Riv. Mat. Univ. Parma, Vol. **7** (2016), 19-37

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Applications of some exponential sums on prime powers: a survey

Abstract. Let Λ be the von Mangoldt function and $N, \ell \geq 1$ be two integers. We will see some results by the author and Alessandro Zaccagnini obtained using the original Hardy & Littlewood circle method function, *i.e.*

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha),$$

where $e(x) = \exp(2\pi i x)$, instead of $S_\ell(\alpha) = \sum_{n=1}^{N} \Lambda(n) e(n^\ell \alpha)$. We will also motivate why, for some short interval additive problems, the approach using $\widetilde{S}_\ell(\alpha)$ gives sharper results than the ones that can be obtained with $S_\ell(\alpha)$. The final section of this paper is devoted to correct an oversight occurred in [**17**] and [**19**].

Keywords. Waring-Goldbach problem, Hardy-Littlewood method, Laplace transforms, Cesàro averages.

Mathematics Subject Classification (2010): Primary 11P32; Secondary 11P55, 11P05, 44A10, 33C10.

1 - Introduction

In a series of recent papers in collaboration with Alessandro Zaccagnini, we proved several results about additive problems with primes and prime powers. Our main tool was the circle method in its original form used by Hardy & Littlewood [8], since, for these applications, it lets us get stronger results than the ones obtainable

Received: November 30, 2015; accepted in revised form: May 17, 2016.
This research was partially supported by the grant PRIN2010-11 *Arithmetic Algebraic Geometry and Number Theory*.

by the more recent setting. This means to work with the function

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha), \tag{1}$$

where $N, \ell \geq 1$ are two integers, Λ is the von Mangoldt function and $e(x) = \exp(2\pi i x)$, instead of

$$S_\ell(\alpha) = \sum_{n=1}^{N} \Lambda(n) e(n^\ell \alpha). \tag{2}$$

We see now the statements we were able to prove and we compare them with the known results obtained using (2). Then we will try to give a motivation why, for such kind of problems, using (1) is better than using (2).

2 - Statements

All the following results are asymptotic formulae concerning suitable averages for functions counting the number of ways an integer is representable as a sum of primes, prime powers or powers. It is well known that usually the behaviour of such functions is erratic but averaging them is, in many cases, sufficient to get much more regular problems. We also remark that every asymptotic formula we will write below has, as a consequence, the existence of an integer representable in the way stated in the result in intervals of the mentioned size.

Our main goal is to get, both assuming the Riemann Hypothesis and unconditionally, an asymptotic relation involving zeros of the Riemann zeta-function $\zeta(s)$, which either holds in the shorter known interval or has the best known error estimate.

We start with the most famous additive problem: the Goldbach one. Letting

$$R_G(n) = \sum_{m_1 + m_2 = n} \Lambda(m_1)\Lambda(m_2),$$

we obtained

Theorem 2.1 (Languasco-Zaccagnini [**16**]). *Let N be a sufficiently large integer. Assuming the Riemann Hypothesis (RH) holds, we have*

$$\sum_{n=1}^{N} R_G(n) = \frac{N^2}{2} - 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}\big(N \log^3 N\big), \tag{3}$$

where $\rho = 1/2 + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

Using (2), Bhowmik & Schlage-Puchta [**1**] proved that the error term is $\mathcal{O}\big(N\log^5 N\big)$. Quite recently Goldston & Yang [**7**, Theorem 1], inserting a suitable average in Bhowmik & Schlage-Puchta's approach (see Lemma 5 of [**7**]), were able to give another proof of Theorem 2.1. In [**16**] we also proved a short interval version of Theorem 2.1 and its analogue with the case of representing an integer as a sum of $k \geq 3$ primes.

Inserting the Cesàro (or Riesz) weight in the picture, a tool widely used in Number Theory history, see, *e.g.*, [**2**], we were also able to push forward the investigation about terms of secondary order in Theorem 2.1 thus getting

Theorem 2.2 (Languasco-Zaccagnini [**19**]). *Let N be a sufficiently large integer and $k > 1$ be a real number. Then*

$$\sum_{n\leq N} R_G(n)\,\frac{(1-n/N)^k}{\Gamma(k+1)} = \frac{N^2}{\Gamma(k+3)} - 2\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(\rho+k+2)}\,N^{\rho+1} + \sum_{\rho_1}\sum_{\rho_2}\frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+k+1)}\,N^{\rho_1+\rho_2} + \mathcal{O}_k\big(N\big), \tag{4}$$

where ρ, with or without subscripts, runs over the non-trivial zeros of $\zeta(s)$ and Γ is the Euler Gamma-function.

In the original version of Theorem 2.2 the error term was erroneously written as $\mathcal{O}_k\big(N^{1/2}\big)$; see Section 6 below for the correction.

Goldston & Yang [**7**, Theorem 2], assuming RH, were recently able to prove that

$$\sum_{n\leq N} R_G(n)\Big(1-\frac{n}{N}\Big) = \frac{N^2}{6} - 2\sum_{\rho}\frac{N^{\rho+1}}{\rho(\rho+1)(\rho+2)} + \mathcal{O}(N) \tag{5}$$

which would correspond to what (4) would imply if one could take $k = 1$ in Theorem 2.2.

