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Two remarks on the local behavior of solutions to logarithmically singular diffusion equations and its porous-medium type approximations

Abstract. For the logarithmically singular parabolic equation (1.1) below, we establish a Harnack type estimate in the $L^1_{\rm loc}$ topology, and we show that the solutions are locally analytic in the space variables and differentiable in time. The main assumption is that $\ln u$ possesses a sufficiently high degree of integrability (see (1.3) for a precise statement). These two properties are known for solutions to singular porous medium type equations $(0<m<1)$, which formally approximate the logarithmically singular equation (1.1) below. However, the corresponding estimates deteriorate as $m \to 0$. It is shown that these estimates become stable and carry to the limit as $m \to 0$, provided the indicated sufficiently high order of integrability is in force. The latter then appears as the discriminating assumption between solutions to parabolic equations with power-like singularities and logarithmic singularities to insure such solutions to be regular.

Keywords. Singular parabolic equations, L^1_{loc}-Harnack estimates, analyticity.

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Contents

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Appendix

1 - Main results

We continue here the investigations initiated in [**2, 3**], on the local behavior of non-negative solutions to logarithmically singular parabolic equations of the type

$$
\begin{aligned}
&u \in C_{\rm loc}\big(0,T;L^2_{\rm loc}(E)\big), \quad \ln u \in L^2_{\rm loc}\big(0,T;W^{1,2}_{\rm loc}(E)\big)\\
&u_t - \Delta \ln u = 0, \qquad \text{weakly in } E_T = E\times(0,T]
\end{aligned}
\tag{1.1}
$$

where E is an open set in $\mathbb{R}^N$ and $T>0$. It is assumed throughout that

$$
u \in L^r_{\rm loc}(E_T) \quad \text{for some } r > \max\Big\{1;\frac{N}{2}\Big\}
\tag{1.2}
$$

and that

$$
\ln u \in L^\infty_{\rm loc}\big(0,T;L^p_{\rm loc}(E)\big) \quad \text{for some } p \ge 1.
\tag{1.3}
$$

The modulus of ellipticity of the principal part is u^{-1}. Therefore, the equation is degenerate as $u \to \infty$ and singular as $u \to 0$. It was shown in [**2**] that (1.2) implies that u is locally bounded in E_T, and hence the equation is not degenerate. Likewise if (1.3) holds for some $p > N+2$, then the solution u is locally bounded below, and hence the equation is not singular. As a consequence u is locally, a classical solution to (1.1). This was realized by establishing a local upper and lower bound on u, via a pointwise Harnack-type estimate.

The main results of this note are that if u is a locally bounded, weak solution to (1.1), then:

- **i.** If $\ln u$ satisfies (1.3) for some $p \geq 2$, then u satisfies a local Harnack inequality in the L^1_{loc} topology, as opposed to a pointwise Harnack estimate.
- **ii.** If $\ln u$ satisfies (1.3) for some $p > N+2$, then u is locally analytic in the space variables uniformly in t, and C^∞ in time.

2 - Harnack type estimates in the topology of L^1_{loc}

For $\rho > 0$ let K_ρ be the cube centered at the origin of $\mathbb{R}^N$ and edge ρ, and for $y \in \mathbb{R}^N$ let $K_\rho(y)$ denote the homothetic cube centered at y. Moreover, $Q_\rho(\theta)$ denotes the parabolic cylinder $K_\rho \times (-\theta\rho^2, 0]$. For $0 < s < t \leq T$ and $y \in E$ let ρ be so small that $K_{2\rho}(y) \times (s,t] \subset E_T$. Since $u \in L^\infty_{\mathrm{loc}}(E_T)$ the quantity

$$M = \operatorname*{ess\,sup}_{K_{2\rho}\times(s,t]} u \tag{2.1}$$

is well defined and finite. Also, if (1.3) holds then the quantity

$$\Lambda_p = \operatorname*{ess\,sup}_{s\leq\tau\leq t} \Big(\fint_{K_{2\rho}(y)} \Big| \ln \frac{u(x,\tau)}{M} \Big|^p dx \Big)^{\frac{1}{p}} \tag{2.2}$$

is well defined and finite.

Proposition 2.1. *Let u be a non-negative, local, weak solution to* (1.1) *satisfying in addition* (1.2) *and* (1.3) *for some $p \geq 2$. There exists a positive constant γ depending only on $\{N, r, p\}$ and Λ_1 and Λ_2, such that for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$, there holds*

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \leq \gamma \left(\inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau)dx + \frac{t-s}{\rho^\lambda} \right), \tag{2.3}$$

where

$$\lambda = 2 - N. \tag{2.4}$$

2.1 - *Weak solutions versus distributional solutions*

The $L^1_{\rm loc}$ Harnack type estimate (2.3), continues to hold for merely distributional solutions to the second of (1.1) whereby $\ln u$ is only in $L^1_{\rm loc}(E_T)$. The assumption (1.2) however is still in force, and (1.3) is required to hold only for some $p > 1$. The constant γ depends only on Λ_p for some $p > 1$.

In § 5 we will prove (2.3) first for such distributional solutions. The proof is rather simple due to the linearity of the principal part with respect to $\ln u$. The linearity however is immaterial, as (2.3) is a structural inequality valid for weak solutions to quasilinear parabolic equations with singularity and degeneracy of the same nature as (1.1). To be specific, consider non-negative, local, weak solutions to quasilinear parabolic equations of the type

$$\begin{aligned} &u \in C_{\rm loc}\big(0,T; L^2_{\rm loc}(E)\big), \quad \ln u \in L^2_{\rm loc}\big(0,T; W^{1,2}_{\rm loc}(E)\big)\\ &u_t - \operatorname{div} \boldsymbol{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T. \end{aligned} \tag{2.5}$$

Here the function $\boldsymbol{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\left.\begin{aligned} \boldsymbol{A}(x,t,u,p)\cdot p \ge C_o \frac{|p|^2}{u}\\ |\boldsymbol{A}(x,t,u,p)| \le C_1 \frac{|p|}{u} \end{aligned}\right. \qquad \text{a.e. in } E_T, \tag{2.6}$$

where C_o and C_1 are given positive constants. Assume that u and $\ln u$ satisfy (1.2)-(1.3) and introduce M and Λ_p as in (2.1) and (2.2). Then u satisfies (2.3) with γ depending on the data $\{N, r, p, C_o, C_1\}$ and Λ_1 and Λ_2.

The proof of this fact is more involved and it is given in § 6.

3 - Analyticity of local weak solutions to (1.1)

The precise statement of these results hinges on the notion of "intrinsic neighborhood" of a point (x_o, t_o), as determined by the degeneracy and singularity of the equation in (1.1).

3.1 - *The intrinsic geometry of* (1.1) *and main results from* **[2]**

Let u be a non-negative, local, weak solution to (1.1). Having fixed $(x_o, t_o) \in E_T$, and $K_{8\rho}(x_o) \subset E$, introduce the quantity

$$\theta \stackrel{\text{def}}{=} \varepsilon\Big(\fint_{K_\rho(x_o)} u^q(\cdot, t_o)dx\Big)^{\frac{1}{q}} \tag{3.1}$$

where $\varepsilon \in (0,1)$ is to be chosen, and $q > 1$ is arbitrary. If $\theta > 0$ assume that

$$(x_o, t_o) + Q_{8\rho}(\theta) = K_{8\rho}(x_o) \times (t_o - \theta(8\rho)^2, t_o] \subset E_T. \tag{3.2}$$

These are backward, parabolic cylinders with "vertex" at (x_o, t_o) whose height depends on the solution itself through the quantity θ. In this sense they are intrinsic to the solution itself.

Continue to assume that u satisfies (1.2) and let $\ln u$ satisfy (1.3) for some $p > N + 2$. Then

$$M = \operatorname*{ess\,sup}_{(x_o,t_o)+Q_{8\rho}(\theta)} u \tag{3.3}$$

is well defined and finite. Moreover the dimensionless quantity

$$\eta = \Big[\fint_{K_\rho(x_o)} \Big(\frac{u(x,t_o)}{M}\Big)^q dx\Big]^{\frac{1}{q}\frac{2}{2r-N}} = \Big(\frac{\theta}{\varepsilon M}\Big)^{\frac{2}{2r-N}} \tag{3.4}$$

is well defined and strictly positive. Finally the dimensionless quantity

$$\Lambda_p = \operatorname*{ess\,sup}_{t_o-\theta(8\rho)^2<\tau<t_o} \Big(\fint_{K_{8\rho}(x_o)} \Big| \ln \frac{u(x,\tau)}{M}\Big|^p dx\Big)^{\frac{1}{p}}, \qquad \text{for some } p > N + 2 \tag{3.5}$$

is well defined and finite.

Theorem 3.1 (Pointwise Harnack estimate **[2]**). *Let u be a non-negative, local, weak solution to* (1.1), *satisfying the integrability conditions* (1.2) *and* (1.3) *for some $p > N + 2$, and assume $\theta > 0$. There exist a constant $\varepsilon \in (0,1)$, and a continuous, increasing function $\eta \to f(\eta, \Lambda_p)$ defined in $\mathbb{R}^+$ and vanishing at $\eta = 0$, that can be quantitatively determined apriori only in terms of $\{N, p, q\}$, and Λ_p, such that*

$$\begin{gathered}\inf_{K_{4\rho}(x_o)} u(\cdot, t) \ge f(\eta, \Lambda_p) \sup_{(x_o,t_o)+Q_{2\rho}(\frac{1}{4}\theta)} u \\ \textit{for all } t \in \Big(t_o - \frac{1}{16}\,\theta\rho^2\,,\, t_o\Big].\end{gathered} \tag{3.6}$$

For $\eta \to 0$ and $\Lambda_p \to \infty$, the function $\eta \to f(\eta, \Lambda_p)$ can be taken to be of the form

$$f(\eta) = \exp\Big\{-\frac{\Lambda_p^{C_1}}{\eta^{C_2}}\Big\} \qquad \text{for } 0 \le \eta \ll 1 \quad \text{and } \Lambda_p \gg 1 \tag{3.7}$$

for positive constants C_1 and C_2 that can be determined apriori only in terms of $\{N, p, q\}$. Moreover

$$\varepsilon \to 0 \text{ and } C_1 + C_2 \to \infty \quad \text{as } p \to \infty \quad \text{or } p \to N+2. \tag{3.8}$$

Remark 3.1. In [2] the constant η was given a more general form. For the purpose of this note the definition (3.4) represents the degeneracy of the equation, quantified by $M \to \infty$. The occurrence $\Lambda_p \to \infty$ quantifies, roughly speaking, the singularity of the equation.

3.2 - *Analyticity in the space variable, of solutions to* (1.1) *at* (x_o, t_o)

Theorem 3.2. *Let u be a non-negative, local, weak solution to* (1.1), *satisfying the integrability conditions* (1.2) *and* (1.3) *for some $p > N+2$, and assume $\theta > 0$. There exist two parameters C and H, that have a polynomial dependence on $f(\eta)$, $[f(\eta)]^{-1}$, N, such that for every N-dimensional multi-index α*

$$|D^\alpha u(x_o, t_o)| \le \frac{CH^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}}\, u(x_o, t_o). \tag{3.9}$$

Moreover, for every non-negative integer k

$$\Big|\frac{\partial^k}{\partial t^k} u(x_o, t_o)\Big| \le \frac{CH^{2k}(2k)!}{\rho^{2k}}\, u(x_o, t_o)^{1-k}. \tag{3.10}$$

Remark 3.2. The theorem continues to hold, with the same assumptions, for local, weak solutions to the quasilinear equations (2.5), provided the function $\boldsymbol{A}$ is analytic in all its arguments whenever u is bounded above and below by positive constants.

4 - Approximating (1.1) by porous medium type equations

Consider local, non-negative, weak solutions in E_T to the porous medium equation

$$\begin{aligned} & u \in C_{\rm loc}\big(0, T; L^2_{\rm loc}(E)\big), \quad w \in L^2_{\rm loc}\big(0, T; W^{1,2}_{\rm loc}(E)\big) \\ & u_t - \Delta w = 0, \qquad \text{weakly in } E_T = E \times (0, T] \end{aligned} \tag{4.1}$$

where

$$w = \frac{u^m - 1}{m} \qquad \text{for } 0 < m \ll 1. \tag{4.2}$$

As $m \to 0$, formally (4.1)-(4.2) tend to (1.1). In [**3**] a precise topology was introduced by which such a formal limit is rigorous.

The main tool that allows us to do this, is a pointwise Harnack-type estimate, which is stable for $0 < m \ll 1$. Under this point of view, it is worth mentioning that the Harnack inequality in the range $0 < m \leqslant \frac{(N-2)_+}{N}$ was first proved in [**1**] for the model equation, and then extended to more general operators in [**4, 7**]. However, when one needs a version of such inequality which is stable with respect to m, as it was the case in [**3**] and here too, despite a formal similarity in the statements, assumptions, results and techniques are quite different.

A natural question is whether solutions to (4.1)-(4.2) satisfy a version of the $L^1_{\rm loc}$ Harnack estimate (2.3), which as $m \to 0$ tends, in some appropriate sense to be made precise, to that of Proposition 2.1. A similar issue arises for the local analyticity of Theorem 3.2.

