Riv. Mat. Univ. Parma, Vol. 7 (2016), 1-18

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On some differences between number fields and function fields

Abstract. The analogy between the arithmetic of varieties over number fields and the arithmetic of varieties over function fields is a leading theme in arithmetic geometry. This analogy is very powerful but there are some gaps. In this note we will show how the presence of isotrivial varieties over function fields (the analogous of which does not seem to exist over number fields) breaks this analogy. Some counterexamples to a statement similar to Northcott Theorem are proposed. In positive characteristic, some explicit counterexamples to statements similar to Lang and Vojta conjectures are given.

Keywords. Arithmetic over function fields, height theory, Lang and Vojta conjectures.

Mathematics Subject Classification (2010): 14G40, 14G22, 11G50.

Contents

Received: January 20, 2016; accepted in revised form: June 21, 2016.
Research supported by the FRIAS-USIAS.

1 - Introduction

Since the XIX century an analogy between the arithmetic of a number field and the arithmetic of a field of rational functions of an algebraic curve has been observed. For instance both are fields of fractions of suitable Dedekind domains where a so called product formula holds. This kind of fields is nowadays called a "global field". We expect that the arithmetic theory of the algebraic points of algebraic varieties over global fields may have similar features, thus a similar theory.

More concretely one expects that there should exist a "formal language" with many models. Some of these models are built using varieties over number fields and others are built using varieties over function fields. A statement proved in this language will give theorems in both theories.

Ideas of this type have many interesting applications: for instance the description of class field theory using adèles and idèles is one of the big achievements of this.

The theory of schemes in algebraic geometry also provides a good example of language which can be applied both over function fields and over number fields. Moreover, Arakelov theory pushes forward this analogy to obtain a good intersection theory which, with some caveat, is formally the same.

At the moment the language of the analogy is sufficiently developed in order to allow to formulate common conjectures and ideas. Lang and Vojta conjectures are leading ideas in this context. Over a number field, the Lang conjecture predicts that the rational points of a variety of general type should not be Zariski dense. Over a field of functions in characteristic zero, an analogous conjecture can be stated but one has to exclude varieties which, after a field extension, are birational to varieties defined over the base field (cf. after). One of the aims of this note is to show that, for function fields in positive characteristic, even a weak form of this is false.

Usually, when one wants to prove a theorem on the arithmetic of rational (algebraic) points of varieties over global fields, the situation is more favorable in the function field case. This is principally due to the fact that, over these fields, an horizontal derivation is available (there is a non trivial derivation over the base field). This is why many statements which are still conjectural on varieties over number fields are proved in the analogous situation over function fields. Consequently, it is widely believed that a conjecture in this theory should be checked before over function fields and then, once the proof is well understood there, one should try to attack it for varieties over number fields. We want to show, mainly by examples, that some part of height theory seems to better behave over number fields than over function fields. This, again, is due to the existence of the so called isotrivial varieties (the analogous of which does not seem to exist over number fields).

In the last part of this paper we will construct explicit examples of surfaces over a function field of positive characteristic which are of general type, are not birational to isotrivial surfaces and which are dominated by a surface defined over the base field. These surfaces will provide counterexamples to statements similar to Lang and Vojta conjectures.

The fact that part of the analogy is broken by the existence of isotrivial varieties is, in our opinion, a very important issue which should be analyzed more deeply. A better comprehension of it would probably improve aspects of the analogy and will lead to a development of the common language. This will allow to better formulate the leading conjectures of the theory.

This note is based on the talk I gave at the "Terzo incontro italiano di teoria dei numeri" held in Pisa in September 2015. I would like to thank the organizers, in particular Andrea Bandini and Ilaria Del Corso for the perfect organization of the meeting and for the possibility they gave me to give a talk.

2 - Notations and terminology

In the sequel K will be a global field. Thus K may be either a number field or the field of rational functions of a smooth projective curve B over the complex numbers or the field of rational functions of a smooth projective curve over an algebraically closed field k of positive characteristic. When the base field K is a number field we will say that "we are in the number field case", otherwise we will say that we are dealing with the "function field case".

In both situations we will denote by $\overline{K}$ the algebraic closure of K.

Let L/K be a finite extension. In the function field case, there is a unique smooth projective curve B_L with a finite morphism $\alpha : B_L \to B$. If we denote by g_L the genus of B_L, we will denote by d_L the number $\dfrac{2g_L - 2}{\deg(\alpha)}$. In the number field case, by analogy with the above, we will denote by d_L the logarithm of the absolute value of the relative discriminant of L over K.

We suppose now that we are in the function field case. In this case we will denote by k the field $\mathbf{C}$ or the aforementioned field k.

Let X_K be a smooth projective K–variety. By a model of X_K over B we mean a normal projective k-variety X (even smooth when k is $\mathbf{C}$) with a flat projective morphism $p : X \to B$ such that the following diagram is cartesian

$$\begin{array}{ccc} X_K & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(K) & \longrightarrow & B. \end{array}$$

It is very easy to construct models of X_K: a model of it may be realized as a closed set of $\mathbf{P}^N \times B$. Such a model, in general, won't be regular and not even normal. If we consider the normalization of it (and, in characteristic zero, resolution of singularities of it) one may always construct normal projective models of X_K (and even smooth, in characteristic zero).

If H_K is a line bundle over X_K, by a model of H_K over B we mean a couple (X, H) where X is a model of X_K over B and H is a line bundle over X whose restriction to X_K is H_K. Since every line bundle is the difference of very ample line bundles, models of (X, H) always exist.