A result similar to Theorem 2.2 can be obtained for the problem of representing an integer as a sum of a prime and a square, *i.e.*, the so-called Hardy-Littlewood numbers since in [8] they conjectured that every large non-square integer should be representable in such a way. Letting

$$R_{HL}(n) = \sum_{m_1+m_2^2=n} \Lambda(m_1),$$

we proved

Theorem 2.3 (Languasco-Zaccagnini [**17**]). *Let N be a sufficiently large integer and $k > 1$ be a real number. Then*

$$\begin{aligned}\sum_{n\le N} R_{HL}(n)\,\frac{(1-n/N)^k}{\Gamma(k+1)} &= \frac{\pi^{1/2}}{2}\frac{N^{3/2}}{\Gamma(k+5/2)} - \frac{1}{2}\frac{N}{\Gamma(k+2)} \\ &\quad - \frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}\,N^{1/2+\rho} \\ &\quad + \frac{1}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}\,N^{\rho} + \frac{N^{3/4-k/2}}{\pi^{k+1}}\sum_{\ell\ge1}\frac{J_{k+3/2}(2\pi\ell N^{1/2})}{\ell^{k+3/2}} \\ &\quad - \frac{N^{1/4-k/2}}{\pi^{k}}\sum_{\rho}\Gamma(\rho)\,\frac{N^{\rho/2}}{\pi^{\rho}}\sum_{\ell\ge1}\frac{J_{k+1/2+\rho}(2\pi\ell N^{1/2})}{\ell^{k+1/2+\rho}} + \mathcal{O}_k\big(N^{1/2}\big),\end{aligned}$$

where ρ runs over the non-trivial zeros of $\zeta(s)$, Γ is the Euler Gamma-function and $J_\nu(u)$ denotes the Bessel function of complex order ν and real argument u.

In the original version of Theorem 2.3 the error term was erroneously written as $\mathcal{O}_k(1)$; see Section 6 below for the correction. We recall here the Sonine representation of such a Bessel function:

$$J_\nu(u) := \frac{(u/2)^\nu}{2\pi i}\int_{(a)} s^{-\nu-1}e^{s}e^{-u^2/4s}\,\mathrm{d}s, \tag{6}$$

where $a > 0$ and $u, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$.

Theorem 2.3 has no analogue proved using (2). Moreover it seems that it was the first time a complex order Bessel function is involved in additive results. In particular these terms come from the modularity of the complex θ-function $\theta(z) = \sum_{m=-\infty}^{+\infty} e^{-m^2 z}$, $z \in \mathbb{C}$, given by the formula

$$\theta(z) = \Big(\frac{\pi}{z}\Big)^{1/2}\theta\Big(\frac{\pi^2}{z}\Big) \quad \text{for } \Re(z) > 0. \tag{7}$$

We then focused our attention on problems involving prime squares. In 1938, Hua [**9**] considered the problem of representing an integer as a sum of a prime and of two prime squares. Letting $\mathcal{A} = \{n \in \mathbb{N} : n \equiv 1 \bmod 2;\ n \not\equiv 2 \bmod 3\}$, Hua proved that the set of integers $n \in [1, N] \cap \mathcal{A}$ that cannot be representable in such a way has a cardinality $\ll N/\log^A N$ for some $A > 0$. Recently L. Zhao [**33**] improved this estimate to $\ll N^{1/3+\varepsilon}$, for every $\varepsilon > 0$. Letting

$$r(n) = \sum_{p_1+p_2^2+p_3^2=n} \log p_1 \log p_2 \log p_3\,,$$

we proved the following

Theorem 2.4 (Languasco-Zaccagnini [22]). *Assume RH. Then*

$$\sum_{n=N+1}^{N+H} r(n) = \frac{\pi}{4} HN + \mathcal{O}\Big(H^{1/2}N(\log N)^2 + HN^{3/4}(\log N)^3 + H^2(\log N)^{3/2}\Big)$$

as $N \to \infty$*, uniformly for* $\infty\big((\log N)^4\big) \le H \le o\Big(N(\log N)^{-3/2}\Big)$*, where* $f = \infty(g)$ *means* $g = o(f)$*.*

We were also able to prove an unconditional version of Theorem 2.4 that holds just for $H \ge N^{7/12+\varepsilon}$. Using (2), a similar result to Theorem 2.4 holds just for $H \ge N^{1/2+\varepsilon}$ under RH; this is due to a limitation I'll discuss in Section 4 below, see the remark after Lemma 4.4.

Similar, but harder, problems with prime and prime squares are the binary ones. Considering

$$r''_{1,2}(n) = \sum_{p_1+p_2^2=n} \log p_1 \log p_2\,,$$

we got the following

Theorem 2.5 (Languasco-Zaccagnini [20]). *Assume RH. Then*

$$\sum_{n=N+1}^{N+H} r''_{1,2}(n) = HN^{1/2} + \mathcal{O}\Big(\frac{H^2}{N^{1/2}} + N^{3/4}(\log N)^3 + HN^{1/3}(\log N)^2\Big)$$

as $N \to \infty$ *uniformly for* $\infty(N^{1/4}(\log N)^3) \le H \le o(N)$*.*

We also obtained that the error term $\mathcal{O}\Big(H\exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$ unconditionally holds uniformly for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$. Recently, Suzuki [30] proved that this result holds in $N^{1/2}\exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big) \le H \le N^{1-\varepsilon}$.