4.1 - *Harnack type estimates in the topology of* $L^1_{\rm loc}$, *for weak solutions to* (4.1)-(4.2)

A first statement in this direction is that u satisfies

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \le \gamma \left[\inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau)dx + \left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{1-m}} \right], \tag{4.3}$$

where

$$\lambda = N(m-1) + 2. \tag{4.4}$$

Here γ depends upon N and m and $\gamma(m) \to \infty$ as $m \to 0$. Thus, one cannot formally recover (2.3) by letting $m \to 0$ in (4.3). However, (4.3) is rather general as it continues to hold for non-negative, local weak solutions to general quasi-linear version of (4.1). Precisely

$$\begin{aligned} &u \in C_{\rm loc}\big(0,T; L^2_{\rm loc}(E)\big), \quad w \in L^2_{\rm loc}\big(0,T; W^{1,2}_{\rm loc}(E)\big) \\ &u_t - \operatorname{div} \boldsymbol{A}(x,t,u,Du) = 0, \qquad \text{weakly in } E_T \end{aligned} \tag{4.5}$$

where the function $\boldsymbol{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\left.\begin{aligned} \boldsymbol{A}(x,t,u,p)\cdot p &\ge C_o u^{m-1}|p|^2 \\ |\boldsymbol{A}(x,t,u,p)| &\le C_1 u^{m-1}|p| \end{aligned}\right\} \qquad \text{a.e. in } E_T, \tag{4.6}$$

where C_o and C_1 are given positive constants. In such a case the constant γ in (4.3) depends also on these structural constants. The proof of these statements is in [**5**], Appendix B.

A major difference between (2.3) and (4.3) is that in the latter u is not required to be locally bounded, nor does γ depend on some analogue of the quantity Λ_p as defined in (2.2). This raises the question as to whether (4.3) holds with γ independent of m but dependent on some analogue of Λ_p.

Henceforth we assume

$$u \in L^r_{\rm loc}(E_T) \quad \text{for some } r > \max\Big\{1; \frac{N}{2}(1-m)\Big\} \tag{4.7}$$

and that

$$w \in L^\infty_{\rm loc}\big(0,T; L^p_{\rm loc}(E)\big) \quad \text{for some } p \geq 1. \tag{4.8}$$

It was shown in [**5**] that (4.7) implies that $u \in L^\infty_{\rm loc}(E_T)$ and hence the corresponding quantity M defined as in (2.1) is well defined and finite. Set

$$\Lambda_{m,p} = \operatorname*{ess\,sup}_{s<\tau<t}\bigg(\int_{K_{2\rho}(y)} \Big(\frac{M^m - u(x,\tau)^m}{m\,M^m}\Big)^p dx\bigg)^{\frac{1}{p}}. \tag{4.9}$$

This is the analogue of (2.2), and, if (4.8) holds, it is well defined and finite.

Proposition 4.1. *Let u be a non-negative, local, weak solution to the singular equations* (4.5)-(4.6), *in E_T. There exists a positive constant γ depending only on the data N, C_o, C_1, and $\Lambda_{\frac{m}{2},1}$, $\Lambda_{\frac{m}{2},2}$ and independent of m, such that* (4.3)-(4.4) *holds true, for all cylinders $K_{2\rho}(y)\times[s,t]\subset E_T$.*

As a consequence, (2.3) can be recovered from (4.3), with the indicated dependences, as $m \to 0$, provided proper conditions are placed, that insure the pointwise convergence of the solutions to (4.1)-(4.2) to solutions to (1.1). These conditions are identified in [**3**] and we will touch on them briefly in the next subsection.

4.2 - *Analyticity in the space variable, of solutions to* (4.1) *at* (x_o, t_o)

Having fixed $(x_o,t_o)\in E_T$ and $K_{8\rho}(x_o)\subset E$, the intrinsic geometry of (4.1)-(4.2) is determined by

$$\theta_m = \varepsilon\Big(\fint_{K_\rho(x_o)} u^q(x,t_o)dx\Big)^{\frac{1-m}{q}}. \tag{4.10}$$

The intrinsic cylinders are as in (3.2) with θ replaced by θ_m. The analogues of η in (3.4) are

$$\sigma = \Big[\fint_{K_\rho(x_o)} \Big(\frac{u(x,t_o)}{M}\Big)^q dx\Big]^{\frac{1}{q}\frac{2}{\lambda_r}} \tag{4.11}$$

where $r \geq 1$ is any number such that

$$\lambda_r = N(m-1) + 2r > 0. \tag{4.12}$$

In [**5**] a Harnack estimate of the form of (3.6) was proved for these solutions with $f(\,\cdot\,)$ depending only on σ and of the form

$$f(\sigma) = \frac{\sigma^\beta}{\gamma(m)} \tag{4.13}$$

where $\gamma(m) \to \infty$ as $m \to 0$. The constant β depends on λ_r and $\beta(\lambda_r) \to \infty$ as $\lambda_r \to 0$. It was observed in [**5**] § 21.5.3 that, for each fixed $m \in (0,1)$, such an estimate implies the local analyticity of the solutions in the space variables about (x_o, t_o), and at least the Lipschitz continuity in time.

Because of the indicated dependence of $\gamma(m)$ on m in (4.13), such a regularity does not directly carry to the limit as $m \to 0$. In [**3**] we established a Harnack estimate of the form (3.6) for solutions to (4.1)-(4.2) and its quasi-linear versions (4.5)-(4.6), with $f(\,\cdot\,)$ depending on σ, as defined in (4.11), and $\Lambda_{m,p}$ as defined in (4.9) provided $p > N+2$. The form of such $f(\,\cdot\,)$ is the same as that in (3.7) with the proper change in symbolism. The new feature of such an $f(\,\cdot\,)$ is that, while depending on the quantities σ and $\Lambda_{m,p}$, each quantifying the degeneracy and the singularity of the equation, is independent of m and hence is "stable" as $m \to 0$, provided σ and $\Lambda_{m,p}$ are uniformly estimated with respect to m.

As a consequence, the analyticity estimates of Theorem 3.2, can be recovered from the analogous ones for solutions to (4.1)-(4.2) whenever solutions $\{u_m\}$ to the latter converge pointwise to solutions to the former. In [**3**] it was shown that this occurs if there exists $m_{**} \in (0,1)$ such that

$$\begin{aligned} &u_m \in L^\infty_{\rm loc}\big(0,T; L^r_{\rm loc}(E)\big) \qquad \text{for some } r > \max\Big\{1;\ \frac{1}{2}N\Big\} \\ &w_m \in L^\infty_{\rm loc}(0,T; L^p_{\rm loc}(E)\big) \qquad \text{for some } p > N+2 \end{aligned} \tag{4.14}$$

uniformly in $m \in (0, m_{**})$. It is also required that there exists an open set $E_o \subset E$ and a positive number $\sigma_{E_o;T}$ such that

$$\int_{E_o} u_m(\cdot, T)dx \geq \sigma_{E_o;T} \quad \text{uniformly in } m. \tag{4.15}$$

5 - Proof of Proposition 2.1 for distributional solutions to (1.1)

The proof is a local version of an argument of [8] for global solutions to the porous medium equation for $0<m<1$. Let $\zeta_1\in C_o^\infty(\mathbb{R}^N)$ be such that

$$\begin{cases} 0\le\zeta_1\le 1\\ \zeta_1=1 \quad \text{in} \quad K_\rho\\ \zeta_1=0 \quad \text{in} \quad \mathbb{R}^N\backslash K_{2\rho}.\end{cases} \tag{5.1}$$

Then, by the divergence theorem

$$\int_{\mathbb{R}^N} \Delta\zeta_1 dx=\int_{K_{2\rho}} \Delta\zeta_1\, dx=\int_{\partial K_{2\rho}} \frac{\partial\zeta_1}{\partial n}\, ds=0. \tag{5.2}$$

By (5.2), for any positive constant M, any ζ_1 as in (5.1), and any non-negative function v such that $\Delta\zeta_1\ln v$ is integrable, we have

$$\int_{K_{2\rho}} \Delta\zeta_1 \ln v dx=\int_{K_{2\rho}} \Delta\zeta_1 \ln\Big(\frac{v}{M}\Big)dx. \tag{5.3}$$

Now consider $\zeta_2\in C_o^\infty(0,+\infty)$ and ζ_1 as in (5.1), such that

$$|D\zeta_1|\le\frac{C_1(N)}{\rho},\qquad |\Delta\zeta_1|\le\frac{C_2(N)}{\rho^2}.$$

By the previous notation, with u a solution to (1.1), we have

$$-\int_0^\infty\int_{\mathbb{R}^N}\zeta_2'\zeta_1 u dx dt=\int_0^\infty\int_{\mathbb{R}^N}\zeta_2\Delta\zeta_1 \ln u dx dt.$$

Taking into account (5.3), for any positive constant M

$$\frac{d}{dt}\int_{K_{2\rho}}\zeta_1 u dx=\int_{K_{2\rho}}\Delta\zeta_1\ln\Big(\frac{u}{M}\Big)dx \qquad \text{in} \qquad \mathcal{D}'(0,T),$$

and also in $L^1_{loc}(0,T)$. Therefore

$$\Big|\frac{d}{dt}\int_{K_{2\rho}}\zeta_1 u dx\Big|=\Big|\int_{K_{2\rho}}\Delta\zeta_1\ln\Big(\frac{u}{M}\Big)dx\Big|\le\int_{K_{2\rho}}|\Delta\zeta_1|\Big|\ln\Big(\frac{u}{M}\Big)\Big|dx.$$

By the definition of Λ_p and from the previous estimate,

$$\left|\frac{d}{dt}\int_{K_{2\rho}}\zeta_1 u dx\right| \le \Lambda_p\left(\int_{K_{2\rho}}|\Delta\zeta_1|^{p'}dx\right)^{\frac{1}{p'}}|K_{8\rho}|^{\frac{1}{p}}$$

$$\le \Lambda_p\frac{C(N)}{\rho^2}\rho^{\frac{N}{p'}}\rho^{\frac{N}{p}} = \frac{C(\Lambda_p, N)}{\rho^\lambda},$$

where $\lambda = 2 - N$. Taking into account the size of the support of ζ_1, for any $0 < s < t < T$ we conclude

$$\int_{K_\rho} u(x,t)dx \le C(\Lambda_p, N)\left(\int_{K_{2\rho}} u(x,s)dx + \frac{t-s}{\rho^\lambda}\right).$$

□

6 - Proof of Proposition 2.1 for weak solutions to (2.5)-(2.6)

6.1 - *An auxiliary lemma*

Lemma 6.1. *Let u be a non-negative, local, weak solution to the quasi-linear singular equations* (2.5)-(2.6), *in E_T. There exist two positive constants γ_1, γ_2 depending only on the data $\{N, C_o, C_1\}$, such that for all cylinders $K_{4\rho}(y)\times[s,t]\subset E_T$, and all $\sigma\in(0,1)$,*

$$\int_s^t\int_{K_\rho(y)}\frac{|Du|^2}{u^2}\zeta^2 dx d\tau \le \gamma_1(1+\Lambda_1)\mathcal{S}_\sigma + \frac{\gamma_2}{\sigma^2}(\Lambda_1^2+\Lambda_2^2)\left(\frac{t-s}{\rho^\lambda}\right),$$

where

$$\mathcal{S}_\sigma = \sup_{s<\tau<t}\int_{K_{(1+\sigma)\rho}(y)} u(\cdot,\tau)dx.$$

Proof. Assume $(y,s) = (0,0)$, fix $\sigma\in(0,1)$, and let $x\to\zeta(x)$ be a non-negative piecewise smooth cutoff function in $K_{(1+\sigma)\rho}$ that vanishes outside $K_{(1+\sigma)\rho}$, equals one on K_ρ, and such that $|D\zeta|\le(\sigma\rho)^{-1}$. Let $s_1\in[0,t]$ be such that

$$\mathcal{S}_\sigma = \sup_{0<s<t}\int_{K_{(1+\sigma)\rho}(y)} u(\cdot,s)dx = \int_{K_{(1+\sigma)\rho}(y)} u(\cdot,s_1)dx.$$

We also set

$$\bar{\mathcal{S}}_\sigma \overset{\text{def}}{=} \frac{\mathcal{S}_\sigma}{\rho^N}.$$

In the weak formulation of (2.5)-(2.6), take the test function

$$\varphi=\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)_+\zeta^2$$

and integrate over $Q=K_{(1+\sigma)\rho}\times(0,t]$, to obtain

$$\begin{aligned}0&=\iint_Q\frac{\partial}{\partial\tau}u\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)_+\zeta^2dx\,d\tau+\iint_Q\boldsymbol{A}(x,\tau,u,Du)\cdot D\Big[\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)_+\zeta^2\Big]dx\,d\tau\\&=I_1+I_2.\end{aligned}$$

We estimate these two terms separately.

$$\begin{aligned}I_1&=\iint_Q\frac{\partial}{\partial\tau}u\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)_+\zeta^2dx\,d\tau=\iint_{Q\cap[u<\bar{S}_\sigma]}\frac{\partial}{\partial\tau}u\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)\zeta^2dx\,d\tau\\&=\int_{K_{(1+\sigma)\rho}\cap[u<\bar{S}_\sigma]}\zeta^2(x)\Big(u\ln\frac{\bar{S}_\sigma}{u}+u\Big)(x,t)\,dx\\&\quad-\int_{K_{(1+\sigma)\rho}\cap[u<\bar{S}_\sigma]}\zeta^2(x)\Big(u\ln\frac{\bar{S}_\sigma}{u}+u\Big)(x,0)\,dx.\end{aligned}$$

