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A few remarks on a Manin-Mumford conjecture in function field arithmetic and generalized Pila-Wilkie estimates

Abstract. We present here the natural extension of our Pila-Wilkie type estimates on the number of rational points of the trascendent part of a compact analytic subset of $\mathbb{F}_q((1/T))^n$ (see [**D1**]) to analogous subsets of K^n, where K is a general local field of any characteristic. That would integrate the analogous estimate provided by F. Loeser, G. Comte and R. Cluckers in [**CCL**, Theorem 4.1.6]. We remind in the first two sections the main ideas of our construction by correcting two minor mistakes we made in [**D1**]. We then generalize the strategy to any local field.

Keywords. $T-$modules, rational points, Manin-Mumford conjecture, local fields.

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1 - Introduction

Most of the notations we use come from [**D2**] and [**D1**]. In particular we put $A := \mathbb{F}_q[T]$ and $k := \mathbb{F}_q(T)$. The completion of k with respect to the place at infinity, represented by the $1/T-$adic valuation, is then $k_\infty := \mathbb{F}_q((1/T))$. By choosing $\overline{k_\infty}$ an algebraic closure of k_∞ and calling $\mathcal{C} := (\overline{k_\infty})_\infty$, we let $\overline{k}$ be the algebraic closure of k in $\mathcal{C}$.

Definition 1.1. A $T-$module of dimension m and degree $\widetilde{d}$ is a couple $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$, where $\mathbb{G}_a$ is the algebraic additive group over $\mathcal{C}$ and Φ the $\mathbb{F}_q-$algebra

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homomorphism defined from $\mathbb{F}_q[T]$ to $\overline{k}^{m,m}\{\tau\}$ (see [**D2**, Definition 1.5]) such that

$$\Phi(T)(\tau) = a_0 + a_1\tau + ... + a_{\tilde{d}}\tau^{\tilde{d}} \tag{1}$$

where $a_0, ..., a_{\tilde{d}} \in \overline{k}^{m,m}$ and $a_0 = T1_m + N$, for some nilpotent matrix $N \in \overline{k}^{m,m}$.

The constant coefficient matrix a_0 is called the **differential** of $\Phi(T)$ and it is denoted as $d\Phi(T)$. We define similarly $d\Phi(a(T))$ for any $a(T) \in A$. Therefore it is clear that $d(\Phi(a(T))) = a(T)1_m + N_{a(T)}$ for a nilpotent matrix $N_{a(T)}$. We also remind that a $T-$module $\mathcal{A}$ is **abelian** if its **rank** (see [**D2**, Definition 1.12]) is finite. We also denote by $k(\Phi)$ the *definition field* of $\mathcal{A}$: it is the finite field extension of k in $\overline{k}$ generated by the entries of the coefficients of $\Phi(T)$.

Definition 1.2. Let $\mathcal{A} = (\mathbb{G}_a^{m_\mathcal{A}}, \Phi)$ be a $T-$module. A connected reduced algebraic subgroup $\mathcal{B}$ of $\mathcal{A}$ of dimension $m_\mathcal{B} < m_\mathcal{A}$, isomorphic to $\mathbb{G}_a^{m_\mathcal{B}}$ is called a **sub**$-T-$**module** of $\mathcal{A}$ if $\Phi(T)(\mathcal{B}) \subseteq \mathcal{B}$. If B is a subring of A and $\Phi(a(T))(\mathcal{B}) \subseteq \mathcal{B}$ for every $a(T) \in B$, we call $\mathcal{B}$ a **sub**$-B-$**module** of $\mathcal{A}$ (see [**D2**] for more details).

The main purpose of our work in [**D1**] was to outline a possible strategy for proving a general adaptation of the Manin-Mumford conjecture to the context of abelian and **uniformizable** $T-$modules (we introduce such a special class of $T-$modules in the next section). We showed in [**D2**] that a naïve version of this statement is false in general and we exhibited several counterexamples by remarking firstly that a theoretical statement must involve, as analogues of abelian subvarieties, all the possible sub$-B-$modules of the given $\mathcal{A}$ (see [**D2**, Proposition 2.5]). Moreover, as we showed in [**D2**, Proposition 2.12], one may still find counterexamples produced by an insufficient "degree of abelianity" of $\mathcal{A}$ (see the discussion after [**D2**, Proposition 2.12]). We had therefore to strenghten the hypotheses on the finiteness of the rank of the $T-$motive associated to $\mathcal{A}$ (see [**D2**, Theorem 2.13] and [**D2**, Proposition 2.15]). The final statement we propose is the following one.