As far as we know, Theorem 2.5 is the first short interval result for this problem; we recall that the method in Plaksin [27] (Lemma 11 there) gives long interval results only and that an adaptation his idea to the short interval $[N, N+H]$ leads to weaker results. We also remark that using (2) we can only reach the level $H \ge N^{1/2+\varepsilon}$. Finally we recall that in [20] we also have results for the $p + m^2 = n$ case.

Even harder problems are the ones involving two prime squares, or a prime square and a square. As far as we know, for these problems just long interval averages were available, see the Introduction of the paper by Daniel [3]. Letting

$$r''_{2,2}(n) = \sum_{p_1^2+p_2^2=n} \log p_1 \log p_2 \quad\text{and}\quad r'_{2,2}(n) = \sum_{p^2+m^2=n} \log p\,,$$

we obtained the following

Theorem 2.6 (Languasco-Zaccagnini [**21**]). *Assume RH. Then*

$$\sum_{n=N+1}^{N+H} r''_{2,2}(n) = \frac{\pi}{4} H + \mathcal{O}\Big(\frac{H^2}{N} + H^{1/2}N^{1/4}(\log N)^{3/2}\Big)$$

as $N \to \infty$ *uniformly for* $\infty(N^{1/2}(\log N)^3) \le H \le o(N)$. *Moreover we also have that*

$$\sum_{n=N+1}^{N+H} r'_{2,2}(n) = \frac{\pi}{4} H + \mathcal{O}\Big(\frac{H^2}{N} + \frac{H \log\log N}{(\log N)^{1/2}}\Big)$$

as $N \to \infty$ *uniformly for* $\infty(N^{1/2}(\log N)^2) \le H \le o(N)$.

We also obtained that the error term $\mathcal{O}\Big(H \exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$ unconditionally holds uniformly for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$ in both cases.

3 - Settings for Theorems 2.1 and 2.4-2.6

The proofs of Theorems 2.1 and 2.4-2.6 all work with similar settings but, for simplicity, here we just write the Goldbach case in short interval.

Let $N, H > 0$, $H = o(N)$, be integers,

$$z = 1/N - 2\pi i\alpha \tag{8}$$

and $M_1(\alpha) = 1/z$ be the expected main term for the involved exponential sum, see Lemma 4.1 below. Moreover let

$$U(\alpha, H) = \sum_{1 \le m \le H} e(m\alpha).$$

We have

$$\begin{aligned} \sum_{n=N+1}^{N+H} e^{-n/N} R_G(n) &= \int_{-1/2}^{1/2} M_1(\alpha)^2 U(\alpha, H)\, d\alpha \\ &+ \int_{-1/2}^{1/2} (\widetilde{S}_1(\alpha)^2 - M_1(\alpha)^2) U(\alpha, H)\, d\alpha. \end{aligned} \tag{9}$$

The first term on the right hand side will lead to the expected main term while the second one is connected with a L^2-estimate for $\widetilde{S}_1(\alpha) - M_1(\alpha)$.

In fact, using the algebraic relation $f^2 - g^2 = (f-g)^2 + 2g(f-g)$, the Cauchy-Schwarz inequality and $U(\alpha, H) \ll \min(H; |\alpha|^{-1})$, an upper bound for the second term on the right hand side of (9) is

$$H \int_{-1/H}^{1/H} |\widetilde{S}_1(\alpha) - M_1(\alpha)|^2 \, \mathrm{d}\alpha + \int_{1/H}^{1/2} |\widetilde{S}_1(\alpha) - M_1(\alpha)|^2 \, \frac{\mathrm{d}\alpha}{\alpha}, \tag{10}$$

since a mean-square estimate of $\widetilde{S}_1(\alpha)$, needed to estimate the mixed term in the previously mentioned agebraic relation, can be obtained using the Prime Number Theorem. A partial integration reveals that the key quantity is then

$$\int_{-\xi}^{\xi} |\widetilde{S}_1(\alpha) - M_1(\alpha)|^2 \, \mathrm{d}\alpha$$

with $0 < \xi \leq 1/2$. Hence we clearly need results on computing the main term and on such a truncated mean-square. For this second object, it is clear that the larger ξ-uniformity we have, the stronger result on H we get.

In the next section we will see that, for $\ell \geq 2$, the ξ-uniformity for the $\widetilde{S}_\ell(\alpha)$ truncated L^2-average is, at least in the conditional case, much larger than the corresponding one for $S_\ell(\alpha)$.

We finally remark that in many cases it is easy to remove the contribution of the exponential weight $e^{-n/N}$ from (9) since $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$. For proving the result stated in Theorem 2.1 such a relation for $e^{-n/N}$ was not good enough and we had to modify the previously described setting to be able to detect the secondary order terms in (3), see eq. (3.1) of the proof of Theorem 3.1 in [**16**].