Next,

$$\begin{aligned}I_2&=\iint_Q\boldsymbol{A}(x,\tau,u,Du)\cdot D\Big[\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)_+\zeta^2\Big]dx\,d\tau\\&=\iint_{Q\cap[u<\bar{S}_\sigma]}\boldsymbol{A}(x,\tau,u,Du)\cdot D\Big[\Big(\ln\frac{\bar{S}_\sigma}{u}\Big)\zeta^2\Big]dx\,d\tau\\&=-\iint_{Q\cap[u<\bar{S}_\sigma]}\zeta^2\boldsymbol{A}(x,\tau,u,Du)\frac{Du}{u}dx\,d\tau\\&\quad+2\iint_{Q\cap[u<\bar{S}_\sigma]}\zeta\,\ln\frac{\bar{S}_\sigma}{u}\boldsymbol{A}(x,\tau,u,Du)D\zeta dx\,d\tau\\&\le-C_o\iint_{Q\cap[u<\bar{S}_\sigma]}\zeta^2\frac{|Du|^2}{u^2}dx\,d\tau\\&\quad+2C_1\iint_{Q\cap[u<\bar{S}_\sigma]}\zeta\,\ln\frac{\bar{S}_\sigma}{u}\frac{|Du|}{u}|D\zeta|dx\,d\tau\\&\le-\frac{C_o}{2}\iint_{Q\cap[u<\bar{S}_\sigma]}\zeta^2\frac{|Du|^2}{u^2}dx\,d\tau+\frac{\gamma}{\sigma^2\rho^2}\iint_{Q\cap[u<\bar{S}_\sigma]}\Big|\ln\frac{\bar{S}_\sigma}{u}\Big|^2dx\,d\tau,\end{aligned}$$

where $\gamma = 2\,\dfrac{C_1^2}{C_o}$. Therefore, we conclude that

$$\begin{aligned}
\frac{C_o}{2} \iint\limits_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2 \frac{|Du|^2}{u^2}\,dx\,d\tau &\le \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\Big(u \ln \frac{\bar{\mathcal{S}}_\sigma}{u} + u\Big)(x,t)\,dx \\
&+ \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \Big|\ln \frac{\bar{\mathcal{S}}_\sigma}{u}\Big|^2 dx\,d\tau \\
&= J_1 + J_2\,.
\end{aligned}$$

We have

$$\begin{aligned}
J_1 &= \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\Big(u \ln \frac{\bar{\mathcal{S}}_\sigma}{u} + u\Big)(x,t)\,dx \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \Big(u \ln \frac{\bar{\mathcal{S}}_\sigma}{u} + u\Big)(x,t)\,dx \\
&= \bar{\mathcal{S}}_\sigma \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \frac{u}{\bar{\mathcal{S}}_\sigma} \ln \frac{\bar{\mathcal{S}}_\sigma}{u}(x,t)\,dx + \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} u(x,t)\,dx \\
&\le \bar{\mathcal{S}}_\sigma \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \ln \frac{\bar{\mathcal{S}}_\sigma}{u}(x,t)\,dx + \mathcal{S}_\sigma \\
&\le \gamma \mathcal{S}_\sigma \fint\limits_{K_{(1+\sigma)\rho}} \ln \frac{M}{u}(x,t)\,dx + \mathcal{S}_\sigma \le \gamma \varLambda_1 \mathcal{S}_\sigma + \mathcal{S}_\sigma = \gamma(1+\varLambda_1)\mathcal{S}_\sigma,
\end{aligned}$$

where $\gamma = 2^N$. Moreover,

$$\begin{aligned}
J_2 &= \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \Big|\ln \frac{\bar{\mathcal{S}}_\sigma}{u}\Big|^2 dx\,d\tau \le \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \Big|\ln \frac{M}{u}\Big|^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q} \Big|\ln \frac{M}{u}\Big|^2 dx\,d\tau = \frac{\gamma}{\sigma^2\rho^2} \int_0^t \int\limits_{K_{(1+\sigma)\rho}} \Big|\ln \frac{M}{u}\Big|^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2} \sup_{0<\tau<t} \fint\limits_{K_{(1+\sigma)\rho}} \Big|\ln \frac{M}{u}\Big|^2 dx\, t\rho^N \le \frac{\gamma}{\sigma^2} \varLambda_2^2 \Big(\frac{t}{\rho^\lambda}\Big).
\end{aligned}$$

Hence, we have

$$\iint\limits_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2 \frac{|Du|^2}{u^2}\,dx\,d\tau \le \gamma(\varLambda_1+1)\mathcal{S}_\sigma + \frac{\gamma}{\sigma^2}\varLambda_2^2\Big(\frac{t}{\rho^\lambda}\Big). \tag{6.1}$$

Now, if we take the test function

$$\varphi = \left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right)_+ \zeta^2$$

in the weak formulation of (2.5)-(2.6) and integrate over $Q = K_{(1+\sigma)\rho} \times (0, t]$, we obtain

$$\begin{aligned} 0 &= \iint_Q \frac{\partial}{\partial \tau} u \left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right)_+ \zeta^2 dx\, d\tau + \iint_Q \boldsymbol{A}(x, \tau, u, Du) \cdot D\left[\left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right)_+ \zeta^2 \right] dx\, d\tau \\ &= I_3 + I_4. \end{aligned}$$

We estimate these two terms separately.

$$\begin{aligned} I_3 &= \iint_Q \frac{\partial}{\partial \tau} u \left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right)_+ \zeta^2 dx\, d\tau = \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \frac{\partial}{\partial \tau} u \left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right) \zeta^2 dx\, d\tau \\ &= \int_{K_{(1+\sigma)\rho} \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta^2(x) \left(u \ln \frac{u}{\bar{\mathcal{S}}_\sigma} - u \right)(x, t)\, dx \\ &\quad - \int_{K_{(1+\sigma)\rho} \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta^2(x) \left(u \ln \frac{u}{\bar{\mathcal{S}}_\sigma} - u \right)(x, 0)\, dx. \end{aligned}$$

Next,

$$\begin{aligned} I_4 &= \iint_Q \boldsymbol{A}(x, \tau, u, Du) \cdot D\left[\left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right)_+ \zeta^2 \right] dx\, d\tau \\ &= \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \boldsymbol{A}(x, \tau, u, Du) \cdot D\left[\left(\ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right) \zeta^2 \right] dx\, d\tau \\ &= \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta^2 \boldsymbol{A}(x, \tau, u, Du) \frac{Du}{u} dx\, d\tau \\ &\quad + 2 \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta \ln \frac{u}{\bar{\mathcal{S}}_\sigma} \boldsymbol{A}(x, \tau, u, Du) D\zeta dx\, d\tau \\ &\geq C_o \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta^2 \frac{|Du|^2}{u^2} dx\, d\tau \\ &\quad - 2C_1 \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta \ln \frac{u}{\bar{\mathcal{S}}_\sigma} \frac{|Du|}{u} |D\zeta| dx\, d\tau \\ &\geq \frac{C_o}{2} \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \zeta^2 \frac{|Du|^2}{u^2} dx\, d\tau - \frac{\gamma}{\sigma^2 \rho^2} \iint_{Q \cap [u > \bar{\mathcal{S}}_\sigma]} \left| \ln \frac{u}{\bar{\mathcal{S}}_\sigma} \right|^2 dx\, d\tau, \end{aligned}$$

where again $\gamma = 2\dfrac{C_1^2}{C_o}$. Therefore, we conclude that

$$\begin{aligned}
\frac{C_o}{2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 \frac{|Du|^2}{u^2} dx\,d\tau &\le \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\left(u \ln \frac{u}{\bar{\mathcal{S}}_\sigma}\right)(x,0)\,dx \\
&\quad + \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2(x) u(x,t)\,dx + \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \left|\ln \frac{u}{\bar{\mathcal{S}}_\sigma}\right|^2 dx\,d\tau \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\left(u \ln \frac{u}{\bar{\mathcal{S}}_\sigma}\right)(x,0)\,dx + \mathcal{S}_\sigma \\
&\quad + \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \left|\ln \frac{u}{\bar{\mathcal{S}}_\sigma}\right|^2 dx\,d\tau \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\left(u \ln \frac{M}{\bar{\mathcal{S}}_\sigma}\right)(x,0)\,dx + \mathcal{S}_\sigma + \frac{\gamma}{\sigma^2\rho^2}\left|\ln \frac{M}{\bar{\mathcal{S}}_\sigma}\right|^2 t\rho^N \\
&\le \mathcal{S}_\sigma + \left(\ln \frac{M}{\bar{\mathcal{S}}_\sigma}\right)\mathcal{S}_\sigma + \frac{\gamma}{\sigma^2}\left|\ln \frac{M}{\bar{\mathcal{S}}_\sigma}\right|^2 \left(\frac{t}{\rho^\lambda}\right).
\end{aligned}$$

We need to evaluate $\ln(M/\bar{\mathcal{S}}_\sigma)$. As in the interval $(0,1]$ the function $f(s) = -\ln s$ is convex, Jensen's inequality yields

$$\begin{aligned}
\ln \frac{M}{\bar{\mathcal{S}}_\sigma} &= \ln \frac{M}{\displaystyle\fint_{K_{(1+\sigma)\rho}} u(x,s_1)dx} = -\ln \fint_{K_{(1+\sigma)\rho}} \frac{u(x,s_1)}{M} dx \\
&\le \fint_{K_{(1+\sigma)\rho}} -\ln \frac{u(x,s_1)}{M} dx = \fint_{K_{(1+\sigma)\rho}} \ln \frac{M}{u(x,s_1)} dx \\
&\le \gamma \varLambda_1,
\end{aligned}$$

where $\gamma = 2^N$. Hence, we have

$$\iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 \frac{|Du|^2}{u^2} dx\,d\tau \le \gamma(\varLambda_1 + 1)\mathcal{S}_\sigma + \frac{\gamma}{\sigma^2}\varLambda_1^2\left(\frac{t}{\rho^\lambda}\right). \tag{6.2}$$

The lemma follows by combining estimates (6.1) and (6.2).

The use of $\left(\ln \dfrac{\bar{\mathcal{S}}_\sigma}{u}\right)_+ \zeta^2$ as test function can be justified using $\left(\ln \dfrac{\bar{\mathcal{S}}_\sigma}{u+\varepsilon}\right)_+$ and then letting $\varepsilon \to 0$. $\square$

Corollary 6.1. *Let u be a non-negative, local, weak solution to the singular equations* (2.5)-(2.6), *in E_T. There exists a positive constant γ depending only on the data $\{N, C_o, C_1\}$, such that for all cylinders $K_{4\rho}(y)\times[s,t]\subset E_T$, and all $\sigma\in(0,1)$,*

$$\frac{1}{\rho}\int_s^t\int_{K_\rho(y)}|\boldsymbol{A}(x,\tau,u,Du)|dx\,d\tau$$

$$\le\gamma\max\Big\{(1+\Lambda_1)^{\frac12};(\Lambda_1^2+\Lambda_2^2)^{\frac12}\Big\}\Big[\mathcal{S}_\sigma+\frac{1}{\sigma^2}\Big(\frac{t-s}{\rho^\lambda}\Big)\Big]^{\frac12}\Big(\frac{t-s}{\rho^\lambda}\Big)^{\frac12}.$$

Proof. Assume $(y,s)=(0,0)$, and let $Q=K_\rho\times(0,t]$. By the structure conditions of $\boldsymbol{A}$

$$\begin{aligned}\frac{1}{\rho}\int_0^t\int_{K_\rho}|\boldsymbol{A}(x,\tau,u,Du)|dx\,d\tau&\le\frac{C_1}{\rho}\iint_Q\frac{|Du|}{u}dx\,d\tau\\&\le\frac{C_1}{\rho}\bigg(\iint_Q\frac{|Du|^2}{u^2}dx\,d\tau\bigg)^{\frac12}\rho^{\frac N2}t^{\frac12}\\&=C_1\bigg(\iint_Q\frac{|Du|^2}{u^2}dx\,d\tau\bigg)^{\frac12}\Big(\frac{t}{\rho^\lambda}\Big)^{\frac12}.\end{aligned}$$

By Lemma 6.1 we conclude. □

6.2 - *Proof of Proposition* 2.1

Assume $(y,s)=(0,0)$. For $n=0,1,2\dots$ set

$$\rho_n=\sum_{j=1}^n\frac{1}{2^j}\rho,\quad K_n=K_{\rho_n};\qquad\tilde\rho_n=\frac{\rho_n+\rho_{n+1}}{2},\quad\tilde K_n=K_{\tilde\rho_n}$$

and let $x\to\zeta_n(x)$ be a non-negative, piecewise smooth cutoff function in $\tilde K_n$ that equals one on K_n, and such that $|D\zeta_n|\le 2^{n+2}/\rho$. In the weak formulation of (2.5)-(2.6) take ζ_n as a test function, to obtain

$$\begin{aligned}\int_{\tilde K_n}u(x,\tau_1)\zeta_n dx&\le\int_{\tilde K_n}u(x,\tau_2)\zeta_n dx+\frac{2^{n+2}}{\rho}\bigg|\int_{\tau_1}^{\tau_2}\int_{\tilde K_n}|\boldsymbol{A}(x,\tau,u,Du)|dx\,d\tau\bigg|\\&\le\int_{\tilde K_n}u(x,\tau_2)\zeta_n dx+\gamma\,2^n\mathcal{S}_{n+1}^{\frac12}\Big(\frac{t}{\rho^\lambda}\Big)^{\frac12}+\gamma\,4^n\Big(\frac{t}{\rho^\lambda}\Big),\end{aligned}$$

where

$$\mathcal{S}_n = \sup_{0\le\tau\le t} \int_{K_n} u(\cdot,\tau)dx.$$

Since the time levels τ_1 and τ_2 are arbitrary, choose τ_2 one for which

$$\int_{K_{2\rho}} u(\cdot,\tau_2)dx = \inf_{0\le\tau\le t} \int_{K_{2\rho}} u(\cdot,\tau)dx \stackrel{\text{def}}{=} \mathcal{I}.$$

With this notation, the previous inequality takes the form

$$\mathcal{S}_n \le \mathcal{I} + \gamma\big(\text{data}, \Lambda_1, \Lambda_2\big)\, 4^n \Big(\frac{t}{\rho^\lambda}\Big) + \gamma\big(\text{data}, \Lambda_1, \Lambda_2\big)\, 2^n \mathcal{S}_{n+1}^{\frac{1}{2}} \Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{2}}.$$

By Young's inequality, for all $\varepsilon_o \in (0,1)$

$$\mathcal{S}_n \le \varepsilon_o \mathcal{S}_{n+1} + \gamma\big(\text{data}, \Lambda_1, \Lambda_2, \varepsilon_o\big) 4^n \Big[\mathcal{I} + \Big(\frac{t}{\rho^\lambda}\Big)\Big].$$

From this, by iteration

$$\mathcal{S}_o \le \varepsilon_o^n \mathcal{S}_n + \gamma(\text{data}, \Lambda_1, \Lambda_2, \varepsilon_o)\Big[\mathcal{I} + \Big(\frac{t}{\rho^\lambda}\Big)\Big] \sum_{i=0}^{n-1} (4\varepsilon_o)^i.$$

Choose ε_o so that the last term is majorized by a convergent series, and let $n \to \infty$. □

The proof of Proposition 4.1 for weak solutions to the porous medium type quasilinear equations (4.5)-(4.6), is similar, with the obvious modifications, and we confine it to Appendix B.