Suppose that $p \in X_K(L)$ is an L-rational point and X is a model of X_K over B. By the valuative criterion of properness, there is an unique k–morphism $P : B_L \to X$ such that $p \circ P = \alpha$ and the following diagram is cartesian

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & B_L \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle P} \\ X_K & \longrightarrow & X. \end{array}$$

We will say that P is the model of the point p over X.

Suppose that X_K is a variety. We will say that X_K is *isotrivial* if we can find a variety X_0 *defined over* k and an isomorphism $X_K \times_K \mathrm{Spec}\,(\overline{K}) \simeq X_0 \times_k Spec\,(\overline{K})$.

For instance, the projective space $\mathbf{P}_N$ is isotrivial (and the isomorphism may be defined over K). If $K = k(t)$ and X_K is the curve $\{y^2z = x^3 + tz^3\} \subset \mathbf{P}^2$; then X_K is isotrivial but it is not defined over k : it will be isomorphic to $y^2z = x^3 + z^3$ over the field $k(t^{1/6})$.

Suppose that X_K is a smooth variety, let $f : X \to B$ be a model of it. If we restrict f to an open set U of B, we may suppose that the morphism f is smooth. The restriction to the generic fibre of the canonical exact sequence of differentials associated to f gives rise to an extension

$$0 \longrightarrow \mathscr{O}_X \longrightarrow E \longrightarrow \Omega^1_{X_K/K} \longrightarrow 0 \qquad (2.1)$$

which gives a class $KS(X_K) \in H^1(X_K, (\Omega^1_{X_K/K})^\vee)$, called the *Kodaira Spencer class of* X_K. It is independent on the model X. The following important fact holds:

Fact 2.1. *Let X_K be a smooth variety over a function field (of any characteristic). If the Kodaira Spencer class of X_K is non zero, then X_K is not isotrivial.*

Let's sketch why Fact 2.1 holds: suppose that there exists a smooth projective variety X_0 defined over k such that $X_0 \times_k K \simeq X_K$ (the isomorphism is defined over K), then one easily sees that $X_0 \times B$ is a model of X_K and the exact sequence (2.1) is split. If K'/K is a finite extension, denote by X' the K' variety $X_K \times_K K'$. One easily checks that one has an isomorphism $H^1(X_K, (\Omega^1_{X_K/K})^\vee) \otimes K' \simeq H^1(X_{K'}, (\Omega^1_{X_{K'}/K'})^\vee)$ and the image of $KS(X_K) \otimes 1$ via this isomorphism is $KS(X_{K'})$. Thus, if there exists a finite extension K'/K and an isomorphism $X_0 \times_k K' \simeq X_0 \times_K K'$, then $KS(X_K) \otimes 1 = 0$ and consequently $KS(X_K) = 0$.

One of the leading conjecture on arithmetic of varieties over global fields is the Lang conjecture: we recall that if X is a smooth projective variety defined over a field and K_X is the canonical bundle of it, then X_K is said to be *of general type* if $h^0(X_K, K_X^n) \sim_n n^{\dim(X)}$.

Conjecture 2.2 (Lang). *Let K be a global field of characteristic zero and X_K be a smooth projective variety of general type defined over K. If K is a function field, then we also suppose that X_K is not birational to an isotrivial variety. Then $X(K)$ is not Zariski dense.*

In the last section of this paper we will show that the hypothesis on the characteristic of the field is necessary.

3 - Height theories and remarks on Northcott theorem

Suppose that K is a global field as before. If X_K is a projective variety, we denote by $FUB(X_K)$ the group of functions $f : X_K(\overline{K}) \to \mathbf{R}$ modulo bounded functions.

The main properties of height theory for varieties over number fields may be resumed by the following statements:

suppose that K is a number field. There is a unique map of groups

$$h : Pic(X_K) \longrightarrow FUB(X_K)$$
$$L \longrightarrow h_L(\,\cdot\,)$$

(we will say that $h_L(\,\cdot\,)$ is the height associated to L), such that:

(i) it is functorial in X_F, i.e., if $\varphi : X_F \to Y_F$ is a morphism of varieties, then, for every $L \in Pic(Y_K)$ and every $p \in X_K(\overline{K})$ we have $h_L(\varphi(p)) = h_{\varphi^*(L)}(p)$;

(ii) if X_F is the projective space $\mathbf{P}_N$ and $L = \mathscr{O}(1)$, then the standard Weil height is in the class of $h_L(\,\cdot\,)$.

Moreover the following properties are verified:

(a) if D is an effective divisor on X_K and $L = \mathscr{O}_{X_K}(D)$, then $h_L \geq O(1)$ on $(X_K - D)(\overline{K})$.

(b) (Northcott Theorem) Let L_K be an ample line bundle over X_K and let $h_L(\,\cdot\,)$ be a function representing the height with respect to L_K. Suppose that A and B are positive constants. Then the set

$$\left\{p \in X_K(\overline{K}) \text{ s.t. } [K(p) : K] \leq B \ \text{ and } h_L(p) \leq A\right\}$$

is finite.

When K is a function field, a theory formally similar to height theory is available.