4 - General results on $\widetilde{S}_\ell(\alpha)$ and $S_\ell(\alpha)$ and motivations

For (1) we have an explicit formula which can be proved via a Mellin transform argument. It generalizes and slightly sharpens what Linnik [**24**]-[**25**] proved for $\ell = 1$. Recall that $z = 1/N - 2\pi i \alpha$.

Lemma 4.1 (Lemma 2 of [**22**]). *Let $\ell \geq 1$ be an integer, $N \geq 2$ and $\alpha \in [-1/2, 1/2]$. Then*

$$\widetilde{S}_\ell(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) + \mathcal{O}_\ell(1), \tag{11}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

This is in fact an explicit formula for $\widetilde{S}_\ell(\alpha)$ and plays a key role in every result listed in Section 2; unfortunately a direct analogue of such a result for (2) is unknown. Lemma 4.1 means that the expected main term comes form the $\dfrac{\Gamma(1/\ell)}{\ell z^{1/\ell}}$ part. In fact it can be usually obtained via the following

Lemma 4.2 (Lemma 4 of [**22**]). *Let N be a positive integer and $\mu > 0$. Then*

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha)\,\mathrm{d}\alpha = e^{-n/N}\,\frac{n^{\mu-1}}{\Gamma(\mu)} + \mathcal{O}_\mu\Big(\frac{1}{n}\Big),$$

uniformly for $n \geq 1$.

The proof of Lemma 4.2 directly follows from the Laplace formula (14).

As we have seen in Section 3 we now need information on some truncated L^2 average. For (1) we have the following result (the case $\ell = 1$ under RH is in [**11**]).

Lemma 4.3 (Lemma 3 of [**22**] and Lemma 1 of [**21**]). *Let ε be an arbitrarily small positive constant, $\ell \geq 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\xi}^{\xi} \Big|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}\Big|^2 \mathrm{d}\alpha \ll_\ell N^{2/\ell-1} \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big)$$

uniformly for $0 \leq \xi < N^{-1+5/(6\ell)-\varepsilon}$. Assuming RH we get

$$\int_{-\xi}^{\xi} \Big|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}\Big|^2 \mathrm{d}\alpha \ll_\ell N^{1/\ell}\xi L^2$$

uniformly for $0 \leq \xi \leq 1/2$.

The proof of Lemma 4.3, even if it is a direct one, is quite lengthy due to some tedious computation to be performed for handling the contribution of complex powers of the z^ρ-type.

In Theorems 3.1-3.2 of [**18**] (see also Lemma 1 of [**22**] and the Theorem in Languasco [**10**]) we proved a direct analogue of the previous lemma for (2). Letting

$$T_\ell(\alpha) = \sum_{n=1}^{N} e(n^\ell \alpha)\,,$$

which now plays the role of the expected main term for $S_\ell(\alpha)$, we have

Lemma 4.4. *Let ε be an arbitrarily small positive constant, $\ell > 0$ be a real number, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\xi}^{\xi} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell N^{2/\ell-1}\Big(\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) + \frac{L^2}{\xi N}\Big),$$

uniformly for $1/(2N) \le \xi < N^{-1+5/(6\ell)-\varepsilon}$. *Assuming RH we get*

$$\int_{-\xi}^{\xi} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell N^{1/\ell}\xi L^2 + \frac{N^{2/\ell-2}L^2}{\xi},$$

uniformly for $1/(2N) \le \xi \le N^{1/\ell-1}/2$. *For* $\ell = 1$ *the last estimate can be replaced by* $N\xi L^2 + \min\Big(\dfrac{L^2}{\xi(\log 3\xi)^2}; N\xi L^4\Big)$ *uniformly for* $1/(2N) \le \xi \le 1/2$.

We remark that the ξ-uniformity here is, at least in the conditional case and for $\ell \ge 2$, worse than the one obtained in Lemma 4.3. This is in fact a key limitation for the kind of problems we are working on. We try now to explain why we encountered it. The proof of Lemma 4.4 is obtained via Gallagher's lemma [**6**] which gives

$$\begin{aligned}
\int_{-\xi}^{\xi} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha &= \int_{-\xi}^{\xi} \Big| \sum_{N \le n^\ell \le 2N} (\Lambda(n) - 1) e(n^\ell \alpha) \Big|^2 \mathrm{d}\alpha \\
&\ll \xi^2 \int_{N}^{2N-H} \Big(\sum_{\substack{x \le n^\ell \le x+H \\ N \le n^\ell \le 2N}} (\Lambda(n) - 1) \Big)^2 \mathrm{d}x + \mathcal{O}\big(H^3 N^{2/\ell-2} L^2 + H\big),
\end{aligned} \tag{12}$$

where we defined $H = 1/(2\xi)$. From $x \le n^\ell \le x + H$ one immediately gets $x^{1/\ell} \le n \le (x+H)^{1/\ell}$ and hence, to have $(x+H)^{1/\ell} - x^{1/\ell} \gg 1$, we need $Hx^{1/\ell-1} \gg 1$, that is, essentially, $\xi \ll N^{1/\ell-1}$.