7 - Analyticity in the space variables, of solutions to (1.1)

Let u be a non-negative, local, weak solution to (1.1), satisfying the integrability conditions (1.2) and (1.3) for some $p > N + 2$.

Fix $(x_o, t_o) \in E_T$, assume that $K_{8\rho}(x_o) \subset E$, and assume that the quantity θ defined in (3.1) is positive. The cylinder $(x_o, t_o) + Q_{8\rho}(\theta)$ is assumed to be contained in the domain of definition of u as in (3.2). The quantities M, η and Λ_p are defined as in (3.3)-(3.5).

From the Harnack-type inequality (3.6),

$$[f(\eta)]\, u(x_o, t_o) \le u(x,t) \le [f(\eta)]^{-1} u(x_o, t_o) \tag{7.1}$$

for any (x,t) within the cylinder

$$Q \equiv K_{2\rho}(x_o) \times \Big(t_o - \frac{1}{16}\theta\rho^2, t_o\Big]. \tag{7.2}$$

Equation (1.1) can be rewritten as

$$u_t - \operatorname{div}\Big(\frac{1}{u}\,Du\Big) = 0. \tag{7.3}$$

By (7.1) this can be regarded as a particular instance of a linear parabolic equation with bounded and measurable coefficients. By known results (for example, [**10**], Chapter II) local, weak solutions to (7.3) are locally bounded and locally Hölder continuous. Consequently, (7.3) can be regarded as a linear parabolic equation with bounded, and Hölder continuous coefficients. Again by classical theory (see [**10**], Chapter III), one can conclude that local, weak solutions are indeed C^∞ with respect to the space variable.

By (7.1) the quantity θ can be estimated as

$$\begin{aligned} &\theta \le \varepsilon \sup_{K_\rho(x_o)} u(\cdot, t_o) \le \varepsilon[f(\eta)]^{-1} u(x_o, t_o)\\ &\theta \ge \varepsilon \inf_{K_\rho(x_o)} u(\cdot, t_o) \ge \varepsilon[f(\eta)]\, u(x_o, t_o). \end{aligned} \tag{7.4}$$

Let $\delta = \delta(\eta) \overset{\text{def}}{=} \varepsilon[f(\eta)]$ and introduce the change of variables

$$x \to \frac{x - x_o}{\rho}, \quad t \to \frac{t - t_o}{u(x_o, t_o)\rho^2}, \quad v = \frac{u}{u(x_o, t_o)}.$$

It maps Q onto to

$$\tilde{Q} \overset{\text{def}}{=} K_2 \times \Big(-\frac{1}{16}\frac{\theta}{u(x_o,t_o)}, 0\Big] \supset Q_\delta \overset{\text{def}}{=} K_2 \times \Big(-\frac{\delta}{16}, 0\Big], \tag{7.5}$$

and within Q_δ the function v satisfies

$$v_t - \frac{1}{v}\Delta v = -\frac{|Dv|^2}{v^2}, \tag{7.6}$$

with

$$f(\eta) \le v \le f(\eta)^{-1}.$$

By a result of [**9**], there exist constants $0<\sigma<1$, C and H such that

$$\begin{aligned} &\sup_{Q_{\sigma\delta}} |D^\alpha v| \le C H^{|\alpha|} |\alpha|!,\\ &\sup_{Q_{\sigma\delta}} \Big|\frac{\partial^k}{\partial t^k} v\Big| \le C H^{2k} (2k)! \end{aligned} \tag{7.7}$$

where $Q_{\sigma\delta} = K_{2\sigma} \times \Big(-\frac{1}{16}\sigma\delta, 0\Big]$. Tracing the dependence of constants gives

$$C = \gamma_1 C_o, \quad H = \gamma_2 \max\{C_o[f(\eta)]^{-1}, [f(\eta)]^{-2}\} \tag{7.8}$$

where γ_1 and γ_2 are constants independent of v, and C_o is a function of $f(\eta)$ and satisfies

$$\left|\frac{\partial^k}{\partial t^k} D^\alpha v\right| \le C_o \quad \text{in } Q_\delta \text{ for } |\alpha| + 2k \le 4\Big[\frac{N}{2}\Big] + 16,$$

where $[a]$ denotes the integer part of a. Thus in particular an upper bound on these derivatives up to the indicated order, gives their analyticity as signified by (7.7). Assuming such an upper bound for the moment, we return to the original coordinates to get

$$\begin{aligned} |D^\alpha u(x_o, t_o)| &= |D^\alpha v(0,0)| \frac{u(x_o, t_o)}{\rho^{|\alpha|}} \le \frac{CH^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}} u(x_o, t_o), \\ \left|\frac{\partial^k}{\partial t^k} u(x_o, t_o)\right| &= \left|\frac{\partial^k}{\partial t^k} v(0,0)\right| \frac{u(x_o, t_o)^{1-k}}{\rho^{2k}} \le \frac{CH^{2k}(2k)!}{\rho^{2k}} u(x_o, t_o)^{1-k}. \end{aligned} \tag{7.9}$$

The proof is concluded, once the dependence of C_o on $f(\eta)$ is determined. This estimation can be achieved by local DeGiorgi's or Moser's estimates. While the method is known, it is technically involved and reported in detail in Appendix A.

The analogous analyticity estimates for solutions to the porous medium type equation (4.1)-(4.2) are similar, with the obvious changes, and we omit the details.

Appendix

A - Analyticity in the space variables, of solutions to (1.1). Estimating the first $4\left[\frac{N}{2}\right] + 16$ derivatives of v.

We will use expressions such as $w^4 f(w)$, $w^5 f'$ and similar ones, but we only have at our disposal the notion of weak solution, and therefore, such a way of working does not seem justified. However, by the Harnack estimate of Theorem 3.1, solutions are classical, and in these calculations we are turning the *qualitative* information of u being classical into the *quantitative* information of u being analytic.

With respect to the previous sections, we use a different notation for cylinders, and we let $Q(\rho, \theta) = K_\rho \times (-\theta, 0]$.

A.1 - *An estimate of* $\|Dv\|_\infty$

Take D_{x_i} of the logarithmic diffusion equation and set $w_i = v_{x_i}$ to get

$$\partial_t w_i - \operatorname{div}\Big(\frac{1}{v} Dw_i - \frac{1}{v^2} w_i Dv\Big) = 0.$$

Setting $w=(w_1,\dots,w_N)$, yields

$$w_t - \operatorname{div}\Big(\frac{1}{v}Dw - \frac{1}{v^2}w\otimes w\Big) = 0. \tag{A.1}$$

For all derivations below we stipulate that $\lambda<1$, $\Lambda=\lambda^{-1}>1$, $\lambda<\dfrac{1}{v}<\Lambda$, $\theta<1$ and $\rho<1$.

Proposition A.1. *Let w be a solution to* (A.1) *and ζ be a cutoff function in $Q=Q(\rho,\theta)$. Then*

$$\begin{aligned}
&\sup_{-\theta<t<0}\int_{K_\rho}\int_0^{|w|} sf(s)ds\zeta^2\,dx \\
&\qquad+\frac{\lambda}{2}\iint_Q |Dw|^2 f(|w|)\zeta^2\,dxdt+\frac{\lambda}{2}\iint_Q |w||D|w||^2 f'(|w|)\zeta^2\,dxdt\\
&\le \frac{2\Lambda^2}{\lambda}\iint_Q f(|w|)|w|^2|D\zeta|^2\,dxdt+\frac{2\Lambda^4}{\lambda}\iint_Q |w|^4 f(|w|)\zeta^2\,dxdt\\
&\qquad+\frac{\Lambda^4}{2\lambda}\iint_Q |w|^5 f'(|w|)\zeta^2\,dxdt+\iint_Q\int_0^{|w|} sf(s)ds2\zeta\zeta_t\,dxdt,
\end{aligned}$$

where $f:\mathbb{R}_+\to\mathbb{R}_+$ is a bounded, non-negative, non-decreasing Lipschitz continuous function.

Proof. Multiply (A.1) by $wf(|w|)\zeta^2$ and integrate by parts to get

$$0=\iint_Q\frac{\partial}{\partial t}\int_0^{|w|} sf(s)ds\zeta^2\,dxdt+I$$

where

$$\begin{aligned}
I&=\iint_Q\Big(\frac{1}{v}Dw-\frac{1}{v^2}w\otimes w\Big)\cdot\Big[Dwf(|w|)\zeta^2+wf'(|w|)D|w|\zeta^2+\ wf(|w|)2\zeta D\zeta\Big]dxdt\\
&=\iint_Q\Big[\frac{1}{v}f(|w|)|Dw|^2\zeta^2+\frac{1}{v}wf'(|w|)Dw\cdot D|w|\zeta^2+\frac{1}{v}wf(|w|)2\zeta Dw\cdot D\zeta\\
&\quad-\frac{1}{v^2}w\otimes w\cdot Dwf(|w|)\zeta^2-\frac{1}{v^2}w\otimes w\cdot wf'(|w|)D|w|\zeta^2\\
&\quad-\frac{1}{v^2}w\otimes w\cdot wf(|w|)2\zeta D\zeta\Big]dxdt
\end{aligned}$$

$$\geq \iint_Q \Big[\frac{1}{v}|Dw|^2 f(|w|)\zeta^2 + \frac{1}{v}|w||D|w||^2 f'(|w|)\zeta^2 + \frac{1}{v}|w|f(|w|)2\zeta D|w|\cdot D\zeta$$
$$- \frac{1}{v^2}|w|wD|w|f(|w|)\zeta^2 - \frac{1}{v^2}|w|^2 wD|w|f'(|w|)\zeta^2 - \frac{1}{v^2}|w|^2 wD\zeta f(|w|)2\zeta]dxdt.$$

Observing that $|D|w|| \leq |Dw|$ the previous inequality yields

$$\iint_Q \left[\frac{\partial}{\partial t}\int_0^{|w|} sf(s)ds\zeta^2 + \lambda|Dw|^2 f(|w|)\zeta^2 + \lambda|w||D|w||^2 f'(|w|)\zeta^2 \right] dxdt$$
$$\leq \iint_Q \Big[\Lambda|w|f(|w|)2\zeta D|w|\cdot D\zeta + \Lambda^2|w|^2 D|w|f(|w|)\zeta^2$$
$$+ \Lambda^2|w|^3 D|w|f'(|w|)\zeta^2 + \Lambda^2|w|^3|D\zeta|f(|w|)2\zeta\Big] dxdt$$
$$\leq \frac{\lambda}{2}\iint_Q |Dw|^2 f(|w|)\zeta^2\, dxdt + \frac{\Lambda^2}{\lambda}\iint_Q f(|w|)|w|^2|D\zeta|^2\, dxdt$$
$$+ \frac{\Lambda^4}{\lambda}\iint_Q |w|^4 f(|w|)\zeta^2\, dxdt + \Lambda^2\iint_Q |w|^3|D\zeta|f(|w|)2\zeta\, dxdt$$
$$+ \frac{\lambda}{2}\iint_Q |w||D|w||^2 f'(|w|)\zeta^2\, dxdt + \frac{\Lambda^4}{2\lambda}\iint_Q |w|^5 f'(|w|)\zeta^2\, dxdt.$$

□

We will use this energy estimate and Moser's iteration to derive a bound for $\|Dv\|_\infty$.

Proposition A.2. *There exists a positive parameter δ that depends only on $\frac{\Lambda}{\lambda}$, such that if $\rho \in (0,\delta]$ then*

$$\|w\|_{\infty,Q(\sigma\rho,\sigma\theta)} \leq \gamma\left(\frac{\Lambda}{\lambda}\right)^{\mu_1}\frac{(1+\theta^{-\mu_2})}{(1-\sigma)^{\mu_2}}$$

where γ, μ_1, and μ_2 are positive parameters that depend only on N.