Suppose now that K is a function field. There is a unique map of groups $h_L : Pic(X_K) \to FUB(X_K)$ which verifies property (i) above and which verifies the following

(ii$'$) if X_F is the projective space $\mathbf{P}_N$ and $L = \mathscr{O}(1)$, then the class h_L is computed as follows: suppose that $p \in X_K(L)$ and $P : B_L \to \mathbf{P}^N$ is the associated morphism; then

$$h_L(p) = \frac{\deg\left(P^*(L)\right)}{[L : K]}\,.$$

It is easy to verify that a property similar to property (a) above holds in this case. Moreover the proof of this is formally the same in the function field and in the number field case.

On the opposite side, property (b) above fails in general.

We will now describe some examples which show the failure of Northcott property of heights over function fields.

Example 3.1. *Suppose that $X_K = \mathbf{P}_N$ and $L = \mathscr{O}(1)$. Then, every point $p \in X_K(k)$ gives rise to a point $p \in X_K(K)$ and it is easy to see that all these points have bounded height (the bound will depend on the model of X_K we choose). Moreover these points are Zariski dense.*

Of course one may object that the example above is isotrivial. But it is not hard to change it in a non isotrivial example:

Example 3.2. *Fix $r > N+4$ and consider r non trivial morphisms $f_i : B \to \mathbf{P}_N$. We suppose that the morphisms f_i are not conjugate under the action of $PGL(N+1)$. Each one of the f_i's defines a point $p_i \in P_N(K)$. None of these points is a point of $\mathbf{P}_N(k)$. Let X_K be the blow up of $\mathbf{P}_N$ in these points. Then:*

(1) *X_K is not isotrivial;*

(2) *The set of K–rational points of bounded height (with respect to an ample line bundle) is Zariski dense.*

Let's explain why (1) and (2) of example above hold.

(1) Let X_K be a K–variety and $X \to B$ a model of it. For every closed point $b \in B(k)$ we denote by X_b the fiber of X over b; it is a projective k-variety. The variety X_K is isotrivial if and only if there is a non empty open set $U \subseteq B$ such that, for every $b \in U(k)$, the variety X_b is k–isomorphic to a fixed k-variety X_0. We observe the following fact: if X_1 is the variety obtained by blowing up $\mathbf{P}_N$ in $N+4$ points in general position and X_2 is the variety obtained by blowing up $\mathbf{P}_N$ in another $N+4$–uple of points in general position (which is not in the $PGL(N+1)$–orbit of the previous one), then X_1 and X_2 are *not* isomorphic (one easily sees that they can be isomorphic if and only if the blown up points are in the same orbit under $PGL(N+1)$).

Consequently if the f_i's are not conjugate under $PGL(N+1)$ and b and b' are two general points of B, then the sets $\{f_i(b)\}$ and $\{f_i(b')\}$ are not conjugate under the same action. Thus the corresponding X_b and $X_{b'}$ are not isomorphic. Thus X_K is not isotrivial.

(2) Each point $q \in \mathbf{P}_N(k)$ rises to a point q_1 of X_K. Denote by $\pi : X_K \to \mathbf{P}_N$ the projection and by L the line bundle $\pi^*(\mathcal{O}(1))$. By functoriality, we have that for each point q_1 as above, $h_L(q_1)$ is bounded independently of q_1. Moreover L is a big bundle, thus we can find an effective divisor D and an ample divisor A on X_K such that, for n sufficiently big we have $nL = A + D$. By property (a) of heights, the height with respect to A of the points q_1 as above which are not in D is bounded from above independently of q_1.

The main criticism for the example above is that the variety X_K is *birational* to an isotrivial variety. If we focus our attention, not on rational points, but on points of bounded degree, even this objection can be abandoned.

Example 3.3. *Let X_K be* any *curve defined over K (isotrivial or not). Let $f : X_K \to \mathbf{P}_1$ be a morphism defined over K. Let L_K be the line bundle $f^*(\mathcal{O}(1))$ (it is an ample line bundle over X_K). Let $d = \deg(f)$. Fix a representative of $h_L(\,\cdot\,)$. Then* *we can find a constant A such that the set*

$$\{p \in X_L(\overline{K}) \ : \ [K(p) : K] \leq d \ \text{ and } \ h_L(p) \leq A\}$$

is infinite (thus Zariski dense).

Indeed, if we take a point $p \in X_K(\overline{K})$ such that $f(p) \in \mathbf{P}_1(k)$ then, by functoriality of the heights, we have that $h_L(p) \leq A$ for a suitable constant A independent of p. Such a p is defined over an extension of K which is of degree less than or equal to d (because, in particular $f(p) \in \mathbf{P}_1(K)$).

We remark that the example above may even be strengthened in characteristic zero (or when d is coprime to the characteristic of k) : a refinement of the argument above gives that there is a constant $B > d_K$ such that

$$\{p \in X_L(\overline{K}) \ : \ d_{K(p)} \leq B \ \text{ and } \ h_L(p) \leq A\}$$

is Zariski dense.

Indeed, extend the morphism f to a morphism F from a model X of X_K to $\mathbf{P}_1 \times B$.

Let R be the branch divisor of F. Let b be the degree of R over $\mathbf{P}_1$. If $p \in X_K(\overline{K})$ is a point such that $f(p) \in \mathbf{P}_1(k)$, then the curve $B_{K(p)}$ is a covering of B of degree at most d and ramified in at most b points. Thus, by Hurwitz formula, the genus of $B_{K(p)}$ is bounded independently of p.