The estimate of the integral at the right hand side of (12) can be performed exploiting density estimates for the non-trivial zeros of the Riemann-zeta function, as in Saffari & Vaughan [**29**]. We further notice that if $\ell \ge 2$ and $\xi \ge N^{1/\ell-1}$, then the bound $\ll \xi^2 N \log N$ (which in this range is worse than $\xi N^{1/\ell} \log^2 N$) follows immediately from the Prime Number Theorem.

For $\ell = 1$ and assuming RH, a slightly more careful reasoning (see Languasco [**10**]) leads to the alternative estimate in Lemma 4.4. This reveals that Lemmas 4.3 and 4.4 in this case have the same the ξ-uniformity and they differ just for some logarithmic factors; in fact the gain in our Theorem 2.1 directly follows from such a slight improvement.[1]

For $\ell \geq 2$ and assuming RH, Lemmas 4.3 and 4.4 have quite different uniformities in the ξ-aspect and this means that additive problems with prime powers summands fit better with the (1)-approach since the larger ξ we get, the shorter H we can reach in (10) and hence in (9).

However, Lemma 4.4 is quite useful in proving Diophantine approximation results using linear forms in prime and prime powers see, *e.g.*, [**12**], [**13**], [**14**], [**15**], [**18**].

For problems involving prime squares and squares we also needed information about some truncated L^2 estimates for $\widetilde{S}_\ell(\alpha)$ and about

$$\omega_\ell(\alpha) = \sum_{m=1}^{\infty} e^{-m^\ell/N} e(m^\ell \alpha) = \sum_{m=1}^{\infty} e^{-m^\ell z}. \tag{13}$$

Using a well-known Montgomery & Vaughan result [**26**], we proved

Lemma 4.5 (Lemma 2 of [**21**]). *Let $0 < \xi \leq 1/2$, $\ell \geq 2$ be an integer, N be a sufficiently large integer and $L = \log N$. Then*

$$\int_{-\xi}^{\xi} |\omega_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \xi N^{1/\ell} + \begin{cases} L & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2 \end{cases}$$

and

$$\int_{-\xi}^{\xi} |\widetilde{S}_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \xi N^{1/\ell} L + \begin{cases} L^2 & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2. \end{cases}$$

Moreover in some cases we also needed information about the fourth moment for $\widetilde{S}_2(\alpha)$. We proved the following

Lemma 4.6 (Lemma 5 of [**22**]). *Let N be a sufficiently large integer and $L = \log N$. We have*

$$\int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)|^4 \, \mathrm{d}\alpha \ll NL^2.$$

[1] Goldston & Yang [**7**], introducing a further external average technique (see their Lemmas 5 and 6), were able to overcome this problem at least for the $\ell = 1$ case.

The proof of Lemma 4.6 exploits cancellation and uses a Rieger's result (see Satz 3 on page 94 of [**28**]) about the number of solutions of the $p_1^2 + p_2^2 = p_3^2 + p_4^2$ problem.

We conclude this section remarking another limitation we encountered using (2). In problems involving two prime squares we have to approximate $T_2(\alpha) = \sum_{1 \le m^2 \le N} e(m^2\alpha)$ with $f_2(\alpha) = (1/2) \sum_{1 \le m \le N} m^{-1/2} e(m\alpha)$; a standard approach in Waring-Goldbach problems. The main term for the $p_1^2 + p_2^2 = n$ problem has order of magnitude $\asymp H$; this comes from the evaluation of the integral of $f_2(\alpha)^2 U(-\alpha, H)$ which can be computed using Lemma 2.9 of Vaughan [**31**]. But we also need to evaluate the quantity

$$\left| \int_{-1/H}^{1/H} (T_2(\alpha)^2 - f_2(\alpha)^2) U(-\alpha, H) e(-N\alpha) \, d\alpha \right|.$$

By Theorem 4.1 of Vaughan [**31**] we know that $|T_2(\alpha) - f_2(\alpha)| \ll (1 + |\alpha| N)^{1/2}$ and hence it is not hard to see that the previous integral is $\ll NH^{-1/2}$. Since the expected order of magnitude of the main term is H, the previous estimate is under control if and only if $H \ge N^{2/3+\varepsilon}$; a result which is weaker than the one we obtained, see the statement of Theorem 2.6 above.

5 - Settings for Theorems 2.2-2.3

The method in this case is based on Laplace transforms. In particular we used a formula due to Laplace [**23**], namely

$$\frac{1}{2\pi i} \int_{(a)} v^{-s} e^{v} \, dv = \frac{1}{\Gamma(s)}, \tag{14}$$

where $\Re(s) > 0$ and $a > 0$, see, e.g., formula 5.4(1) on page 238 of [**5**]. In the following we will need the general case of (14) which, *e.g.*, can be found in [**4**], formulae (8)-(9):

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{iDu}}{(a+iu)^s} \, du = \begin{cases} D^{s-1} e^{-aD} \Gamma(s) & \text{if } D > 0, \\ 0 & \text{if } D < 0, \end{cases} \tag{15}$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$, and

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(a+iu)^s} \, du = \begin{cases} 0 & \text{if } \Re(s) > 1, \\ 1/2 & \text{if } s = 1, \end{cases} \tag{16}$$

for $a \in \mathbb{C}$ with $\Re(a) > 0$ (for the case $s = 1$ the principal value of the integral has to be taken). Formulae (15)-(16) enable us to write averages of arithmetical functions by means of line integrals as we will see below. We recall that Walfisz, see [**32**, Ch. X] used (14) in the particular case where $s \in \mathbb{N}$, $s \geq 2$.