Conjecture 1. *Let $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ be an abelian uniformizable $T-$module, such that there exists $i \in \mathbb{N} - \{0\}$ such that the leading coefficient of the form $\Phi(T^i)$ is an invertible matrix. Let X be an algebraic subvariety of $\mathcal{A}$ not containing translates by torsion points of $\mathcal{A}$ of nontrivial sub$-B-$modules of $\mathcal{A}$ for any B subring of A. Then X contains only finitely many torsion points of $\mathcal{A}$.*

The strategy we follow to prove such a statement is described in [**D1**] and is based essentially on the ideas of U. Zannier and J. Pila (see [**PZ**]) to provide an alternative

proof of the Manin-Mumford conjecture (M. Raynaud's theorem) in the setting of abelian varieties over number fields. As we will see in the next section, there exists a particular class of abelian $T-$modules, called **uniformizable**, which presents several analogies with abelian varieties. In particular, a $T-$module $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ of such a class is endowed with a homeomorphic $\mathbb{F}_q[T]-$module isomorphism between $\mathcal{A}$ and the quotient of $\mathcal{C}^m$ by a $\mathbb{F}_q[T]-$lattice $\Lambda_{\mathcal{A}} := \Lambda$ of rank equal to the rank of $\mathcal{A}$. Such an isomorphism is called the **exponential map** associated to $\mathcal{A}$. The **tangent space** of $\mathcal{A}$ at $\overline{0}$ is the Lie algebra of $\mathbb{G}_a^m$ and is therefore denoted by $Lie(\mathcal{A})$.

In [**D1**] we use the properties we have described right above to establish a relation between the set of torsion points of $\mathcal{A}$ and the set of $k-$rational points of $Lie(\mathcal{A})/\Lambda$. As we have remarked in [**D1**, Proposition 5], if $d\Phi(T)$ is diagonal, we have the analogous situation we know already for abelian varieties, that is the two sets are in bijection via the exponential map. More precisely, we remarked that by assuming the rank of Λ to be d, we have that $Lie(\mathcal{A})$ is, as $k_\infty-$vector space, the direct sum of a $d-$dimensional subspace in which Λ is cocompact and a *free* subspace. Up to a change of basis, we may therefore assume Λ to be A^d and $Lie(\mathcal{A})/\Lambda \simeq (k_\infty/A)^d \oplus Free(k_\infty)$. This means that the exponential map induces a homeomorphism of $A-$modules

$$Lie(\mathcal{A})/\Lambda \simeq \mathcal{A} \tag{2}$$

which puts the torsion points of $\mathcal{A}$ in bijection with the $\overline{z} \in Lie(\mathcal{A})$ such that $d\Phi(a(T)) \cdot \overline{z} \in \Lambda$. Our strategy requires the proof of several intermediate statements (see [**D1**] for more details). The first of them is the main result in [**D1**] and will be recalled in the next section (see Theorem 2.4).

Such a theorem provides an upper bound estimate for the number of $k-$rational points in the transcendent part of an analytic subset of $Lie(\mathcal{A})/\Lambda$ and will be generalized in the present paper to a unified statement holding for a non-Archimedean local field of any characteristic.

We thus call (K, v) such a field. It is well known that K is the completion with respect to v of a global field which is a finite extension of $\mathbb{Q}$ if $char(K) = 0$, or a finite extension of $\mathbb{F}_p(T)$ if $char(K) = p$. Then K is either a finite field extension of $\mathbb{Q}_p$ for some prime number p, or it is isomorphic to a field of the form $\mathbb{F}_q((x))$ for some power q of p, where the variable x is a $\mathbb{F}_q-$rational function of $T^{1/n}$ for some $n \in \mathbb{N} - \{0\}$. Let us call $\mathbb{Z}_K := \mathbb{Z}$ and $\mathbb{Q}_K := \mathbb{Q}$ if $char(K) = 0$ and $\mathbb{Z}_K := \mathbb{F}_q[T]$ and $\mathbb{Q}_K := \mathbb{F}_q(T)$ if $char(K) = p$. Since in all cases $|\mathbb{Z}_K|_v \subseteq \mathbb{N}$, if we define a height function on $\mathbb{Q}_K^n$ ($n \in \mathbb{N} - \{0\}$) as

$$\widetilde{H}(\overline{z}) = \widetilde{H}\Big(\frac{a_1}{b_1}, ..., \frac{a_n}{b_n}\Big) := \max_{i=1,...,n} \{\max\{|a_i|_v, |b_i|_v\}\} \tag{3}$$

then we obviously have $\widetilde{H}(\overline{z}) \in \mathbb{N}$. We then prove the following statement.