At the moment the best result we know in the direction of an analogous of Northcott theorem in the function field case is the following theorem due to Moriwaki [**4**]:

Theorem 3.4. *Let X_K be a projective variety which is, either of general type or does not contain any rational curve. Let L_K be an ample line bundle over X_K and $h_{L_K}(\cdot)$ be a representative of the height with respect to it. Let A be a constant. Suppose that the set*

$$\{p \in X_K(K) \ : \ h_{L_K}(p) \leq A\}$$

is Zariski dense on X_K.

Then X_K is birational to an isotrivial variety.

Of course this theorem very well applies to curves, abelian varieties, geometrically hyperbolic varieties etc. but in our opinion it should be generalized and we should find the most general statement. For instance a statement which is true for varieties of arbitrary Kodaira dimension.

A refinement of the Lang conjecture above is the more ambitious Vojta conjecture:

Conjecture 3.5 (Vojta). *Suppose that K is a global field of characteristic zero, X_K is a smooth projective variety defined over it and K_X is the canonical line bundle of X_K. Then we can find a proper closed subset $Z \subsetneq X_K$ and a positive constant A such that, for every $p \in X_K(\overline{K}) - Z$ we have*

$$h_{K_X}(p) \leq A \cdot d_{K(p)} + O(1). \tag{3.1}$$

We remark that Vojta conjecture above implies Lang conjecture *only in the number field case.* In the function field case it implies some kind of arithmetic statement only if we can couple it with Theorem 3.4. It is known for curves, cf. for instance [**1**] where a stronger version of it holds. This version has been proved by Yamanoi and McQuillan (independently). Vojta conjecture holds also for varieties with ample cotangent bundle [**4**] and for a big class of surfaces [**3**]. In positive characteristic it is false, we show some counterexamples in the next section. Nevertheless one can see [**2**] for the case of curves in positive characteristic.

4 - Explicit counterexamples in positive characteristic

In this section we show that, if K is a function field in positive characteristic, we can always find explicit examples of varieties of general type which are non isotrivial and having a Zariski dense set of K–rational points. We will also show that, in some explicit examples, the set of rational points with bounded height is Zariski dense. Thus the Lang conjecture is false in this case and its statement should be corrected.

Let K be a field of positive characteristic $p > 2$ (algebraically closed in the first part of this section) and let X be a smooth projective variety defined over it. Let L be an ample line bundle over X. We fix a Zariski covering $\{U_i = \operatorname{Spec}(A_i)\}_{i \in I}$ by affine open sets of X and a cocycle $\{g_{ij}\}$ submitted to it and defining L.

Let $s \in H^0(X, L^p)$ be a non zero section. We may suppose that it is locally defined by functions $f_i \in A_i$ submitted to the conditions $f_i = g_{ij}^p f_j$ on $U_i \cap U_j$.

We associate to s an inseparable covering of X as follows: we consider the schemes $\operatorname{Spec}(A_i[z_i]/(z_i^p - f_i))$ glued together over $U_i \cap U_j$ by $z_i = g_{ij} z_j$. This gives rise to a scheme Z_s with a finite, totally inseparable morphism $f_s : Z_s \to X$. We will call Z_s *the inseparable ramified p–covering associated to s.*

Remark that the morphism f_s is actually ramified everywhere, but the name is chosen in analogy with the prime to p case.

The section s defines a global differential $d(s) \in H^0(X, \Omega^1_{X/F} \otimes L^p)$ as follows:

locally, over U_i we define $d(s)|_{U_i} := d(f_i)$. Since $f_i = g_{ij}^p f_j$ we have that $d(f_i) = g_{ij}^p d(f_j)$ over $U_i \cap U_j$. Thus the $d(f_i)$ glue to a global form $d(s) \in H^0(X, \Omega^1_{X/F} \otimes L^p)$.

4.1 - *Regularity of Z_s*

Let $z \in Z_s$ be a closed point and $x = f_s(z)$. Choose, over an algebraic closure $\overline{K}$ of K, an isomorphism between the completion $\widehat{\mathscr{O}_{X,x}}$ of the local ring of X at x and the ring $\overline{K}[\![x_1, \dots, x_n]\!]$. The restriction of $d(s)$ to $\widehat{\mathscr{O}_{X,x}}$ may be written as $h_1 d(x_1) + \cdots + h_n d(x_n)$.

Claim 4.1. *The point z is singular if and only if the ideal $(h_1, \dots, h_n)$ is contained in the maximal ideal of $\widehat{\mathscr{O}_{X,x}}$.*

Proof. The regularity of Z_s may be checked on the completions. Choose i such that $x \in U_i$. Then the restriction of Z_s to $Spf(\widehat{\mathscr{O}_{X,x}})$ is the formal scheme $Spf(\widehat{\mathscr{O}_{X,x}}[z]/(z^p - f_i))$. It is non regular if and only if $\dfrac{\partial}{\partial z}(z^p - f_i)$ and $\dfrac{\partial}{\partial x_j}(z^p - f_i)$ belong to the maximal ideal of $\widehat{\mathscr{O}_{X,x}}[\![z]\!]$ for all j. Since $\dfrac{\partial}{\partial z}(z^p - f_i) = 0$, and the ideal $\left(\dfrac{\partial}{\partial x_1}(z^p - f_i), \dots, \dfrac{\partial}{\partial x_n}(z^p - f_i)\right)$ coincides with the ideal $(h_1, \dots, h_n)$ the claim follows. □

Let $z \in Z_s$ be a closed singular point and $x = f_s(z)$. Suppose that the matrix $\dfrac{\partial h_i}{\partial x_j}(0)$ is non singular. Then we will say that z is a *non degenerate singular point*. One may check that the notion of "non degenerate singular point" depends only on the divisor $div(s)$. In particular it does not depend on the choice of the coordinates around x.