As we did in Section 3, we just write the setting about the Goldbach problem since the other one is similar. We need to slightly change the notation of $\widetilde{S}_\ell(\alpha)$ since now we need to let $\Im(z)$ run over $\mathbb{R}$. Hence from now on we use

$$\widetilde{S}_\ell(z) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell z}, \quad z = 1/N + iy,\ y \in \mathbb{R}.$$

We first recall that the Prime Number Theorem is equivalent, via Lemma 5.1 below, to the statement

$$\widetilde{S}_1(z) \sim \frac{1}{z} \qquad \text{for } N \to \infty\ , \tag{17}$$

for the proof see Lemma 9 in Hardy & Littlewood [8]. By (1) and (8) we have $\widetilde{S}_1(z)^2 = \sum_{n\geq 1} R_G(n) e^{-nz}$. Hence, for $a = 1/N$, $N \in \mathbb{N}$ with $N > 0$, we have

$$\frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1}\, \widetilde{S}_1(z)^2 \,\mathrm{d}z = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \sum_{n\geq 1} R_G(n) e^{-nz} \,\mathrm{d}z. \tag{18}$$

Since $\sum_{n\geq 1} R_G(n) e^{-n/N} = \widetilde{S}_1(1/N)^2 \asymp N^2$ by (17), where $f \asymp g$ means $g \ll f \ll g$, we can exchange the series and the line integral in (18) provided that $k > 0$. Using (15) for $n \neq N$ and (16) for $n = N$, we see that for $k > 0$ the right hand side of (18) is

$$= \sum_{n\geq 1} R_G(n) \left[\frac{1}{2\pi i} \int_{(1/N)} e^{(N-n)z} z^{-k-1} \,\mathrm{d}z \right] = \sum_{n\leq N} R_G(n)\, \frac{(N-n)^k}{\Gamma(k+1)}.$$

Summing up, for $k > 0$ we have

$$\sum_{n\leq N} R_G(n)\, \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1}\, \widetilde{S}_1(z)^2 \,\mathrm{d}z, \tag{19}$$

where $N \in \mathbb{N}$ with $N > 0$. This is the fundamental equation of the method since it lets us insert the Cesàro (or Riesz) weight in the picture.

Clearly we need to work on a vertical line and hence some of the results of Section 4 need to be adapted to this case. In particular we have

Lemma 5.1 (Lemma 4.1 of [**19**]). *Let $z = 1/N + iy$, where $y \in \mathbb{R}$. Then*

$$\widetilde{S}_1(z) = \frac{1}{z} - \sum_\rho z^{-\rho} \Gamma(\rho) + E(y, 1/N)$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(y, 1/N) \ll 1 + |z|^{1/2} \begin{cases} 1 & \text{if } |y| \le 1/N \\ 1 + \log^2(N|y|) & \text{if } |y| > 1/N. \end{cases}$$

In the original statement of Lemma 5.1 the constant term in the estimate for $E(y, 1/N)$ is missing (but the original proof is correct). This oversight affects the size of the the error term estimate for Theorems 2.2 e 2.3; see Section 6 below for the correction.

The remaining part of the proof of Theorem 2.2 follows by inserting Lemma 5.1 into the right hand side of (19), by verifying that we can safely interchange sums and integrals (this leads to the condition $k > 1$) and using again (15)-(16).

For the $p + m^2$ problem, in (19) one copy of $\widetilde{S}_1(z)$ has to be replaced by the function $\omega_2(z) = \sum_{m=1}^{\infty} e^{-m^2 z}$. The use of the modular relation for $\omega_2(z)$, which immediately follows from (7), leads to much more complicated computations than in the Goldbach case; in fact integrals of the type in (6) appeared in several series also involving non-trivial zeros of the Riemann zeta-function. Eventually, after some work, we were able to prove that even in this case all the interchanging between series and integrals are justified if $k > 1$; thus proving Theorem 2.3.

In both Theorems 2.2-2.3 would be very interesting being able to improve the range of k. The most important result would be getting $k > 0$ but, unfortunately, this seems not to be reachable by current technologies even if there are indications that some result of this kind should hold. For example, the term

$$\sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} N^{\rho_1 + \rho_2}$$

involved in the statement of Theorem 2.2, is in fact absolutely convergent for $k > 1/2$, as we proved in Section 7 of [**19**]. Another positive indication of this fact is Theorem 2 of Goldston-Yang [**7**], see (5) before, since assuming RH, they were able to reach $k = 1$ (even if they do not give an explicit expression for such a term, but just an estimate).