Proof. We first estimate $\|w\|_2$. In the weak formulation of (1.1) take the test function $v\zeta^2$. By standard calculations

$$\lambda\iint_{Q(\rho,\theta)} |Dv|^2\zeta^2\, dxdt \leq \gamma\int_{K_\rho\times\{-\theta\}} v^2\zeta^2\, dx + \gamma\iint_{Q(\rho,\theta)} v|D\zeta|^2\, dxdt.$$

Hence

$$\iint\limits_{Q(\sigma\rho,\sigma\theta)} |Dv|^2\,dxdt \le \gamma\frac{\Lambda^2}{\lambda}\frac{\rho^N}{(1-\sigma)^2}\left[1+\frac{\theta}{\rho^2}\right] \le \gamma\left(\frac{\Lambda}{\lambda}\right)^2\frac{1+\theta}{(1-\sigma)^2\rho^2}. \tag{A.2}$$

The energy estimate (A.1) with $f(|w|)=|w|^{2\beta}$ and $\beta\ge 0$, yields

$$\begin{aligned} &\sup_{-\theta<t<0}\int\limits_{K_\rho} |w|^{2\beta+2}\zeta^2(x,t)\,dx + \iint\limits_{Q(\rho,\theta)} |Dw|^2|w|^{2\beta}\zeta^2\,dxdt \\ &\le \frac{\Lambda^4}{\lambda^2}\iint\limits_{Q(\rho,\theta)} |w|^{2\beta+2}[|D\zeta|^2+\zeta_t]\,dxdt + \frac{\Lambda^4}{\lambda^2}\iint\limits_{Q(\rho,\theta)} |w|^{2\beta+4}\zeta^2\,dxdt. \end{aligned} \tag{A.3}$$

Notice that v is locally Hölder continuous with the Hölder norm and its exponent α depending only on $\dfrac{\Lambda}{\lambda}$ and N, that is

$$|v(x,\cdot)-v(0,\cdot)| \le \gamma\Lambda\rho^\alpha.$$

Now we apply an integration by parts and use the Hölder continuity of v to estimate the last term of (A.3). If we freeze the time variable, then

$$\begin{aligned} I &= \int\limits_{K_\rho} |w|^{2\beta+4}\zeta^2\,dx = \int\limits_{K_\rho} |w|^{2\beta+2} w\cdot w\zeta^2\,dx \\ &= \int\limits_{K_\rho} |w|^{2\beta+2}\zeta^2 Dv\cdot D(v-v(0,\cdot))\,dx \\ &= \int\limits_{K_\rho} (v-v(0,\cdot))\Big[\Delta v|w|^{2\beta+2}\zeta^2 + 2|w|^{2\beta+2}\zeta D\zeta\cdot Dv \\ &\quad +(2\beta+2)|w|^{2\beta+1}D|w|Dv\zeta^2\Big]dx \\ &\le \gamma\Lambda\int\limits_{K_\rho} \rho^\alpha\Big[\Delta v|w|^{2\beta+2}\zeta^2 + 2|w|^{2\beta+2}\zeta D\zeta\cdot Dv \\ &\quad +(2\beta+2)|w|^{2\beta+1}D|w|\cdot Dv\zeta^2\Big]dx \\ &\le \frac{3}{4}\int\limits_{K_\rho} |w|^{2\beta+4}\zeta^2\,dx + \gamma\rho^{2\alpha}\Lambda^2\int\limits_{K_\rho} |w|^{2\beta}|Dw|^2\,dx \\ &\quad + \gamma\rho^{2\alpha}\Lambda^2\int\limits_{K_\rho} |w|^{2\beta+2}|D\zeta|^2\,dx. \end{aligned} \tag{A.4}$$

Since all constants are independent of t, combining (A.3)-(A.4) yields

$$\begin{aligned}
&\sup_{-\theta<t<0} \int_{K_\rho} |w|^{2\beta+2}\zeta^2(x,t)\,dx + \iint_{Q(\rho,\theta)} |Dw|^2|w|^{2\beta}\zeta^2\,dxdt \\
&\le \frac{\Lambda^4}{\lambda^2} \iint_{Q(\rho,\theta)} |w|^{2\beta+2}[|D\zeta|^2+\zeta_t]dxdt \\
&\quad + \gamma\frac{\Lambda^6}{\lambda^2} \iint_{Q(\rho,\theta)} \rho^{2\alpha}\Big[|w|^{2\beta}|Dw|^2 + |w|^{2\beta+2}|D\zeta|^2\Big]dxdt.
\end{aligned}$$

Thus, by taking

$$\gamma\frac{\Lambda^6}{\lambda^2}\rho^{2\alpha} = \frac{1}{2} \tag{A.5}$$

we have

$$\begin{aligned}
&\sup_{-\theta<t<0} \int_{K_\rho} |w|^{2\beta+2}\zeta^2(x,t)\,dx + \iint_{Q(\rho,\theta)} |Dw|^2|w|^{2\beta}\zeta^2\,dxdt \\
&\le \gamma\frac{\Lambda^6}{\lambda^2} \iint_{Q(\rho,\theta)} |w|^{2\beta+2}\Big[|D\zeta|^2+\zeta_t+\rho^{2\alpha}|D\zeta|^2\Big]dxdt \\
&\le \gamma\frac{\Lambda^6}{\lambda^2} \iint_{Q(\rho,\theta)} |w|^{2\beta+2}\Big[|D\zeta|^2+\zeta_t\Big]dxdt,
\end{aligned}$$

as $\rho^{2\alpha}<1$ by (A.5). An application of the Sobolev parabolic embeddings gives

$$\iint_{Q(\sigma\rho,\sigma\theta)} |w|^{(2\beta+2)\frac{N+2}{N}}dxdt \le \left\{\gamma\frac{\Lambda^6}{\lambda^2}\iint_{Q(\rho,\theta)} |w|^{2\beta+2}\Big[|D\zeta|^2+\zeta_t\Big]dxdt\right\}^{1+\frac{2}{N}}.$$

Now take

$$\rho_n = \sigma\rho + \frac{1-\sigma}{2^n}\rho, \qquad \theta_n = \sigma\theta + \frac{1-\sigma}{2^n}\theta, \qquad Q_n = Q(\rho_n,\theta_n),$$

let ζ be a standard cutoff function in Q_n, and set

$$a_n = 2\Big(\frac{N+2}{N}\Big)^n \quad \text{and} \quad I_n = \iint_{Q_n} |w|^{a_n}dxdt.$$

Let $b=4^a$ and $a = 1+\dfrac{2}{N}$. Begin from $2\beta+2=a_o$, that is from $\beta=0$, and apply the above estimate recursively up to $2\beta+2=a_n$. This gives

$$I_{n+1}\le C^a b^n I_n^a,\qquad \text{where}\qquad C=\frac{\Lambda^6}{\lambda^6}\left\{\frac{1}{(1-\sigma)^2\rho^2}+\frac{1}{(1-\sigma)\theta}\right\}.$$

Iterating these recursive inequalities yields

$$I_{n+1}\le C^{\sum_{i=0}^n a^{i+1}}b^{\sum_{i=0}^n (n-i)a^i}I_o^{a^{n+1}}\le C^{\gamma(N)a^{n+1}}b^{\gamma(N)a^{n+1}}I_o^{a^{n+1}}.$$

Now take the $\dfrac{1}{a^{n+1}}$ power of both sides and let $n\to\infty$. Taking into account the estimate of I_o in (A.2), we obtain

$$\begin{aligned}\|w\|_{\infty,Q(\sigma\rho,\sigma\theta)}&\le\gamma\left(\frac{\Lambda}{\lambda}\right)^{\mu_1}\frac{(1+\theta^{-\mu_2})}{(1-\sigma)^{\mu_2}}I_o\\&\le\gamma\left(\frac{\Lambda}{\lambda}\right)^{\mu_1}\frac{(1+\theta^{-\mu_2})}{(1-\sigma)^{\mu_2}},\end{aligned}$$

where the constants μ_1 and μ_2 have been properly modified. □

A.2 - *An upper bound for* $\|v_t\|_\infty$

Proposition A.3. *Let v be a classical solution to the logarithmic diffusion equation and assume $0<\lambda\le v^{-1}\le\Lambda$ in $Q(\rho,\theta)$; then*

$$\|v_t\|_{\infty,Q(\sigma\rho,\sigma\theta)}\le\gamma\left(\frac{\Lambda}{\lambda}\right)^{\mu_1}\frac{(1+\theta^{-\mu_2})}{(1-\sigma)^{\mu_2}},$$

where γ, μ_1, and μ_2 are positive parameters that depend only on N.

Proof. Multiply (1.1) by the test function $v_t\zeta^2$ and integrate over the cylinder $Q_1=Q\Big(\dfrac{\rho+\sigma\rho}{2},\dfrac{\theta+\sigma\theta}{2}\Big)$, where $\sigma\in(0,1)$. Here ζ vanishes on the parabolic boundary of Q_1 and takes value 1 in $Q(\sigma\rho,\sigma\theta)$. A standard calculation gives

$$\begin{aligned}0&=\iint_{Q_1}\left[v_t^2\zeta^2dxdt+\frac1v Dv[\zeta^2Dv_t+2v_t\zeta D\zeta]\right]dxdt\\&=\iint_{Q_1}\left[v_t^2\zeta^2dxdt+\frac{1}{2v}\zeta^2\frac{\partial}{\partial t}|Dv|^2+\frac2v v_t\zeta DvD\zeta\right]dxdt\\&=\iint_{Q_1}v_t^2\zeta^2dxdt+\int_{K_{\frac{\rho+\sigma\rho}{2}}\times\{0\}}\frac{1}{2v}|Dv|^2\zeta^2\,dx\\&\quad-\iint_{Q_1}|Dv|^2\left[-\frac{1}{2v^2}v_t\zeta^2+\frac1v\zeta\zeta_t\right]dxdt+\iint_{Q_1}\frac2v v_t\zeta DvD\zeta dxdt.\end{aligned}$$

This gives the estimate

$$\begin{aligned}\iint_{Q_1} v_t^2\zeta^2 dxdt \le &\iint_{Q_1} |Dv|^2 \frac{1}{2v^2}|v_t|\zeta^2 dxdt + \iint_{Q_1} \frac{1}{v}\zeta|\zeta_t||Dv|^2 dxdt\\ &+ \iint_{Q_1} \frac{1}{v}|Dv||v_t|2\zeta|D\zeta|dxdt\\ \le &\frac{1}{2}\iint_{Q_1} v_t^2\zeta^2 dxdt + \iint_{Q_1} \frac{|Dv|^4}{4v^4}\zeta^2 dxdt\\ &+ \iint_{Q_1} \frac{4|Dv|^2}{v^2}|D\zeta|^2 dxdt + \iint_{Q_1} \frac{|Dv|^2}{v}|\zeta_t|dxdt.\end{aligned}$$

Taking into account the estimate for $\|Dv\|_{\infty,Q_1}$ of the previous section, we have

$$\|v_t\|_{2,Q(\sigma\rho,\sigma\theta)} \le \gamma\left(\frac{\Lambda}{\lambda}\right)^{\mu_1}\frac{(1+\theta^{-\mu_2})}{(1-\sigma)^{\mu_2}},$$

for some $\mu_1(N)$, and $\mu_2(N) > 0$.

Now take the time derivative of the logarithmic diffusion equation, and in the corresponding weak formulation use the test function $v_t f(|v_t|)\zeta^2$ where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded, non-decreasing Lipschitz function, and ζ vanishes on the parabolic boundary of $Q = Q(\rho,\theta)$ and takes value 1 in $Q(\sigma\rho,\sigma\theta)$. Let $M = \|Dv\|_{\infty,Q}$. A standard calculation yields

$$\begin{aligned}&\iint_Q \frac{1}{2}\frac{\partial}{\partial t}|v_t|^2 f\zeta^2 dxdt + \lambda\iint_Q |Dv_t|^2 f\zeta^2 dxdt\\ &\quad + \lambda\iint_Q |Dv_t|^2|v_t|f'\zeta^2 dxdt\\ &\le 2\Lambda\iint_Q |Dv_t||v_t|f\zeta|D\zeta|dxdt\\ &\quad + \Lambda^2\iint_Q |v_t||Dv|\Big[|Dv_t|f\zeta^2 + |v_t|f'|Dv_t|\zeta^2 + 2|v_t|f\zeta|D\zeta|\Big]dxdt\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Let us estimate the four terms.

$$I_1 \le \frac{\lambda}{4}\iint_Q |Dv_t|^2 f\zeta^2 dxdt + \frac{16\Lambda^2}{\lambda}\iint_Q |v_t|^2 f|D\zeta|^2 dxdt;$$

$$I_2 \le \frac{\lambda}{4}\iint_Q |Dv_t|^2 f\zeta^2 dxdt + 4\frac{M^2\Lambda^4}{\lambda}\iint_Q |v_t|^2 f\zeta^2 dxdt;$$

$$I_3 \le \frac{\lambda}{2}\iint_Q |Dv_t|^2 |v_t| f'\zeta^2 dxdt + 4\frac{M^2\Lambda^4}{\lambda}\iint_Q |v_t|^3 f'\zeta^2 dxdt;$$

$$I_4 \le M\Lambda^2\iint_Q |v_t|^2 f\zeta^2 dxdt + M\Lambda^2\iint_Q |v_t|^2 f|D\zeta|^2 dxdt.$$

Summarizing we have

$$\begin{aligned}
&\sup_{-\theta<t<0}\int_{K_\rho}\int_0^{|v_t|} sf(s)ds\zeta^2 dx + \lambda\iint_Q |Dv_t|^2 f\zeta^2 dxdt \\
&\quad + \lambda\iint_Q |D|v_t||^2 f'|v_t|\zeta dxdt \\
&\le \gamma\Big(\frac{\Lambda}{\lambda}\Big)^4 [M^2+1]\iint_Q \Big[|v_t|^2 f\zeta^2 + |v_t|^3 f'\zeta^2 + |v_t|^2 f|D\zeta|^2\Big] dxdt \\
&\quad + 2\iint_Q\int_0^{|v_t|} sf(s)ds\,\zeta\zeta_t dxdt.
\end{aligned}$$