4.2 - *Structure and desingularization of Z_s near a non degenerate singular point*

Claim 4.2. *Suppose that the point $z \in Z_s$ is a non degenerate singular point and $x = f_s(z)$. Then there exist formal coordinates $x_1, \dots, x_n$ on $\widehat{\mathscr{O}_{X,x}}$ for which Z_s is given by the equations*

$$z^p = x_1^2 + \cdots + x_n^2.$$

Proof. Locally, near x, the variety Z_s is defined by the equation $z^p = f(x_1, \dots, x_n)$ with $\det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)(x) \neq 0$. Denote by $\mathscr{M}_{X,x}$ the maximal ideal of $\widehat{\mathscr{O}_{X,x}}$. Since z is singular, we have that $f \equiv a_0 + \dfrac{1}{2}\sum_{i,j} a_{ij} x_i x_j \bmod (\mathscr{M}^3_{X,x})$ in $\widehat{\mathscr{O}_{X,x}}$ with $a_{ij} = a_{ji}$; moreover the symmetric matrix (a_{ij}) is non singular because the singularity is non degenerate. The change of variable $z_1 := z - a_0$ gives the new equation $z_1^p = f_1$ for Z_s near x, with $f_1(x) = 0$ and $\dfrac{\partial f_1}{\partial x_j}(x) = 0$. To prove the claim it suffices to prove that we can choose formal coordinates $x_1, \dots, x_n$ such that, for every r we have $f_1(x_1, \dots, x_n) \equiv x_1^2 + \dots + x_n^2 \bmod (\mathscr{M}_{X,x}^r)$. Since we are in characteristic different from two and $\det(a_{ij}) \neq 0$, we may suppose that the bilinear form $\sum_{ij} a_{ij} x_i x_j$ is diagonal. Consequently we may suppose by induction on r, that $f_1 \equiv x_1^2 + \dots + x_n^2 \bmod (\mathscr{M}_{X,x}^{r+2})$. Thus $f_1 \equiv x_1^2 + \dots + x_n^2 + \sum_{|I|=r+2} a_I x^I \bmod (\mathscr{M}_{X,x}^{r+3})$, where $I = (i_1, \dots, i_n)$ is a multi-index. Choose a change of variable $x_i = \tilde{x}_i + \sum_{|J|=r+1} b_J^i \tilde{x}^J$. In the new coordinates we have that

$$f_1(\tilde{x}_1, \dots, \tilde{x}_n) \equiv \tilde{x}_1^2 + \dots + \tilde{x}_n^2 + 2 \sum_{i,J} b_J^i x^J \cdot \tilde{x}_i + \sum_{|I|=r+2} a_I \tilde{x}^I \mod (\mathscr{M}_{X,x}^{r+3}).$$

Thus a suitable choice of the b_J^i's allows to obtain that $f_1(\tilde{x}_1, \dots, \tilde{x}_n) \equiv \tilde{x}_1^2 + \dots + \tilde{x}_n^2 \bmod (\mathscr{M}_{X,x}^{r+3})$. □

We suppose that Z_s has only non degenerate singular points. In this case we remark that the singular points are isolated. We begin by studying the desingularization of an affine hypersurface Z whose equation is

$$z^p = x_1^2 + \dots + x_n^2. \tag{4.1}$$

Proposition 4.3. *The desingularization of the hypersurface* (4.1) *is obtained by performing p blow ups on isolated singular points. Each of these points is of multiplicity two.*

Proof. Let $f : \widetilde{X} \to \mathbb{A}^{n+1}$ be the blow up of the point $(0; 0; \dots; 0)$. The local equations of it are given by $z = vx_i$ and $x_j = u_j x_i$ $(i = 1, \dots, n)$ or by $x_i = w_i z$. We denote by E the exceptional divisor of $\widetilde{X}$.

In the first case the local equation of the strict transform $\widetilde{Z}$ of the hypersurface (4.1) is

$$v^{p-2} x_i^{p-2} = 1 + u_1^2 + \dots + u_n^2 \tag{4.2}$$

(the i–term is not part of the sum). In this case we remark that the local equation is smooth (because the characteristic of the field is not two). In the second case the equation of the strict transform is

$$z^{p-2} = w_1^2 + \dots + w_n^2 \tag{4.3}$$

(to simplify notation we put $x_i = w_i$). Denote by $\widetilde{Z}$ the strict transform of Z. We see that $f^*(\mathscr{O}(Z)) = \mathscr{O}(\widetilde{Z})(2E)$ thus the multiplicity of the singular point is two. If we blow up again the origin of the last chart we obtain that the equation of the strict transform will be $z^{p-4} = w_1^2 + \cdots + w_n^2$ and the multiplicity of the singular point is again two.