6 - Corrigendum of [17] and of [19]

We take this occasion to correct the oversight occurred in Lemma 5.1 which affects both [**17**] and [**19**]. The amended final statements are written before as Theorems 2.2-2.3. We insert here both the new parts using the original sections and equations numbering.

Corrigendum of [17]. In [**17**] the affected parts are Lemma 1 which has to be replaced by Lemma 5.1 as we said before, a part of page 571 and the first part of Section 9 there. These parts should be substituted with what follows.

On page 571 the affected part is the estimate of

$$\int_{(a)} |E(y,a)|\,|e^{Nz}|\,|z|^{-k-1}|\omega_2(z)|\,|\mathrm{d}z|.$$

Taking $a = 1/N$, using (12)-(13) and Lemma 5.1 before, this integral is

$$\ll a^{-1/2}e^{Na}\Big(\int_{-a}^{a} a^{-k-1}\mathrm{d}y + \int_{a}^{+\infty} y^{-k-1}\mathrm{d}y + \int_{a}^{+\infty} y^{-k-1/2}\log^2(y/a)\,\mathrm{d}y\Big)$$

$$\ll_k e^{Na}a^{-1/2}\Big(a^{-k} + a^{-k+1/2}\int_{1}^{+\infty} v^{-k-1/2}\log^2 v\,\mathrm{d}v\Big) \ll_k N^{k+1/2},$$

provided that $k > 1/2$. The previous estimate has to be inserted in (15) and (26) and causes the change in the final error term of Theorem 2.3.

Moreover the first part of Section 9 should be replaced by the following. We need $k > 1$ in this section. We first have to establish the convergence of

$$\sum_{\ell\geq 1}\int_{(1/N)}\Big|\sum_{\rho}\Gamma(\rho)z^{-\rho}\Big|\,|e^{Nz}|\,|z|^{-k-3/2}e^{-\pi^2\ell^2\Re(1/z)}\,|\mathrm{d}z|. \tag{37}$$

Using the Prime Number Theorem and (29), we first remark that

$$\begin{aligned}\Big|\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big| &= \Big|\widetilde{S}(z) - \frac{1}{z} - E\Big(y,\frac{1}{N}\Big)\Big| \ll N + \Big|E\Big(y,\frac{1}{N}\Big)\Big| \\ &\ll N + |z|^{1/2}\log^2(2N|y|).\end{aligned} \tag{38}$$

By (33) and (38), we can write that the quantity in (37) is

$$\begin{aligned}&\ll N\sum_{\ell\geq 1}\int_{0}^{1/N}\frac{e^{-\ell^2 N}}{|z|^{k+3/2}}\mathrm{d}y + N\sum_{\ell\geq 1}\int_{1/N}^{+\infty}\frac{e^{-\ell^2/(Ny^2)}}{|z|^{k+3/2}}\,\mathrm{d}y \\ &\qquad + \sum_{\ell\geq 1}\int_{1/N}^{+\infty}\log^2(2Ny)\,\frac{e^{-\ell^2/(Ny^2)}}{|z|^{k+1}}\,\mathrm{d}y = V_1 + V_2 + V_3,\end{aligned} \tag{39}$$

say. V_1 can be estimated exactly as U_1 in Section 8 and we get $V_1 \ll_k N^{k+1}$. For V_2

we can work analogously to U_2 thus obtaining

$$V_2 \ll N \sum_{\ell \geq 1} \int_{1/N}^{+\infty} \frac{e^{-\ell^2/(Ny^2)}}{y^{k+3/2}} \,\mathrm{d}y \ll N^{k/2+5/4} \sum_{\ell \geq 1} \frac{1}{\ell^{k+1/2}} \int_0^{\ell^2 N} u^{k/2-3/4} e^{-u} \,\mathrm{d}u$$

$$\leq \Gamma\Big(\frac{2k+1}{4}\Big) N^{k/2+5/4} \sum_{\ell \geq 1} \frac{1}{\ell^{k+1/2}} \ll_k N^{k/2+5/4},$$

provided that $k > 1/2$, where we used the substitution $u = \ell^2/(Ny^2)$. Hence, we have

$$V_1 + V_2 \ll_k N^{k+1}, \tag{40}$$

provided that $k > 1/2$. Using the substitution $u = \ell^2/(Ny^2)$, we obtain

$$V_3 \ll \sum_{\ell \geq 1} \int_{1/N}^{+\infty} \log^2(2Ny) \frac{e^{-\ell^2/(Ny^2)}}{y^{k+1}} \,\mathrm{d}y$$

$$= \frac{N^{k/2}}{8} \sum_{\ell \geq 1} \frac{1}{\ell^k} \int_0^{\ell^2 N} u^{k/2-1} \log^2\Big(\frac{4\ell^2 N}{u}\Big) e^{-u} \,\mathrm{d}u.$$

Hence, a direct computation shows that

$$\begin{aligned} V_3 &\ll N^{k/2} \sum_{\ell \geq 1} \frac{\log^2(\ell N)}{\ell^k} \int_0^{\ell^2 N} u^{k/2-1} e^{-u} \,\mathrm{d}u \\ &\quad + N^{k/2} \sum_{\ell \geq 1} \frac{1}{\ell^k} \int_0^{\ell^2 N} u^{k/2-1} \log^2(u)\, e^{-u} \,\mathrm{d}u \\ &\ll_k \Gamma(k/2) N^{k/2} \sum_{\ell \geq 1} \frac{\log^2(\ell N)}{\ell^k} + N^{k/2} \ll_k N^{k/2} \log^2 N \end{aligned} \tag{41}$$

provided that $k > 1$. Inserting (40)-(41) into (39) we get, for $k > 1$, that the quantity in (37) is $\ll N^{k+1}$.