Theorem 1.1. *Let W be an irreducible $K-$analytic subset of K^m for some $m \in \mathbb{N} - \{0,1\}$. We define, for any $t \in \mathbb{N} - \{0\}$, the following number*

$$N(W - W^{alg.}, t) := |\{\overline{z} \in (W - W^{alg.})(\mathbb{Q}_K), \widetilde{H}(\overline{z}) \leq t\}|. \tag{4}$$

For each real number $\varepsilon > 0$ there exists $c = c(W, \varepsilon) > 0$ such that

$$N(W - W^{alg.}, t) \leq ct^{\varepsilon}. \tag{5}$$

We prove Theorem 1.1 in the third and last section. The next one will be devoted to correct two minor mistakes we made in [**D1**], giving us the opportunity to recall the main tools and ideas on which our arguments in [**D1**] were based and to present a few remarks and examples related to the intrinsic nature of the algebraic and arithmetic objects involved in this setting.

2 - Manin-Mumford conjecture in function field arithmetic

We begin by recalling the crucial definition of the **exponential map**.

Theorem 2.1. *Let $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ be an abelian $T-$module. We call $Lie(\mathcal{A})$ the* **tangent space** *associated to $\mathcal{A}$. Now, given a sub$-T-$module $\mathcal{B}$ of $\mathcal{A}$, having dimension $m_{\mathcal{B}} \leq m$ (so not necessarily different from $\mathcal{A}$), let $Lie(\mathcal{B})$ be the vector subspace of $Lie(\mathcal{A})$ defined as the usual Lie algebra of $\mathcal{B}$. Then there exists a unique $\mathbb{F}_q-$linear $k(\Phi)-$analytic map*

$$\overline{e}_{\mathcal{B}} : Lie(\mathcal{B}) \to \mathcal{B} \tag{6}$$

such that

$$\Phi(a(T))(\overline{e}_{\mathcal{B}}(\overline{z})) = \overline{e}_{\mathcal{B}}(d\Phi(a(T)) \cdot \overline{z}) \tag{7}$$

for all $a(T) \in A$ and for all $\overline{z} \in Lie(\mathcal{B})$. Such a map is called the **exponential function** *of $\mathcal{B}$.*

Proof. The proof of this very important result was provided by G. Anderson and it can be found in [**Goss**, Chapter 5, Section 9]. □

Proposition 2.1. *The exponential map $\overline{e}_{\mathcal{A}} : Lie(\mathcal{A}) \to \mathcal{A}$ of an abelian $T-$module $\mathcal{A}$ restricts to $Lie(\mathcal{B})$ to the exponential map $\overline{e}_{\mathcal{B}} : Lie(\mathcal{B}) \to \mathcal{B}$ for any $\mathcal{B}$ sub$-T-$module of $\mathcal{A}$.*

Proof. By [**D2**, Proposition 1.14], we know that $\overline{e}_{\mathcal{A}}(Lie(\mathcal{B})) \subseteq \mathcal{B}$. As $\overline{e}_{\mathcal{A}}$ and $\overline{e}_{\mathcal{B}}$ respect the same properties we previously listed in Theorem 2.1, the statement follows from the uniqueness of the exponential map. □

Corollary 2.1. *Given an abelian $T-$module $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$, the vector subspace $Lie(\mathcal{B})$ of $Lie(\mathcal{A})$ is* **invariant** *under the differential $d\Phi(T)$ of $\mathcal{A}$ for every $\mathcal{B}$ sub$-T-$module of $\mathcal{A}$. In other words*

$$d\Phi(T) \cdot Lie(\mathcal{B}) \subseteq Lie(\mathcal{B}).$$

Proof. As $\overline{e}_{\mathcal{B}}(d\Phi(T)\overline{z}) = \overline{e}_{\mathcal{A}}(d\Phi(T)\overline{z})$ for all $\overline{z} \in Lie(\mathcal{B})$ and $\overline{e}_{\mathcal{B}}$ is defined on $Lie(\mathcal{B})$ only (while on the contrary $\overline{e}_{\mathcal{A}}$ takes values in $\mathcal{B}$ on the a priori bigger set $\overline{e}_{\mathcal{A}}^{-1}(\mathcal{B})$), it follows that for all $\overline{z} \in Lie(\mathcal{B})$ one has that $d\Phi(T)\overline{z} \in Lie(\mathcal{B})$. Indeed otherwise the exponential map $\overline{e}_{\mathcal{B}}$ could not be defined on such values, as it has to since $\overline{e}_{\mathcal{B}}(d\Phi(T)\overline{z}) = \Phi(T)(\overline{e}_{\mathcal{B}}(\overline{z}))$. □