Thus after $\frac{p-1}{2}$ blow ups, the local equation of the strict transform is

$$z = w_1^2 + \cdots + w_n^2 \tag{4.4}$$

which is smooth and again the multiplicity of the last singular point is two. □

As a corollary of the proof we obtain the following:

Corollary 4.4. *Let X be a smooth variety and $Z \subset X$ be an hypersurface on it. Suppose that Z has an isolated singular point P and the local formal equation of Z near it is of the form* (4.1). *Let $X_1 \to X$ be the blow up of X in P, Z_1 be the strict transform of Z and E_1 be the exceptional divisor of X_1. Recursively, let $X_i \to X_{i-1}$ be the blow up of X_{i-1} in the singular point of Z_{i-1}, denote by Z_i the strict transform of Z_{i-1} and by E_i the exceptional divisor of X_i. By abuse of notation, for $j < i$, we denote by E_j the pull back of the divisor E_j to X_i. Then:*

(*a*) *$Z_{(p-1)/2}$ is smooth;*

(*b*) *if $f : X_{(p-1)/2} \to X$ is the projection, then*

$$f^*(\mathscr{O}(Z)) = \mathscr{O}(Z_{(p-1)/2})\Big(-\sum_{i=1}^{\frac{p-1}{2}} E_i\Big). \tag{4.5}$$

4.3 - *Inseparable ramified covering of general type*

Suppose now that X is a smooth projective variety of dimension N and L a very ample line bundle on it. Let $s \in H^0(X, L^{np})$ ($n > 0$ sufficiently big) be a global section such that $\operatorname{div}(s)$ is smooth and $f : Z_s \to X$ the inseparable ramified covering associated to it. We suppose that Z_s has only non degenerate singular points.

Proposition 4.5. *In the hypotheses above, let $\widetilde{Z_s} \to Z_s$ be its desingularization* (*it exists by Corollary* 4.4). *If n is sufficiently big then the variety $\widetilde{Z_s}$ is a smooth projective variety of general type.*

Proof. The variety Z_s is a divisor inside the smooth projective variety $Y := \mathbf{P}(\mathscr{O}_X \oplus L^n)$. The variety $\widetilde{Z_s}$ is obtained as the strict transform of Z_s in the variety $g : \widetilde{Y} \to Y$ obtained by taking consecutive blow ups at smooth closed points. Denote by E_{ij} the exceptional divisors of $\widetilde{Y}$.

The canonical line bundle of $\widetilde{Y}$ will be $g^*(K_Y) + N\sum_{ij} E_{ij} = g^*(\mathscr{O}_{\mathbf{P}}(-2) + L^n + K_X) + N\sum_{ij} E_{ij}$ (we adopt the abuse of notation of Corollary 4.4).

The class of Z_s in Pic(Y) will be $\mathscr{O}_{\mathbf{P}}(p) + L^{np}$. Thus it is ample on Y. The class of $\widetilde{Z_s}$ in Pic$(\widetilde{Y})$ will be (cf. 4.4) $g^*(\mathscr{O}_{\mathbf{P}}(p) + L^{np}) - 2\sum_{ij} E_{ij}$. Consequently, by adjunction formula, we have that

$$K_{\widetilde{Z_s}} = (K_{\widetilde{Y}} + \widetilde{Z_s})|_{\widetilde{Z_s}} = (g^*(\mathscr{O}_{\mathbf{P}}(p-2) + L^{np+1} + K_X) + (N-2)\sum_{ij} E_{ij})|_{\widetilde{Z_s}}. \tag{4.6}$$

As soon as n is sufficiently big, the line bundle $g^*(\mathscr{O}_{\mathbf{P}}(p-2) + L^{np+1} + K_X)$ is ample on Z_s. Thus, for n sufficiently big, the restriction of $g^*(K_Y + L^{np})$ is a big and nef line bundle on $\widetilde{Z_s}$. The divisor $(N-2)\sum_{ij} E_{ij}$ is effective. Since an effective divisor plus a big and nef is big, the conclusion follows. □

We show now that, if $s \in H^0(X, L^{np})$ is sufficiently generic and n is sufficiently big, then the associated inseparable ramified covering Z_s has only non degenerate singular points:

Proposition 4.6. *Suppose that, L is very ample and for every $x \in X$ the restriction map*

$$\alpha : H^0(X, L^{np}) \longrightarrow L^{np} \otimes \mathscr{O}_X / I_x^3 \tag{4.7}$$

is surjective (I_x being the ideal sheaf of x). Then for a generic $s \in H^0(X, L^{np})$, the inseparable ramified covering Z_s has only non degenerate singular points.

Proof. Let x be a point of X and $s \in H^0(X, L^{np})$. If we fix (formal) local coordinates $z_1, \ldots, z_N$ and a local trivialization f of s around x, then $\alpha(s) = f(x) + \sum_i f_{z_i}(x) z_i + \frac{1}{2}\Big(\sum_{ij} f_{z_i,z_j}(x) z_i z_j\Big)$. Since the map (4.7) is surjective, for generic s, the divisor div(s) will be smooth and the quadratic form associated to the matrix (f_{z_i,z_j}) will be non degenerate. In this case the associated inseparable ramified covering Z_s will have non degenerate singular points over x. We thus see that the set of $s \in H^0(X, L^{np})$ for which the associated inseparable ramified covering Z_s has a singularity which is degenerate at x, is a closed set of codimension $N+2$ which we will denote by S_x. Indeed the elements of the vector space $\mathscr{O}_X / I_x^3$ for which the associated quadratic form is degenerate is a closed subvariety of codimension N+2. We will denote again by S_x the image of S_x in $\mathbf{P}(H^0(X, L^{np}))$; it will be again a closed set of codimension $N+2$. For a fixed s the set of degenerate singular points of Z_s is a closed set whose projection on X will be denoted by N_s.