The remaining part of Section 9 of [**17**] is left untouched.

Corrigendum of [19]. In [**19**] the affected parts are Lemma 4.1 which has to be replaced by Lemma 5.1 as we said before, a part of page 1949 and the first part of Section 6 there. These parts should be substituted with what follows.

On page 1949 the affected part is the following. Recalling (2.4) and Lemma 5.1, we have for $a = 1/N$ that

$$\int_{(a)} \left|E(y,a)\right|^2 |e^{Nz}|\,|z|^{-k-1}\,|\mathrm{d}z| \ll_k e^{Na}a^{-k} \ll_k N^k$$

for $k > 1$. For $a = 1/N$, by (2.4) and Lemma 5.1, the second remainder term in (3.1) gives a contribution

$$\ll N \int_{(1/N)} |E(y,1/N)||e^{Nz}||z|^{-k-1}\,|\mathrm{d}z| \ll_k N^{k+1}.$$

This term has to be inserted in (3.2)-(3.3) and causes the change in the error term of Theorem 2.3.

Moreover the first part of Section 6 should be replaced by the following. We need $k > 1$ in this section. Arguing as in Section 5 we first need to establish the convergence of

$$\sum_{\rho_1} |\Gamma(\rho_1)| \int_{(1/N)} \Big|\sum_{\rho_2} \Gamma(\rho_2) z^{-\rho_2}\Big|\,|e^{Nz}|\,|z|^{-k-1}\,|z^{-\rho_1}|\,|\mathrm{d}z|. \tag{6.1}$$

Using the Prime Number Theorem and Lemma 5.1 we first remark that

$$\Big|\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big| \ll N + |z|^{1/2}\log^2(2N|y|). \tag{6.2}$$

By symmetry, we may assume that $\gamma_1 > 0$. By (6.2), (2.4) and (4.1), for $y \in (-\infty, 0]$ we are first led to estimate

$$\sum_{\rho_1:\gamma_1>0} \gamma_1^{\beta_1-1/2}\exp\Big(-\frac{\pi}{2}\gamma_1\Big)\Bigg(\int_{-1/N}^{0} N^{k+2+\beta_1}\,\mathrm{d}y + N\int_{-\infty}^{-1/N} \frac{\mathrm{d}y}{|y|^{k+1+\beta_1}}$$
$$+ \int_{-\infty}^{-1/N} \log^2(2N|y|)\,\frac{\mathrm{d}y}{|y|^{k+1/2+\beta_1}}\Bigg) \ll_k N^{k+2},$$

by the same argument used in the proof of Lemma 4.3 with $\alpha = k + 1/2$ and $a = 1/N$. On the other hand, for $y > 0$ we split the range of integration into $(0, 1/N] \cup (1/N, +\infty)$. By (6.2), (2.4) and Lemma 4.3 with $\alpha = k + 1$ and $a = 1/N$, on $[0, 1/N]$ we have

$$N\sum_{\rho_1:\gamma_1>0} \gamma_1^{\beta_1-1/2}\int_{0}^{1/N} \exp\Bigg(\gamma_1\Big(\arctan(Ny) - \frac{\pi}{2}\Big)\Bigg)\frac{\mathrm{d}y}{|z|^{k+1+\beta_1}} \ll_k N^{k+2}.$$

On the other interval, again by (2.4), we have to estimate

$$\sum_{\rho_1 : \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^{+\infty} \exp\Bigl(-\gamma_1 \arctan\frac{1}{Ny}\Bigr) \frac{N + y^{1/2}\log^2(2Ny)}{y^{k+1+\beta_1}}\,\mathrm{d}y$$
$$= N^k \sum_{\rho_1 : \gamma_1 > 0} N^{\beta_1} \gamma_1^{\beta_1 - 1/2} \int_{1}^{+\infty} \exp\Bigl(-\gamma_1 \arctan\frac{1}{u}\Bigr) \frac{N + u^{1/2}N^{-1/2}\log^2(2u)}{u^{k+1+\beta_1}}\,\mathrm{d}u.$$

Lemma 4.2 with $\alpha = k + 1/2$ shows that the last term is $\ll_k N^{k+2}$. This implies that the integral in (6.1) is $\ll_k N^{k+2}$ provided that $k > 1$ and hence we can exchange the first summation with the integral in this case.

The remaining part of Section 6 of [**19**] is left untouched.

Acknowledgments. I wish to thank the anonymous Referee for his comments and suggestions. I also wish to thank Alessandro Zaccagnini and the organisers of the "Terzo Incontro Italiano di Teoria dei Numeri", Scuola Normale Superiore, Pisa, 21-25 September 2015.

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