The kernel of the exponential function of an abelian $T-$module $\mathcal{A}$ is an $A-$lattice Λ in $\mathcal{C}^m$. We know (see [**D2**, Lemma 1.15]) that if the exponential function is **surjective**, then the rank of Λ is precisely the rank d of $\mathcal{A}$. An abelian $T-$module $\mathcal{A}$ respecting such a condition is called **uniformizable**.

As announced in the introduction, we shall now correct two mistakes we made in [**D1**]. The first of them is related to the uniformization properties of the sub$-B-$modules of an abelian uniformizable $T-$module. By the cathegorical anti-equivalence between $T-$modules and $T-$motives given by the contravariant functor $Hom(\cdot, \mathbb{G}_a)$ (see [**D1**]) the abelianity easily holds for sub$-B-$modules of an abelian $T-$module. But the uniformization requires, as we are going to see, more attention. We start by stating the following important result of J. Yu.

Theorem 2.2. *Let $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ be an abelian $T-$module defined over $\overline{k}$. Let $\overline{u} \in Lie(\mathcal{A})(\overline{k_\infty})$ such that $\overline{e}(\overline{u}) \in \mathcal{A}(\overline{k})$. Then the smallest vector subspace of $Lie(\mathcal{A})(\overline{k_\infty})$ defined over $\overline{k}$ which contains $\overline{u}$ and it is invariant under the action of the matrix $d\Phi(T)$ is the tangent space at the origin of a sub$-T-$module of $\mathcal{A}$.*

Proof. See [**Yu**, Theorem 3.3]. □

Lemma 2.1 (Correction of [**D1**, Remark 3]). *Let $\mathcal{A}$ be an abelian uniformizable $T-$module of dimension $m_{\mathcal{A}}$ and let $\mathcal{B}$ be a sub$-T-$module of $\mathcal{A}$ of dimension $m_{\mathcal{B}}$. Then $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are abelian and $\mathcal{B}$ is uniformizable.*

Proof. The fact that $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are abelian has already been proved in [**D1**, Remark 3]. As the exponential map is continuous (see the proof of [**D1**, Lemma 3]) and $Lie(\mathcal{B})$ is a vector space, we have that $\overline{e}(Lie(\mathcal{B}))$ is closed with respect to the $1/T-$adic metric. By assuming that $\mathcal{B} - \overline{e}(Lie(\mathcal{B})) \neq \emptyset$, it follows that this complement set cannot contain isolated points in $\mathcal{B}$. Moreover, since k is dense in k_∞, $\overline{k}$ is dense in $\overline{k_\infty}$ and in $\mathcal{C}$, we might assume without loss of generality that there exists $\overline{x} \in \mathcal{B}(\overline{k})$ such that $\overline{x} \notin \overline{e}(Lie(\mathcal{B}))$. On the other hand, even though $\overline{e}(Lie(\mathcal{B}))$ is not a sub$-T-$module of $\mathcal{A}$ (not being an algebraic variety in principle), it is an $A-$module and one can see that in particular $\mathcal{A}/\overline{e}(Lie(\mathcal{B}))$ is still a $T-$module, and it is abelian by the same argument of [**D1**, Remark]. Therefore, since $\mathcal{A}$ is uniformizable, there exists $\overline{u} \in \overline{e}^{-1}(\mathcal{B})$ such that its coset $[\overline{u}] \in Lie(\mathcal{A}/\overline{e}(Lie(\mathcal{B}))) - \{\overline{0}\}$ satisfies $[\overline{e}]([\overline{u}]) = [\overline{x}] \in (\mathcal{B}/\overline{e}(Lie(\mathcal{B})))(\overline{k})$ (note that $[\overline{e}]$ is, by the uniqueness property, the exponential map associated to $\mathcal{A}/\overline{e}(Lie(\mathcal{B}))$). Therefore by Theorem 2.2 there exists a sub$-T-$module $\mathcal{B}'$ of $\mathcal{B}$ such that

$$\bigoplus_{i=0}^{j(\mathcal{A})-1} \{\mathcal{C} \cdot (d\Phi(T^i)[\overline{u}])\} = Lie(\mathcal{B}'/\overline{e}(Lie(\mathcal{B}))) \tag{8}$$