Now take $f(s)=s^\beta$ for $\beta\ge 0$; then

$$\begin{aligned}
&\frac{1}{\beta+2}\sup_{-\theta<t<0}\int_{K_\rho} |v_t|^{\beta+2}\zeta^2 dx + \iint_Q |D|v_t||^2|v_t|^\beta\zeta^2 dxdt \\
&\le \gamma\Big(\frac{\Lambda}{\lambda}\Big)^5 [M^2+1]\iint_Q |v_t|^{\beta+2}\Big[1+\beta+|D\zeta|^2+|\zeta_t|\Big] dxdt.
\end{aligned}$$

Let $a = 1+\dfrac{2}{N}$,

$$C = \gamma\Big(\frac{\Lambda}{\lambda}\Big)^5 [M^2+1]\Big[1+\frac{1}{(1-\sigma)^2\rho^2}+\frac{1}{(1-\sigma)\theta}\Big],$$

and let $w \overset{\text{def}}{=} v_t$. An application of the Sobolev embedding yields

$$\begin{aligned}
&\iint\limits_{Q(\sigma\rho,\sigma\theta)} |w|^{(\beta+2)\frac{N+2}{N}} dxdt \\
&\le \left(\sup_{-\theta<t<0} \int\limits_{K_\rho} \| w|^{\frac{\beta+2}{2}} \zeta|^2 dx \right)^{\frac{2}{N}} \iint\limits_{Q} |D[|w|^{\frac{\beta+2}{2}}\zeta]|^2 dxdt \\
&\le C^a (1+\beta)^{3a} \left(\iint\limits_{Q} |w|^{\beta+2} dxdt \right)^a .
\end{aligned}$$

Take Q_n as before and define

$$\beta_o = 0, \quad \beta_{n+1} + 2 = (\beta_n + 2)\frac{N+2}{N} \quad \Rightarrow \quad \beta_n = 2\left(\frac{N+2}{N}\right)^n - 2,$$

and

$$I_n = \iint\limits_{Q_n} |w|^{\beta_n + 2} dxdt.$$

It then follows that

$$I_{n+1} \le C^a b^n I_n^a$$

for some positive constant b depending only on N. A standard iteration gives

$$\begin{aligned}
I_{n+1} &\le C^{\sum_{i=1}^{n+1} a^i} b^{\sum_{i=0}^{n} (n-i)a^i} I_o^{a^{n+1}} \\
&\le C^{\gamma(N)a^{n+1}} b^{\gamma(N)a^{n+1}} I_o^{a^{n+1}} .
\end{aligned}$$

Therefore, taking the $\frac{1}{a^{n+1}}$ power of both sides and letting $n \to \infty$, we have

$$\|v_t\|_{\infty, Q(\sigma\rho,\sigma\theta)} \le C^\gamma b^\gamma \iint\limits_{Q} |v_t|^2 dxdt.$$

To conclude this section, bound the right hand side using the estimate for $\|v_t\|_2$ on an intermediate cylinder. □

A.3 - *An upper bound for* $\frac{\partial^k}{\partial t^k} D^\alpha v$

Differentiating the logarithmic diffusion equation successively we have

$$\begin{aligned}
&\frac{\partial^k}{\partial t^k} D^\alpha v_t - \operatorname{div} \left(\frac{1}{v} D \frac{\partial^k}{\partial t^k} D^\alpha v + \sum_{j<k} \binom{k}{j} \frac{\partial^{k-j}}{\partial t^{k-j}} \frac{1}{v} \frac{\partial^j}{\partial t^j} D D^\alpha v \right. \\
&\left. + \sum_{|\beta|<|\alpha|} \binom{\alpha}{\beta} \sum_{j\le k} \binom{k}{j} \frac{\partial^{k-j}}{\partial t^{k-j}} D^{\alpha-\beta} \frac{1}{v} \frac{\partial^j}{\partial t^j} D^\beta D v \right) = 0,
\end{aligned} \tag{A.6}$$

where $k \in \mathbb{N}$ and α is a multi-index. For an integer $n > 0$ let

$$|w|^2 = \sum_{k+|\alpha|=n} \left| \frac{\partial^k}{\partial t^k} D^\alpha v \right|^2 .$$

We have the following bound for general derivatives of the logarithmic diffusion equation.

Proposition A.4. *Let v be a classical solution to the logarithmic diffusion equation in $Q(\rho, \theta)$ and fix $\sigma \in (0,1)$. Assume $0 < \lambda \le v^{-1} \le \Lambda$ in $Q(\rho, \theta)$. There exists a positive parameter δ that depends only $\frac{\Lambda}{\lambda}$, such that if $\rho \in (0, \delta]$, then in $Q(\sigma\rho, \sigma\theta)$*

$$\|w\|_{\infty, Q(\sigma\rho, \sigma\theta)} \le \gamma \left(\frac{\Lambda}{\lambda} \right)^{\mu_1} \frac{(1+\theta^{-\mu_2})}{(1-\sigma)^{\mu_2}}, \tag{A.7}$$

where γ, μ_1 and μ_2 are positive parameters that depend only on N and n.

Proof. Multiply (A.6) by the test function $\frac{\partial^k}{\partial t^k} D^\alpha v f(|w|)\zeta^2$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded, non decreasing Lipschitz function. Here ζ vanishes on the parabolic boundary of $Q_2 = Q\Big(\frac{(1+\sigma)\rho}{2}, \frac{(1+\sigma)\theta}{2}\Big)$ and takes value 1 in $Q(\sigma\rho, \sigma\theta)$. Standard calculations and a sum over $k + |\alpha| = n$ give

$$\begin{aligned}
&\iint_{Q_2} \zeta^2 \frac{\partial}{\partial t} \int_0^{|w|} s f(s) ds \, dxdt + \lambda \iint_{Q_2} |w| |D|w||^2 f'(|w|) \zeta^2 dxdt \\
&+ \lambda \sum_{k+|\alpha|=n} \iint_{Q_2} |D \frac{\partial^k}{\partial t^k} D^\alpha v|^2 f(|w|) \zeta^2 dxdt \le I,
\end{aligned}$$

where

$$\begin{aligned}
I = &- \sum_{k+|\alpha|=n} \iint_{Q_2} \frac{2}{v} \zeta f(|w|) \frac{\partial^k}{\partial t^k} D^\alpha v \, D \frac{\partial^k}{\partial t^k} D^\alpha v D\zeta dxdt \\
&- \sum_{k+|\alpha|=n} \iint_{Q_2} \Bigg[\sum_{j<k} \binom{k}{j} \frac{\partial^{k-j}}{\partial t^{k-j}} \frac{1}{v} \frac{\partial^j}{\partial t^j} D D^\alpha v \\
&+ \sum_{|\beta|<|\alpha|} \binom{\alpha}{\beta} \sum_j \binom{k}{j} \frac{\partial^{k-j}}{\partial t^{k-j}} D^{\alpha-\beta} \frac{1}{v} \frac{\partial^j}{\partial t^j} D^\beta D v \Bigg] \\
&\times \Bigg[f(|w|) \zeta^2 D \frac{\partial^k}{\partial t^k} D^\alpha v + \zeta^2 \frac{\partial^k}{\partial t^k} D^\alpha v f'(|w|) D|w| + 2\zeta f(|w|) \frac{\partial^k}{\partial t^k} D^\alpha v D\zeta \Bigg] dxdt.
\end{aligned}$$

Notice that

$$\left|\sum_{j<k}\binom{k}{j}\frac{\partial^{k-j}}{\partial t^{k-j}}\frac{1}{v}\frac{\partial^j}{\partial t^j}DD^\alpha v\right| + \left|\sum_{|\beta|<|\alpha|}\binom{\alpha}{\beta}\sum_{j\le k}\binom{k}{j}\frac{\partial^{k-j}}{\partial t^{k-j}}D^{\alpha-\beta}\frac{1}{v}\frac{\partial^j}{\partial t^j}D^\beta Dv\right|$$
$$\le P[|w|+1]$$

for some polynomial P with variables $\left\{\|\frac{\partial^k}{\partial t^k}D^\alpha v\|_{\infty,Q_2} \text{ for } k+|\alpha|<n; \frac{\Lambda}{\lambda}\right\}$. Thus

$$\begin{aligned}
I \le{}& \frac{\lambda}{4}\sum_{k+|\alpha|=n}\iint_{Q_2}|D\frac{\partial^k}{\partial t^k}D^\alpha v|^2 f(|w|)\zeta^2 dxdt \\
&+\frac{\Lambda^2}{2\lambda}\sum_{k+|\alpha|=n}\iint_{Q_2}|\frac{\partial^k}{\partial t^k}D^\alpha v|^2 f(|w|)|D\zeta|^2 dxdt \\
&+\frac{\lambda}{4}\sum_{k+|\alpha|=n}\iint_{Q_2}|D\frac{\partial^k}{\partial t^k}D^\alpha v|^2 f(|w|)\zeta^2 dxdt \\
&+\frac{\gamma}{\lambda}\iint_{Q_2}P^2(|w|^2+1)f(|w|)\zeta^2 dxdt \\
&+\frac{\lambda}{2}\iint_{Q_2}|D|w||^2 f'(|w|)|w|\zeta^2 dxdt + \frac{\gamma}{\lambda}\iint_{Q_2}P^2(|w|^2+1)|w|f'(|w|)\zeta^2 dxdt \\
&+\iint_{Q_2}P(|w|+1)|w|f(|w|)\zeta|D\zeta|dxdt.
\end{aligned}$$

We obtain the following energy estimate

$$\begin{aligned}
&\sup_{-\frac{\theta+\sigma\theta}{2}<t<0}\int_{K_{\frac{\rho+\sigma\rho}{2}}}\int_0^{|w|} sf(s)ds\zeta^2 dx + \frac{\lambda}{2}\iint_{Q_2}|w||D|w||^2 f'(|w|)\zeta^2 dxdt \\
&\quad+\frac{\lambda}{2}\sum_{k+|\alpha|=n}\iint_{Q_2}|D\frac{\partial^k}{\partial t^k}D^\alpha v|^2 f(|w|)\zeta^2 dxdt \\
&\le \gamma\left(\frac{\Lambda}{\lambda}\right)^2\left[\iint_{Q_2}P^2(|w|^2+1)f(|w|)\zeta^2 dxdt + \iint_{Q_2}|w|^2 f(|w|)|D\zeta|^2 dxdt\right. \\
&\quad\left.+\iint_{Q_2}P^2(|w|^2+1)|w|f'(|w|)\zeta^2 dxdt\right] + 2\iint_{Q_2}\int_0^{|w|} sf(s)ds\,\zeta|\zeta_t|dxdt,
\end{aligned}$$

and also

$$\sup_{-\frac{\theta+\sigma\theta}{2}<t<0}\int_{K_{\frac{\rho+\sigma\rho}{2}}}\int_0^{|w|} sf(s)ds\zeta^2dx+\frac{\lambda}{2}\iint_{Q_2}|w||D|w||^2f'(|w|)\zeta^2dxdt$$
$$+\frac{\lambda}{2}\iint_{Q_2}|D|w||^2f(|w|)\zeta^2dxdt$$
$$\le\gamma\Big(\frac{\Lambda}{\lambda}\Big)^2\Big[\iint_{Q_2}P^2(|w|^2+1)f(|w|)\zeta^2dxdt+\iint_{Q_2}|w|^2f(|w|)|D\zeta|^2dxdt$$
$$+\iint_{Q_2}P^2(|w|^2+1)|w|f'(|w|)\zeta^2dxdt\Big]+2\iint_{Q_2}\int_0^{|w|} sf(s)ds\,\zeta|\zeta_t|dxdt.$$

Now we have at our disposal the sup-estimates for Dv and v_t in terms of Λ/λ only. Next we assume the supremum of $\dfrac{\partial^k}{\partial t^k}D^\alpha v$ is estimated for all $k+|\alpha|<n$ by a similar quantity as the right hand side of (A.7). By Moser's method, the above energy estimate will yield a bound for the case $k+|\alpha|=n$. These will depend on the L^2 norms of $\dfrac{\partial^k}{\partial t^k}D^\alpha v$ and a polynomial with variables $\Big\{\Big\|\dfrac{\partial^k}{\partial t^k}D^\alpha v\Big\|_{\infty,Q_2}$ for $k+|\alpha|<n;\dfrac{\Lambda}{\lambda}\Big\}$.