Let $W \subset X \times \mathbf{P}(H^0(X, L^{np}))$ be the universal divisor and N_W the corresponding closed set of non degenerate singular points. For every $x \in X$, the restriction $(N_W)_x$ of N_W to $\{x\} \times \mathbf{P}(H^0(X, L^{np}))$ will be S_x. Thus the dimension of N_W is $h^0(X, L^{np}) - 1 - (N+2) + N = h^0(X, L^{np}) - 3$. This means that N_W does not dominate $\mathbf{P}(H^0(X, L^{np}))$. Consequently, for generic $s \in \mathbf{P}(H^0(X, L^{np}))$, the corresponding Z_s has only non degenerate singular points. □

4.4 - *Non isotrivial inseparable ramified coverings*

Suppose now that K is a function field of positive characteristic p. Suppose that X is a variety defined over the base field k and L is an ample line bundle over it. Let $s \in H^0(X, L^{np})$ be a smooth section and $g : Z_s \to X$ the associated inseparable ramified covering. Denote by Y_s the divisor $\mathrm{div}(s)$. We are going to relate the Kodaira–Spencer class of Y_s with the Kodaira–Spencer class of $\widetilde{Z_s}$:

$\widetilde{Z_s}$ is a divisor in a blow up of the projective bundle $\mathbf{P} := \mathbf{P}(\mathcal{O}_X \oplus L^n)$. Let $\mathcal{O}_{\mathbf{P}}(1)$ be the tautological line bundle of $\mathbf{P}$.

We fix formal coordinates $x_1, \ldots, x_n$ of X and a local equation $f = 0$ of s around a point of Y_s. Thus a local equation for Z_s is $z^p = f$.

(a) The sheaf of differentials $\Omega^1_{Y_s/K}$ is given by $(\oplus_{i=1}^n \mathcal{O}_{Y_s} dx_i)/df$.

(b) The sheaf of differentials $\Omega^1_{Z_s/K}$ is given by $(\mathcal{O}_{Z_s} dz \oplus_{i=1}^n \mathcal{O}_{Z_s} dx_i)/df$ (observe that the relations do not contain dz).

(c) Let W_s be the divisor pre image of Y_s in Z_s. Its local equation in Z_s is $f = 0$. Denote by $g_s : W_s \to Y_s$ the restriction of g to W_s. From (a) and (b) above we see that the natural map

$$(\Omega^1_{Z_s/K})|_{W_s} \longrightarrow \Omega^1_{W_s/K} \tag{4.8}$$

is an isomorphism.

(d) Locally the sheaf $\mathcal{O}_{Y_s}$ is $A/(f)$ and the local sheaf of W_s is $(A/(f)[z])/(z^p)$. Thus the natural inclusion $\mathcal{O}_{Y_s} \to g_{s,*}(\mathcal{O}_{W_s})$ is split (remark that no singular point of Z_s is located on W_s). This, together with (c) above implies that the natural map

$$\alpha_{Y_s} : H^1(Y_s, (\Omega^1_{Y_s/K})^\vee) \longrightarrow H^1(W_s, g_s^*(\Omega^1_{Y_s/K})^\vee) \tag{4.9}$$

is an inclusion.

(e) Again, by the descriptions in (a), (b) and (c) above we get an exact sequence

$$0 \to f_s^*(\Omega^1_{Y_s/K}) \longrightarrow \Omega^1_{W_s/K} \longrightarrow \mathcal{O}(1) \otimes L^{np} \to 0. \tag{4.10}$$

This exact sequence, together with (d) gives rise to *an inclusion*

$$\alpha_{Y_s} : H^1(Y_s, (\Omega^1_{Y_s/K})^\vee) \longrightarrow H^1(W_s, (\Omega^1_{W_s/K})^\vee). \tag{4.11}$$

(f) From the descriptions above and taking duals we get natural maps

$$H^1(\widetilde{Z}_s, (\Omega^1_{\widetilde{Z}_s/K})^\vee) \xrightarrow{\alpha_{Z_s}} H^1(W_s, (\Omega^1_{W_s/K})^\vee) \xleftarrow{\alpha_{X_s}} H^1(Y_s, (\Omega^1_{Y_s/K})^\vee). \tag{4.12}$$

A simple (but tedious) diagram chasing gives $\alpha_{Z_s}(KS(\widetilde{Z}_s)) = \alpha_{X_s}(KS(Y_s))$.

Thus we deduce the following statement:

Proposition 4.7. *The non vanishing of the of Kodaira Spencer class of Y_s implies the non vanishing of the Kodaira Spencer class of the variety $\widetilde{Z}_s$.*

From the constructions above we get the following theorem:

Theorem 4.8. *Suppose that X is a smooth projective surface defined over the base field k and L is a sufficiently ample line bundle over it. Let X_K be the base change of X to K and $s \in H^0(X_K, L^{np})$ be a non isotrivial smooth divisor. Then the associated inseparable ramified covering Z_s is not birational to an isotrivial surface.*

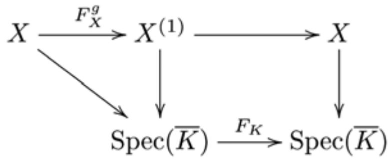
Proof. From Proposition 4.7 and Fact 2.1 we get that $\widetilde{Z}_s$ is not isotrivial. Formula (4.6) computes the canonical line bundle of $\widetilde{Z}_s$. Thus we get that $\widetilde{Z}_s$ is of general type and minimal. Since two minimal surfaces of general type are isomorphic if and only if they are birationally equivalent, the theorem follows. □

Remark 4.9. *In higher dimension we can only conclude that the variety $\widetilde{Z}_s$ is not defined over k. It is possible that a finer study, using MMP, may allow to deduce that $\widetilde{Z}_s$ is not birational to a variety defined over k.*