being $j(\mathcal{A})$ the smallest positive integer such that $d\Phi(T^{j(\mathcal{A})}) = T^{j(\mathcal{A})}\mathbf{1}$ (see [**D2**, Theorem 2.7]). It follows that

$$u \in Lie(\mathcal{B}') \subseteq Lie(\mathcal{B}) \tag{9}$$

contradiction. □

Theorem 2.3. *Let $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ be an abelian $T-$module and let $\mathcal{B}$ be a sub$-T-$module of $\mathcal{A}$. Then*

$$\overline{e}^{-1}(\mathcal{B}) = \Lambda + Lie(\mathcal{B}). \tag{10}$$

Proof. By classical theorems of group homomorphisms one has that

$$\overline{e}(Lie(\mathcal{B})) \simeq Lie(\mathcal{B})/(\Lambda \cap Lie(\mathcal{B})) \simeq (\Lambda + Lie(\mathcal{B}))/\Lambda \simeq \overline{e}^{-1}(\overline{e}(Lie(\mathcal{B})))/\Lambda. \tag{11}$$

By Lemma 2.1 we have now that

$$\overline{e}^{-1}(\overline{e}(Lie(\mathcal{B}))) = \overline{e}^{-1}(\mathcal{B}) \tag{12}$$

which proves the statement. □

We remark that these two apparently innocuous results (which are in fact equivalent: indeed it is easy to see that Theorem 2.3 implies Lemma 2.1) can not rely just on the fact that the exponential map is a local homeomorphism, as one may expect, but actually require a deeper proof. While for a number field K, endowed

with the usual Archimedean absolute value, it is very well known that two homeomorphic analytic sets defined over K have to have the same dimension, this is no longer true in non-Archimedean settings. This is an immediate consequence of the fact that the E. L. J. Brouwer's Fixed Point Theorem is false for local fields, as all polydiscs are closed, compact, convex and invariant under translation by one of their elements, which is clearly a continuous injective map. We present here an example of this phenomenon, which is in particular a counterexample to the Invariance of Domain Theorem, by using G. Anderson's theory of $T-$modules.

Remark 2.1. Given $c \in \overline{k}$ such that $|c|_{1/T} < 1$ and $T = c^{-1} + c$, we consider the $T-$module $\mathcal{A} = (\mathbb{G}_a^2, \Phi)$ such that

$$\Phi(T)(\tau) := T \cdot \mathbf{1}_2 + \begin{pmatrix} 0 & 1 - c^{q+1} \\ 1 - c^q & 0 \end{pmatrix} \tau + \begin{pmatrix} c^{1+q+q^2} & 0 \\ 0 & c^q \end{pmatrix} \tau^2. \tag{13}$$

Then the two-dimensional $\mathcal{C}-$vector space $Lie(\mathcal{A})$ is homeomorphic via the exponential map to a $\overline{k}-$entire subset of $\mathcal{C}^2$ of dimension 1 (see [**D1**, Definition 16]).

Proof. The $T-$module $\mathcal{A}$ we have chosen is a non uniformizable nontrivial abelian $T-$module, and the exponential map associated to it has to be injective (a very elegant example provided by G. Anderson and R. Coleman, see [**Goss**, Example 5.9.9]). In such a situation, $Lie(\mathcal{A})$ is homeomorphic via the exponential map to a proper subset of $\mathcal{A}$. Since the exponential map is invertible and the logarithm is $\overline{k}-$analytic as well, such a subset $\overline{e}(Lie(\mathcal{A}))$ is a proper $\overline{k}-$entire subset of $\mathcal{C}^2$. Thus its dimension is the same at each point in which $\overline{e}(Lie(\mathcal{A}))$ is a $\overline{k}-$**analytic space** (see [**D1**, Definition 13]). By Tate theory, it is easy to see that this dimension is 1. □

