36] S.-W. Zhang, *Gross–Zagier formula for* GL_2, Asian J. Math. **5** (2001), no. 2, 183-290.

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