Take $f(s)=s^\beta$ for $\beta\ge 0$; then the energy estimate yields

$$\frac{1}{\beta+2}\sup_{-\frac{\theta+\sigma\theta}{2}<t<0}\int_{K_{\frac{\rho+\sigma\rho}{2}}}|w|^{\beta+2}\zeta^2dx+\iint_{Q_2}|D|w||^2|w|^\beta\zeta^2dxdt$$
$$\le\gamma\Big(\frac{\Lambda}{\lambda}\Big)^3P^2(1+\beta)\Big[1+\frac{1}{(1-\sigma)^2\rho^2}+\frac{1}{(1-\sigma)\theta}\Big]\iint_{Q_2}[|w|^{\beta+2}+|w|^\beta]dxdt.$$

Let $a=1+\dfrac{2}{N}$ and $C=\gamma\Big(\dfrac{\Lambda}{\lambda}\Big)^3\Big[1+\dfrac{1}{(1-\sigma)^2\rho^2}+\dfrac{1}{(1-\sigma)\theta}\Big]$ and assume $|Q_2|<1$; then an application of the Sobolev embedding yields

$$\iint_{Q(\sigma\rho,\sigma\theta)}|w|^{(\beta+2)\frac{N+2}{N}}dxdt$$
$$\le\Big(\sup_{-\frac{\theta+\sigma\theta}{2}<t<0}\int_{K_{\frac{\rho+\sigma\rho}{2}}}||w|^{\frac{\beta+2}{2}}\zeta|^2dx\Big)^{\frac{2}{N}}\iint_{Q_2}|D[|w|^{\frac{\beta+2}{2}}\zeta]|^2dxdt$$
$$\le C^aP^{2a}(1+\beta)^{3a}\Big[\iint_{Q_2}(|w|^{\beta+2}+1)dxdt\Big]^a$$
$$\le C^aP^{2a}(1+\beta)^{3a}\Big(\iint_{Q_2}|w|^{\beta+2}dxdt\Big)^a+C^aP^{2a}(1+\beta)^{3a},$$

after a proper adjustment of the constant γ in the definition of C. Take

$$\rho_n = \sigma\rho + \frac{1-\sigma}{2^{n+1}}\rho, \qquad \theta_n = \sigma\theta + \frac{1-\sigma}{2^{n+1}}\theta, \qquad Q_n = Q(\rho_n, \theta_n).$$

Define

$$\beta_o = 0, \quad \beta_{n+1} + 2 = (\beta_n + 2)\frac{N+2}{N} \quad \Rightarrow \quad \beta_n = 2\left(\frac{N+2}{N}\right)^n - 2,$$

and

$$I_n = \iint\limits_{Q_n} |w|^{\beta_n+2} dxdt.$$

We have that

$$I_{n+1} \leq C^a P^{2a} b^n I_n^a + C^a P^{2a} b^n,$$

where the constant b depends only on N. A standard iteration and a proper adjustment of P give

$$\begin{aligned} I_{n+1} &\leq 2^{\sum_{i=1}^n (a^i-1)} (CP^2)^{\sum_{i=1}^{n+1} a^i} b^{\sum_{i=0}^n (n-i)a^i} I_o^{a^{n+1}} \\ &\quad + 2^{\sum_{i=1}^n (a^i-1)} (CP^2)^{\sum_{i=1}^{n+1} a^i} b^{\sum_{i=0}^n (n-i)a^i} \\ &\leq (CP^2)^{\gamma(N)a^{n+1}} I_o^{\gamma(N)a^{n+1}} + (CP^2)^{\gamma(N)a^{n+1}}. \end{aligned}$$

Take the $\frac{1}{a^{n+1}}$ power of both sides and let $n \to \infty$ to obtain

$$\|w\|_{\infty, Q(\sigma\rho,\sigma\theta)} \leq (CP^2)^\gamma \iint\limits_{Q_2} |w|^2 dxdt + (CP^2)^\gamma. \tag{A.8}$$

Remember that $Q_2 = Q\left(\frac{\rho+\sigma\rho}{2}, \frac{\theta+\sigma\theta}{2}\right)$; in order to conclude the proof of Proposition A.4, we only need to estimate $\|w\|_{2,Q_2}$. □

A.3.1 - An estimate of $\|w\|_{2,Q_2}$

It is enough to give an estimate of $\|w\|_{2,Q(\sigma\rho,\sigma\theta)}$. Replace k in (A.6) by $k-1$ and assume $k \geq 1$. We can rewrite (A.6) as

$$\frac{\partial^{k-1}}{\partial t^{k-1}} D^\alpha v_t - \operatorname{div}\left(\frac{1}{v} D \frac{\partial^{k-1}}{\partial t^{k-1}} D^\alpha v\right) = \operatorname{div} f,$$

where

$$f = \sum_{j<k-1} \binom{k-1}{j} \frac{\partial^{k-1-j}}{\partial t^{k-1-j}} \frac{1}{v} \frac{\partial^j}{\partial t^j} DD^\alpha v$$
$$+ \sum_{|\beta|<|\alpha|} \binom{\alpha}{\beta} \sum_{j\le k-1} \binom{k-1}{j} \frac{\partial^{k-1-j}}{\partial t^{k-1-j}} D^{\alpha-\beta} \frac{1}{v} \frac{\partial^j}{\partial t^j} D^\beta D v.$$

If ζ is a smooth function in Q_2 and takes value 1 in $Q(\sigma\rho, \sigma\theta)$, then the standard L^2 estimate for the linear parabolic equations gives

$$\begin{aligned} &\| \frac{\partial^k}{\partial t^k} D^\alpha v\|^2_{2,Q(\sigma\rho,\sigma\theta)} \\ &\le \gamma \left[\|\mathrm{div} f\|^2_{2,Q_2} + \left(\frac{1}{(1-\sigma)\theta} + \frac{1}{(1-\sigma)^2\rho^2} \right) \| \frac{\partial^{k-1}}{\partial t^{k-1}} D^\alpha v\|^2_{2,Q_2} \right]. \end{aligned} \tag{A.9}$$

Let us denote P as a polynomial of variables $\left\{ \| \frac{\partial^k}{\partial t^k} D^\alpha v\|_{\infty,Q_2} \text{ for } k+|\alpha|<n; \frac{\Lambda}{\lambda} \right\}$. Observe that

$$\|\mathrm{div} f\|^2_{2,Q_2} \le \gamma P[I_1 + I_2 + I_3 + 1]$$

where

$$I_1 = \iint_{Q_2} \left| \frac{\partial^{k-1}}{\partial t^{k-1}} D v \right|^2 dxdt;$$
$$I_2 = \iint_{Q_2} \left| \frac{\partial^{k-2}}{\partial t^{k-2}} D^{|\alpha|+2} v \right|^2 dxdt;$$
$$I_3 = \iint_{Q_2} \left| \frac{\partial^{k-1}}{\partial t^{k-1}} D^{|\alpha|+1} v \right|^2 dxdt.$$

Here for an integer l

$$|D^l v|^2 = \sum_{|\beta|=l} |D^\beta v|^2.$$

These quantities can all be estimated in the same way. Indeed, they all contain spatial derivatives and we can use the principal part of the differentiated (1.1) to estimate them. Precisely, we write (A.6) as

$$\frac{\partial^{k-s}}{\partial t^{k-s}} D^\eta v_t - \mathrm{div} \left(\frac{1}{v} D \frac{\partial^{k-s}}{\partial t^{k-s}} D^\eta v + \sum_{j<k-s} \binom{k-s}{j} \frac{\partial^{k-s-j}}{\partial t^{k-s-j}} \frac{1}{v} \frac{\partial^j}{\partial t^j} DD^\eta v \right.$$
$$\left. + \sum_{|\beta|<|\eta|} \binom{\eta}{\beta} \sum_{j\le k-s} \binom{k-s}{j} \frac{\partial^{k-s-j}}{\partial t^{k-s-j}} D^{\eta-\beta} \frac{1}{v} \frac{\partial^j}{\partial t^j} D^\beta D v \right) = 0,$$

where $1 \leq s \leq k$ and $|\eta| = |\alpha| + s$. Take the test function

$$\frac{\partial^{k-s}}{\partial t^{k-s}} D^{\eta} v \zeta^2$$

where ζ vanishes on the parabolic boundary of $Q_3 = Q\Big(\frac{3(1+\sigma)\rho}{4}, \frac{3(1+\sigma)\theta}{4}\Big)$ and takes value 1 in Q_2. Integrating in Q_3, a standard calculation yields

$$\begin{aligned}
&\iint\limits_{Q_3} \frac{1}{v} |D \frac{\partial^{k-s}}{\partial t^{k-s}} D^{\eta} v|^2 \zeta^2 \, dxdt \\
&= -\frac{1}{2} \iint\limits_{Q_3} \frac{\partial}{\partial t} \left| \frac{\partial^{k-s}}{\partial t^{k-s}} D^{\eta} v \right|^2 \zeta^2 \, dxdt \\
&\quad - \iint\limits_{Q_3} \Big[\sum_{j<k-s} \binom{k-s}{j} \frac{\partial^{k-s-j}}{\partial t^{k-s-j}} \frac{1}{v} \frac{\partial^j}{\partial t^j} D D^{\eta} v \\
&\quad + \sum_{|\beta|<|\eta|} \binom{\eta}{\beta} \sum_{j \leq k-s} \binom{k-s}{j} \frac{\partial^{k-s-j}}{\partial t^{k-s-j}} D^{\eta-\beta} \frac{1}{v} \frac{\partial^j}{\partial t^j} D^{\beta} D v \Big] \\
&\quad \times \Big[D \frac{\partial^{k-s}}{\partial t^{k-s}} D^{\eta} v \zeta^2 + 2 \frac{\partial^{k-s}}{\partial t^{k-s}} D^{\eta} v \zeta D \zeta \Big] \\
&\leq \frac{\lambda}{2} \iint\limits_{Q_3} |D \frac{\partial^{k-s}}{\partial t^{k-s}} D^{\eta} v|^2 \zeta^2 \, dxdt + \gamma \Big[\frac{1}{(1-\sigma)\theta} + \frac{1}{(1-\sigma)^2 \rho^2} \Big] P.
\end{aligned}$$

This together with (A.9) gives

$$\| \frac{\partial^k}{\partial t^k} D^{\alpha} v \|^2_{2, Q(\sigma\rho, \sigma\theta)} \leq \gamma \Big[\frac{1}{(1-\sigma)\theta} + \frac{1}{(1-\sigma)^2 \rho^2} \Big] P. \tag{A.10}$$

Now the only remaining case is $|\alpha| = n$. For this, we consider the equation (A.6) with $k = 0$ and assume $|\alpha| = n - 1$. Take the test function $\zeta^2 D^{\alpha} v$, where ζ vanishes on the parabolic boundary of Q_3 and takes 1 in $Q(\sigma\rho, \sigma\theta)$.

$$\begin{aligned}
\iint\limits_{Q_3} \frac{1}{v} |D D^{\alpha} v|^2 \zeta^2 dxdt &= -\iint\limits_{Q_3} \frac{1}{2} \frac{\partial}{\partial t} (D^{\alpha} v)^2 \zeta^2 dxdt \\
&\quad - \iint\limits_{Q_3} \frac{1}{v} D D^{\alpha} v D^{\alpha} v 2 \zeta D \zeta dxdt \\
&\quad - \sum_{|\beta|<|\alpha|} \iint\limits_{Q_3} \binom{\alpha}{\beta} D^{\alpha-\beta} \frac{1}{v} D D^{\beta} v D D^{\alpha} v \zeta^2 dxdt
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{|\beta|<|\alpha|} \iint\limits_{Q_3} \binom{\alpha}{\beta} D^{\alpha-\beta}\frac{1}{v} DD^{\beta}vDD^{\alpha}vD^{\alpha}2\zeta D\zeta dxdt \\
&\le \frac{\lambda}{2}\iint\limits_{Q_3} |DD^{\alpha}v|^2\zeta^2 dxdt + \left[\frac{1}{(1-\sigma)\theta}+\frac{1}{(1-\sigma)^2\rho^2}\right]P.
\end{aligned}
$$

Summing over all $|\alpha| = n-1$ actually gives

$$
\sum_{|\alpha|=n} \iint\limits_{Q_3} |D^{\alpha}v|^2\zeta^2 dxdt \le \left[\frac{1}{(1-\sigma)\theta}+\frac{1}{(1-\sigma)^2\rho^2}\right]P.
$$

If we take into consideration an intermediate cylinder, then this, together with (A.10) in (A.8), yields

$$
\|w\|_{\infty,Q(\sigma\rho,\sigma\theta)} \le P^{\gamma}\left[\frac{1}{(1-\sigma)\theta}+\frac{1}{(1-\sigma)\rho}\right]^{\gamma} \tag{A.11}
$$

for some γ depending only on N. The induction hypothesis and the definition of ρ in (A.5) imply that

$$
\|w\|_{\infty,Q(\sigma\rho,\sigma\theta)} \le \gamma_1(N,n)\left(\frac{\Lambda}{\lambda}\right)^{\mu_1}\frac{1+\theta^{-\mu_2}}{(1-\sigma)^{\mu_2}}. \tag{A.12}
$$

□

B - Proof of Proposition 4.1 for weak solutions to equations (4.5)-(4.6)

An auxiliary lemma

Lemma B.1. *Let u be a non-negative, local, weak solution to the singular equations (4.5)-(4.6), in E_T. There exist two positive constants γ_1, γ_2 depending only on the data $\{N, C_o, C_1\}$, such that for all cylinders $K_{4\rho}(y)\times[s,t]\subset E_T$, and all $\sigma\in(0,1)$,*

$$
\int_s^t \int\limits_{K_\rho(y)} \frac{|Du|^2}{u^{2-\frac{m}{2}}}\zeta^2 dx\, d\tau \le \gamma_1(1+\Lambda_{\frac{m}{2},1})\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}} + \frac{\gamma_2}{\sigma^2\rho^2}(\Lambda^2_{\frac{m}{2},1}+\Lambda^2_{\frac{m}{2},2})\mathcal{S}_\sigma^{\frac{m}{2}}(t-s)\rho^{N(1-\frac{m}{2})},
$$

where

$$
\mathcal{S}_\sigma = \sup_{s<\tau<t}\int\limits_{K_{(1+\sigma)\rho}(y)} u(\cdot,\tau)dx.
$$

Proof. In the following we restrict to $0<m<\frac{2}{3}$, since we are mainly interested in proving the stability of the estimates as $m \to 0^+$. For $m \in \left(\frac{1}{3}, 1\right)$ similar arguments hold, provided a slightly different test function φ is chosen (see [**5**], § B.1.1 for more details).