We now correct a second mistake we made in [**D1**], which led us to uncorrectly use the isomorphism given by the exponential map between an abelian uniformizable $T-$module $\mathcal{A}$ and the quotient $Lie(\mathcal{A})/\Lambda$ in counting the torsion points of $\mathcal{A}$ as $k-$rational points of $Lie(\mathcal{A})/\Lambda$. By calling $j(\mathcal{A}) \in \mathbb{N} - \{0\}$ the smallest natural number such that $d\Phi(T^{j(\mathcal{A})}) = T^{j(\mathcal{A})}\mathbf{1}_m$ (see [**D2**, Theorem 2.7]), it is clear that the set of $T^{j(\mathcal{A})}-$torsion points of $\mathcal{A}$ is therefore in bijection with the set of $\mathbb{F}_q[T^{j(\mathcal{A})}]-$rational points of $Lie(\mathcal{A})/\Lambda$. Our results in [**D1**] still apply by considering $\mathcal{A}$ as a $T^{j(\mathcal{A})}-$module or, equivalently, for the $\mathbb{F}_q[T^{j(\mathcal{A})}]-$torsion points of $\mathcal{A}$. By the way, even if this was not explicitely written and [**D1**, Proposition 5] is false for non diagonal differentials $d\Phi(a(T))$, it is easy to see that the estimate we provided in [**D1**, Theorem 10] still applies to all the $a(T)-$torsion points of an algebraic subvariety X of $\mathcal{A}$ exactly in the way we intended, without any real change. We firstly recall here the statement of such a theorem, which is the main result in [**D1**]. Let L be a finite field extension of

k_∞ with degree n, defined as in [**D1**, Theorem 8]. For a subset S of $Lie(\mathcal{A})(L) \simeq k_\infty^{nm}$ and $a(T) \in A$ we put

$$S(k, a(T)) := \{\overline{z} \in S(k),\ \widetilde{H}(\overline{z}) \leq |a(T)|_{1/T}\} \tag{14}$$

where $\widetilde{H}$ is defined in equation (3).

Theorem 2.4. *Let X be an algebraic subvariety of a $T-$module $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ abelian and uniformizable. Let $Y := \overline{e}^{-1}(X)$. Let $W := Y(L) \cap \{(z_1, ..., z_{nm}) \in k_\infty^{nm}, z_{d+1} = ... = z_{nm} = 0\}$. Let us call $W - W^{alg.}$ the trascendent part of W (see* [**PW**] *for a detailed definition). For each real number $\varepsilon > 0$, there exists $c = c(Y, \varepsilon) > 0$ such that, for each $a(T) \in A - \mathbb{F}_q$, we have*

$$|(W - W^{alg.})(k, a(T))| \leq c|a(T)|_{1/T}^{\varepsilon}. \tag{15}$$

Proof. See [**D1**, Theorem 10]. □

Because of a misinterpretation of the action of an element $a(T) \in A - \mathbb{F}_q$ on $Lie(\mathcal{A})$ via the associated differential $d\Phi(a(T))$ and the trivial multiplication by $a(T)$ of the elements $\overline{z} \in Lie(\mathcal{A})$, we have to restrict the statement of [**D1**, Proposition 5] to the diagonal case.

Proposition 2.2 (Correction 1 of [**D1**, Proposition 5]). *Let $a(T) \in A - \mathbb{F}_q$. Let X be an algebraic subvariety of $\mathcal{A}$. We assume that $d\Phi(a(T)) = a(T) \cdot \mathbf{1}_m$. Then the set $X \cap \mathcal{A}[a(T)]$ of the $a(T)-$torsion points in X is in bijection with the following set*

$$W(k, [a(T)]) := \{\overline{z} \in W[a(T)],\ \widetilde{H}(\overline{z}) \leq |a(T)|_{1/T}\} \tag{16}$$

where

$$W[a(T)] := \left\{\overline{z} = \left(\frac{\alpha_1}{\beta_1}, ..., \frac{\alpha_d}{\beta_d}\right) \in W(k),\ lcm_{i=1,...,d}\{\beta_i\} | a(T)\right\}. \tag{17}$$

Proof. The proof of [**D1**, Proposition 5] can be repeated without problems as by our assumption on $d\Phi(a(T))$ the action of this matrix on the points of Y is precisely the scalar multiplication by $a(T)$. □

We remark that

$$|W(k, [a(T)])| \leq |W(k, a(T))| \tag{18}$$

for every $a(T) \in A - \mathbb{F}_q$.