4.5 - *Inseparable ramified coverings and Frobenius*

We recall here some standard facts about the Frobenius morphism of a variety. Let $\overline{K}$ be the algebraic closure of K. If X is a variety over $\overline{K}$, we denote by $F_X : X \to X$ the Frobenius morphism (it is the identity on the topological space and $f \to f^p$ on functions). The Frobenius morphism fits inside a diagram

(4.13)



where, F_K is the Frobenius morphism of K, the square on the right is cartesian and F_X^g is a $\overline{K}$ morphism called *the geometric Frobenius*. Suppose now that X is a smooth projective $\overline{K}$ variety and $\overline{K}(X)$ is the field of rational functions of it. If $\overline{K}(X^{(1)}) = \overline{K}(x_1, \ldots, x_r)$ then the field morphism associated to F_X^g is

$$\overline{K}(x_1, \ldots, x_r) \xrightarrow{F_X^g} \overline{K}(x_1, \ldots, x_r)[T_1, \ldots, T_r]/_{(T_1^p - x_1, \ldots, T_r^p - x^r)} = \overline{K}(X).$$

Suppose now that $f : \widetilde{Z_s} \to X^{(1)}$ is an inseparable ramified morphism associated to a global section of a line bundle over $X^{(1)}$. Then the field of rational functions of $\widetilde{Z_s}$ is $\overline{K}(\widetilde{Z_s}) = \overline{K}(X)[z]/(z^p - h)$ where h is a suitable rational function over $X^{(1)}$. Write $h = \sum a_I x^I$ where I is a multi-index $(i_1, \ldots, i_r)$, $a_I \in \overline{K}$ and $x^I := x_1^{i_1} x_2^{i_2} \ldots x_r^{i_r}$. For every I let $b_I \in \overline{K}$ be such that $b_I^p = a_I$. Thus we obtain an inclusion $\overline{K}(\widetilde{Z_s}) \hookrightarrow \overline{K}(X)$ by sending z to $\sum b_I T^I$.

Consequently we get the following:

Proposition 4.10. *Let X be a smooth projective variety defined over K and $f : \widetilde{Z_s} \to X^{(1)}$ be an inseparable ramified covering associated to a section of a suitable line bundle on it. Then there exists a finite extension K' of K, a blow up $\widetilde{X} \to X$ and a dominant (inseparable) morphism $h : \widetilde{X} \to \widetilde{Z_s}^{(1)}$.*

4.6 - *Inseparable ramified coverings and arithmetic over function fields*

Let K be a function field of one variable over an algebraically closed field k of characteristic $p > 0$. From the construction above we see that, given a smooth projective surface X_0 defined over the base field k, we can construct surfaces $\widetilde{Z_s}^{(1)}$ over K such that:

(a) $\widetilde{Z_s}^{(1)}$ is smooth, projective and of general type.

(b) $\widetilde{Z_s}^{(1)}$ is not birational to an isotrivial surface.

(c) There is a blow up $\widetilde{X_0}$ of $X_0 \otimes_k K$ and a dominant (non separable) morphism $f : \widetilde{X} \otimes_k K \to \widetilde{Z_s}^{(1)}$.

To prove (c) just remark that if Y is a variety, then Y is defined over k if and only if $Y^{(1)}$ is.

We list now two important consequences of this:

(1) *The image by f of each k point of $\widetilde{X}_0 \otimes_k K$ is a K-rational point of $\widetilde{Z_s}^{(1)}$.*

Consequence: The set of $K-$rational points of bounded height in $\widetilde{Z_s}^{(1)}$ is Zariski dense.

(2) *Suppose that $X_0 = \mathbf{P}_2$. Then every form of Vojta inequality fails for $\widetilde{Z_s}^{(1)}$.*

Let's give some details about the proof of consequence (2): in this case a model of X_0 over B is $\mathbf{P}_2 \times B$. Fix a normal projective model $\overline{Z} \to B$ of $\widetilde{Z_s}^{(1)}$. Then (up to an extension of K if necessary), we can find a proper closed set $W \subset \mathbf{P}_2 \times B$ of codimension at least two such that, if $X_1 \to \mathbf{P}_2 \times B$ is the blow up of it, we have a dominant map $h : X_1 \to \overline{Z}$. The lemma below tells us that we can find a Zariski dense set of points $p \in X_1(K)$ having constant discriminant d_p and unbounded height with respect to an (any) ample line bundle. Indeed the pre image in $\mathbf{P}_2 \times B$ of almost every line in $\mathbf{P}_2$ will intersect W in only finitely many points.

The image via h of these points is a set of points which violates Vojta inequality. □

Lemma 4.11. *Let B be a smooth projective curve and W be a finite set of points in $B \times \mathbf{P}_1$, then there are infinitely many sections $g : B \to B \times \mathbf{P}_1$ which do not intersect W.*

Proof. It suffices to observe that we can find a line bundle L on B such that $M := p_{\mathbf{P}_1}^*(\mathscr{O}_{\mathbf{P}}(1)) \otimes p_B^*(L)$ is very ample on $B \times \mathbf{P}_1$. Every smooth global section of M which avoids W satisfies the conclusion of the lemma. □

Consequences (1) and (2) above show that a "naive" version of Lang and Vojta conjectures are definitely false in positive characteristic. Once again this is due to the existence of isotrivial varieties (which in positive characteristic are even more mysterious then in characteristic zero).

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