Remark 2.2 (Correction 2 of [**D1**, Proposition 5]). Let X be an algebraic subvariety of an abelian uniformizable $T-$module $\mathcal{A}$ as before. For each $\varepsilon > 0$ there exists $c = c(X, \mathcal{A}, \varepsilon) > 0$ such that for each $a(T) \in A - \mathbb{F}_q$ we have that

$$|X \cap \mathcal{A}[a(T)]| \leq c|a(T)|_{1/T}^{\varepsilon}. \tag{19}$$

Proof. Let $j(\mathcal{A})$ be the smallest natural number such that $d\Phi(T)^{j(\mathcal{A})}$ is a diagonal matrix (see [**D2**, Theorem 2.7]). Such a number is a power of the characteristic p. Therefore, up to taking $j(\mathcal{A})$ to be a power of q, we may assume that $a(T^{j(\mathcal{A})}) = a(T)^{j(\mathcal{A})}$. Now, a torsion point of $\mathcal{A}$ is clearly a $\mathbb{F}_q[T^{j(\mathcal{A})}]-$torsion point too. Therefore, the set of the $a(T)-$torsion points of X is contained in the set of the $a(T^{j(\mathcal{A})})-$torsion points of X. Since $d\Phi(a(T))^{j(\mathcal{A})} = a(T^{j(\mathcal{A})}) \cdot \mathbf{1}_m$, by Proposition 2.2 we have that

$$\begin{aligned} |X \cap \mathcal{A}[a(T)]| &\leq |X \cap \mathcal{A}[a(T^{j(\mathcal{A})})]| \\ &\leq |W(k, [a(T^{j(\mathcal{A})})])| \leq |W(k, a(T^{j(\mathcal{A})}))| = |W(k, a(T)^{j(\mathcal{A})})|. \end{aligned} \tag{20}$$

By Theorem 2.4 we therefore have that

$$|X \cap \mathcal{A}[a(T)]| \leq c|a(T)|_{1/T}^{j(\mathcal{A})\varepsilon}. \tag{21}$$

Up to taking $\varepsilon/j(\mathcal{A})$ instead of ε the statement is proved. □

3 - Pila-Wilkie estimates for general non-Archimedean fields

The aim of this section is to prove Theorem 1.1. We start by showing a general Implicit Function Theorem for a local non-Archimedean valued field K.

Theorem 3.1. *Let $\overline{F} : K^{n+m} \to K^m$ be a vector of analytic functions on some open set of K^{n+m}, such that its Jacobian matrix $J_{\overline{z}_0}(\overline{F})$ at some point $\overline{z}_0 \in Z(\overline{F})$ has rank m. Up to a permutation of the columns we can divide such a matrix in two blocks as follows*

$$J_{\overline{z}_0}(\overline{F}) = (J_{n,m}(\overline{F}(\overline{z}_0))|J_{m,m}(\overline{F}(\overline{z}_0))) \tag{22}$$

with $J_{m,m}(\overline{F}(\overline{z}_0))$ a square invertible matrix. Then there exists an open neighborhood $U_{\overline{z}_0} \times V_{\overline{z}_0} \subset K^n \times K^m$ and a vector of analytic functions

$$\overline{f} : U_{\overline{z}_0} \to V_{\overline{z}_0} \tag{23}$$

such that for each $\overline{z}_* \in U_{\overline{z}_0}$, we have

$$\overline{F}(\overline{z}_*, \overline{f}(\overline{z}_*)) = 0. \tag{24}$$

Proof. This is a generalization of [**D1**, Corollary 1]. The proof follows precisely the same steps once one has proved the scalar case $m = 1$. The same systems of equations of hyperderivatives of $\overline{F}$ appear again and the same argument holds, since it only uses the ultrametric nature of the involved norm. The proof of the $m = 1$ case is as in [**D1**, Theorem 4]. The formal construction of the inverse function clearly holds for any possible field. The convergence argument, on the other hand, only uses the fact that the norm is ultrametric and the construction of a bounding series is exactly the same. □

The notion of analytic space, regular point and dimension we provided in [**D1**] hold for any non-Archimedean local field. The arguments and the results of [**D1**, Section 2.3] and [**D1**, Section 2.4] hold here as well. We would like to note that the proof of [**D1**, Theorem 7] is much simpler in characteristic 0 since all the involved field extensions become separable and this removes the main difficulty. We thus have the following result.

Theorem 3.2. *Let X be an irreducible affinoid space in the unit polydisc $B_1^m(K)$ which contains $\overline{0}$. Then, the regular points of X are dense in X.*

Proof. By [**D1**, Theorem 7] we have this statement proved if X is absolutely irreducible in the perfect closure of K. By the same argument of the proof of [**D1**, Theorem 8, part 4] we are able to remove this additional hypothesis. □

As we have seen in the proof of [**D1**, Theorem 10] we may assume without loss of generality that W is contained in the unit polydisc $B_1^m(K)$: indeed for $\overline{z} \in W - B_1^m(K)$ we have that $\widetilde{H}(\overline{z}^{-1}) = \widetilde{H}(\overline{z})$, where the inversion function $\overline{z} \mapsto \overline{z}^{-1}$ is defined in the discussion after [**D1**, Proposition 5]. Moreover, by the same argument at the beginning of the proof of [**D1**, Theorem 10] we see that

$$(W \cap B_1^m(K))^{alg.} \cup (W - B_1^m(K))^{alg.} \subseteq W^{alg.} \tag{25}$$

and

$$\begin{aligned} N(W - W^{alg.}, t) \quad &\leq N((W \cap B_1^m(K)) - (W \cap B_1^m(K))^{alg.}, t) \\ &\quad + N((W - B_1^m(K)) - (W - B_1^m(K))^{alg.}, t). \end{aligned} \tag{26}$$

Therefore we can assume that W is contained in $B_1^m(K)$ to prove Theorem 1.1. Hence we assume W is an analytic space. By Theorem 3.1 and Theorem 3.2 we now easily have a $K-$analytic cover of W (see [**D1**, Definition 19]) and, since W can be assumed to be compact, this allows us to assume W to be the image, by some $K-$analytic map,

of a unit polydisc $B_1^n(K)$ for some given $n < m$ (up to multiplying $c(W, \varepsilon)$ by a positive integer only depending on W).

We now state a result which allows us to treat this situation.

Proposition 3.1. *Let $h < d$ and $\delta \in \mathbb{N} - \{0\}$. There exists a real number $\varepsilon = \varepsilon(h, d, \delta) > 0$ such that, for each analytic function*

$$\Phi : B_1^h(K) \to K^d \tag{27}$$

if

$$S := \Phi(B_1^h(K)) \tag{28}$$

and $t \in \mathbb{N} - \{0\}$, then there exists a real number $C = C(h, d, \delta, B_1^h(K), \Phi) > 0$, such that the set $S(\mathbb{Q}_K, t)$ is contained in the union of at most Ct^{ε} hypersurfaces in K^d having degree at most δ. Moreover, if δ approaches $+\infty$, then ε converges to 0.

Proof. The proofs of [**D1**, Lemma 4] and [**D1**, Proposition 7] apply here with no relevant changes. □

Proof of Theorem 1.1. This is close to the proof of [**D1**, Theorem 10] and we refer to it for the details. For each $\varepsilon > 0$ we choose $\delta > 0$ big enough so that $\varepsilon(\delta) \leq \varepsilon/2$, where $\varepsilon(\delta) > 0$ is associated to δ as described in Proposition 3.1. By our previous discussion we have that W can be assumed to verify the hypotheses we gave on the set S in Proposition 3.1: hence $W(\mathbb{Q}_K, t)$ is contained in the union of at most $C(W, \varepsilon)t^{\varepsilon/2}$ hypersurfaces in K^m having degree at most δ, for some $C(W, \varepsilon) > 0$ only depending on W and ε (as we have chosen δ depending on ε). Let $\mathbb{P}_{v(\delta)}(K)$ be the set which parametrizes the family of all hypersurfaces with degree at most δ in K^m. For each $r \in \mathbb{P}_{v(\delta)}$ we define

$$S_r := W \cap H_r \tag{29}$$

where H_r is the hypersurface in K^m associated to r. By the same induction argument we used in [**D1**, Theorem 10] we have that

$$N(S_r - S_r^{alg.}, t) \leq c(S_r, \varepsilon)t^{\varepsilon/2}. \tag{30}$$

Now, it is clear that $S_r^{alg.} = (W \cap H_r)^{alg.} = W^{alg.} \cap H_r$, so that

$$\begin{aligned} W - W^{alg.} &= \bigcup_{r \in \mathbb{P}_{v(\delta)}(K)} ((W - W^{alg.}) \cap H_r) \\ &= \bigcup_{r \in \mathbb{P}_{v(\delta)}(K)} ((W \cap H_r) - (W \cap H_r)^{alg.}). \end{aligned} \tag{31}$$

Let $\mathcal{S}$ be the set of $r \in \mathbb{P}_{\nu(\delta)}(K)$ which represents the family of hypersurfaces with degree at most δ which cover W as described before. So $\mathcal{S}$ is finite and its cardinality is at most $C(W,\varepsilon)t^{\varepsilon/2}$. Let

$$K(W,\varepsilon) := \max_{r\in\mathcal{S}}\{c(S_r,\varepsilon)\}. \tag{32}$$

By choosing

$$c(W,\varepsilon) := K(W,\varepsilon)C(W,\varepsilon) \tag{33}$$

the statement follows. □

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