Assume $(y,s)=(0,0)$, fix $\sigma \in (0,1)$, and let $x \to \zeta(x)$ be a non-negative piecewise smooth cutoff function in $K_{(1+\sigma)\rho}$ that vanishes outside $K_{(1+\sigma)\rho}$, equals one on K_ρ, and such that $|D\zeta| \le (\sigma\rho)^{-1}$. Let $s_1 \in [0,t]$ be such that

$$\mathcal{S}_\sigma = \sup_{0<s<t} \int\limits_{K_{(1+\sigma)\rho}(y)} u(\cdot,s)dx = \int\limits_{K_{(1+\sigma)\rho}(y)} u(\cdot,s_1)dx,$$

and set

$$\bar{\mathcal{S}}_\sigma \overset{\text{def}}{=} \frac{\mathcal{S}_\sigma}{\rho^N}.$$

In the weak formulation of (4.5)-(4.6) take the test function

$$\varphi = \left(\frac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)_+ \zeta^2,$$

and integrate over $Q = K_{(1+\sigma)\rho} \times (0,t]$, to obtain

$$\begin{aligned} 0 &= \iint\limits_Q \frac{\partial}{\partial\tau} u \left(\frac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)_+ \zeta^2 dx\, d\tau \\ &\quad + \iint\limits_Q \boldsymbol{A}(x,\tau,u,Du) \cdot D\left[\left(\frac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)_+ \zeta^2\right] dx\, d\tau \\ &= I_1 + I_2. \end{aligned}$$

We estimate these two terms separately.

$$\begin{aligned} I_1 &= \iint\limits_Q \frac{\partial}{\partial\tau} u \left(\frac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)_+ \zeta^2 dx\, d\tau \\ &= \iint\limits_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \frac{\partial}{\partial\tau} u \left(\frac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right) \zeta^2 dx\, d\tau \\ &= -\int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2(x) \left(\int_{u(x,t)}^{\bar{\mathcal{S}}_\sigma} \frac{s^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m} ds \right) dx \\ &\quad + \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2(x) \left(\int_{u(x,0)}^{\bar{\mathcal{S}}_\sigma} \frac{s^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m} ds \right) dx. \end{aligned}$$

Next,

$$
\begin{aligned}
I_2 &= \iint_Q \boldsymbol{A}(x,\tau,u,Du)\cdot D\left[\left(\frac{u^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)_+\zeta^2\right]dx\,d\tau\\
&= \iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \boldsymbol{A}(x,\tau,u,Du)\cdot D\left[\left(\frac{u^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)\zeta^2\right]dx\,d\tau\\
&= -\frac{1}{2}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{-\frac{m}{2}-1}\boldsymbol{A}(x,\tau,u,Du)\cdot Du\,dx\,d\tau\\
&\quad +2\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta\left(\frac{u^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)\boldsymbol{A}(x,\tau,u,Du)\cdot D\zeta\,dx\,d\tau\\
&\le -\frac{C_o}{2}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} u^{-\frac{m}{2}-1}u^{m-1}\zeta^2|Du|^2\,dx\,d\tau\\
&\quad +2C_1\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta\left(\frac{u^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)u^{m-1}|Du||D\zeta|dx\,d\tau\\
&\le -\frac{C_o}{4}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{\frac{m}{2}-2}|Du|^2\,dx\,d\tau\\
&\quad +\frac{\gamma}{\sigma^2\rho^2}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{u^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)^2dx\,d\tau,
\end{aligned}
$$

where $\gamma = 4\,\dfrac{C_1^2}{C_o}$. Therefore, we conclude that

$$
\begin{aligned}
&\frac{C_o}{4}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{\frac{m}{2}-2}|Du|^2\,dx\,d\tau\\
&\quad\le \int_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\left(\int_{u(x,0)}^{\bar{\mathcal{S}}_\sigma}\frac{s^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\,ds\right)dx\\
&\qquad +\frac{\gamma}{\sigma^2\rho^2}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{u^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)^2dx\,d\tau\\
&\quad= J_1+J_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_1 &= \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2(x)\left(\int\limits_{u(x,0)}^{\bar{\mathcal{S}}_\sigma} \frac{s^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\,ds\right)dx \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}\left(\int\limits_{u(x,0)}^{\bar{\mathcal{S}}_\sigma} \frac{\left(\frac{\bar{\mathcal{S}}_\sigma}{s}\right)^{\frac{m}{2}}-1}{m}\,d\left(\frac{s}{\bar{\mathcal{S}}_\sigma}\right)\right)dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}\left(\int\limits_{\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^m}^{1} \frac{y^{-\frac{1}{2}}-1}{m}\frac{y^{\frac{1}{m}-1}}{m}\,dy\right)dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \frac{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}}{m^2}\left[\frac{2m}{2-m}y^{\frac{1}{m}-\frac{1}{2}}-my^{\frac{1}{m}}\right]_{\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^m}^{1} dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \frac{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}}{m}\left[\frac{2}{2-m}\left(1-\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^{1-\frac{m}{2}}\right)-\left(1-\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)\right]dx \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \frac{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}}{m}\left[\frac{m}{2-m}-\frac{2}{2-m}\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\left(\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^{-\frac{m}{2}}-1\right)\right]dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}\left[\frac{1}{2-m}+\frac{2}{2-m}\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\left(\frac{1-\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^{-\frac{m}{2}}}{m}\right)\right]dx \\
&\le \frac{1}{2-m}\int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}dx \\
&\quad +\frac{2}{2-m}\int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} u(x,0)\frac{u(x,0)^{-\frac{m}{2}}-\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}dx = J_1'+J_1''.
\end{aligned}
$$

$$
J_1' = \frac{1}{2-m}\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}\int\limits_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} dx \le \frac{\gamma}{2-m}\mathcal{S}_\sigma^{1-\frac{m}{2}}\rho^{N\frac{m}{2}}, \qquad \text{where} \quad \gamma = 2^N.
$$

$$
\begin{aligned}
J_1'' &= \frac{2}{2-m} \int_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} u(x,0)\frac{u(x,0)^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\,dx \\
&= \frac{2}{2-m} \int_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \frac{u(x,0)^{1-\frac{m}{2}}}{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}} \frac{\bar{\mathcal{S}}_\sigma^{\frac{m}{2}} - u(x,0)^{\frac{m}{2}}}{m\bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}\,dx \\
&\le \frac{2}{2-m}\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \int_{K_{(1+\sigma)\rho}\cap[u<\bar{\mathcal{S}}_\sigma]} \frac{M^{\frac{m}{2}} - u(x,0)^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\,dx \\
&\le \frac{2\gamma}{2-m}\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \Lambda_{\frac{m}{2},1}\rho^N = \frac{2\gamma}{2-m}\Lambda_{\frac{m}{2},1}\mathcal{S}_\sigma^{1-\frac{m}{2}}\rho^{N\frac{m}{2}}, \qquad \text{where} \quad \gamma = 2^N.
\end{aligned}
$$

Therefore,

$$
J_1 \le \gamma(1+\Lambda_{\frac{m}{2},1})\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}}.
$$

Moreover,

$$
\begin{aligned}
J_2 &= \frac{\gamma}{\sigma^2\rho^2} \iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)^2 dx\,d\tau \\
&= \frac{\gamma}{\sigma^2\rho^2} \iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{\bar{\mathcal{S}}_\sigma^{\frac{m}{2}} - u^{\frac{m}{2}}}{mu^{\frac{m}{2}}\bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}\right)^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2} \iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} u^{\frac{m}{2}}\left(\frac{M^{\frac{m}{2}} - u^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2}\bar{\mathcal{S}}_\sigma^{\frac{m}{2}} \iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \left(\frac{M^{\frac{m}{2}} - u^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2}\bar{\mathcal{S}}_\sigma^{\frac{m}{2}} t\rho^N \sup_{0<\tau<t} \fint_{K_{(1+\sigma)\rho}} \left(\frac{M^{\frac{m}{2}} - u^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 dx \\
&\le \frac{\gamma}{\sigma^2\rho^2}\Lambda^2_{\frac{m}{2},2}\mathcal{S}_\sigma^{\frac{m}{2}}\rho^{N(1-\frac{m}{2})}\,t.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\frac{C_o}{4}\iint_{Q\cap[u<\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{\frac{m}{2}-2}|Du|^2 dx\,d\tau \\
&\le \gamma(\Lambda_{\frac{m}{2},1}+1)\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}} + \frac{\gamma}{\sigma^2\rho^2}\Lambda^2_{\frac{m}{2},2}\mathcal{S}_\sigma^{\frac{m}{2}}\rho^{N(1-\frac{m}{2})}\,t.
\end{aligned}
\tag{B.1}
$$

Now, if we take the test function

$$\varphi = \left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right)_+ \zeta^2$$

in the weak formulation of (4.5)–(4.6) and integrate over $Q = K_{(1+\sigma)\rho} \times (0, t]$, we obtain

$$\begin{aligned}
0 &= \iint_Q \frac{\partial}{\partial \tau} u \left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right)_+ \zeta^2 dx\, d\tau \\
&\quad + \iint_Q \boldsymbol{A}(x, \tau, u, Du) \cdot D \left[\left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right)_+ \zeta^2 \right] dx\, d\tau \\
&= I_3 + I_4.
\end{aligned}$$

We estimate these two terms separately.

$$\begin{aligned}
I_3 &= \iint_Q \frac{\partial}{\partial \tau} u \left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right)_+ \zeta^2 dx\, d\tau \\
&= \iint_{Q \cap [u > \bar{S}_\sigma]} \frac{\partial}{\partial \tau} u \left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right) \zeta^2 dx\, d\tau \\
&= \int_{K_{(1+\sigma)\rho} \cap [u > \bar{S}_\sigma]} \zeta^2(x) \left(\int_{\bar{S}_\sigma}^{u(x,t)} \frac{\bar{S}_\sigma^{-\frac{m}{2}} - s^{-\frac{m}{2}}}{m}\, ds \right) dx \\
&\quad - \int_{K_{(1+\sigma)\rho} \cap [u > \bar{S}_\sigma]} \zeta^2(x) \left(\int_{\bar{S}_\sigma}^{u(x,0)} \frac{\bar{S}_\sigma^{-\frac{m}{2}} - s^{-\frac{m}{2}}}{m}\, ds \right) dx.
\end{aligned}$$

Next,

$$\begin{aligned}
I_4 &= \iint_Q \boldsymbol{A}(x, \tau, u, Du) \cdot D \left[\left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right)_+ \zeta^2 \right] dx\, d\tau \\
&= \iint_{Q \cap [u > \bar{S}_\sigma]} \boldsymbol{A}(x, \tau, u, Du) \cdot D \left[\left(\frac{\bar{S}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m} \right) \zeta^2 \right] dx\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{-\frac{m}{2}-1} \boldsymbol{A}(x,\tau,u,Du)\cdot Du\,dx\,d\tau \\
&\quad + 2 \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta\left(\frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m}\right) \boldsymbol{A}(x,\tau,u,Du)\cdot D\zeta dx\,d\tau \\
&\ge \frac{C_o}{2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{-\frac{m}{2}-1} u^{m-1} |Du|^2 dx\,d\tau \\
&\quad - 2C_1 \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta\left(\frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m}\right) u^{m-1}|Du||D\zeta| dx\,d\tau \\
&\ge \frac{C_o}{4} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{\frac{m}{2}-2} |Du|^2 dx\,d\tau \\
&\quad - \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - u^{-\frac{m}{2}}}{m}\right)^2 dx\,d\tau,
\end{aligned}
$$

where again $\gamma = 4\,\dfrac{C_1^2}{C_o}$. Therefore, we conclude that

$$
\begin{aligned}
&\frac{C_o}{4} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{\frac{m}{2}-2} |Du|^2 dx\,d\tau \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2(x) \left(\int_{\bar{\mathcal{S}}_\sigma}^{u(x,0)} \frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - s^{-\frac{m}{2}}}{m}\, ds\right) dx \\
&\quad + \frac{\gamma}{\sigma^2\rho^2} \iint\limits_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{u^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{m u^{\frac{m}{2}} \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}\right)^2 dx\,d\tau = J_3 + J_4.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_3 &= \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2(x) \left(\int_{\bar{\mathcal{S}}_\sigma}^{u(x,0)} \frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - s^{-\frac{m}{2}}}{m}\, ds\right) dx \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \left(\int_{\bar{\mathcal{S}}_\sigma}^{u(x,0)} \frac{1-\left(\frac{\bar{\mathcal{S}}_\sigma}{s}\right)^{\frac{m}{2}}}{m}\, d\left(\frac{s}{\bar{\mathcal{S}}_\sigma}\right)\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \left(\int\limits_1^{\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^m} \frac{1-y^{-\frac{1}{2}}}{m} \frac{y^{\frac{1}{m}-1}}{m}\, dy \right) dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \frac{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}}{m^2} \left[m y^{\frac{1}{m}} - \frac{2m}{2-m} y^{\frac{1}{m}-\frac{1}{2}} \right]_1^{\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma}\right)^m} dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \frac{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}}{m} \left[\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma} - 1 \right) - \frac{2}{2-m} \left(\left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma} \right)^{1-\frac{m}{2}} - 1 \right) \right] dx \\
&\le \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \frac{\bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}}}{m} \left[\frac{m}{2-m} + \frac{2}{2-m} \frac{u(x,0)}{\bar{\mathcal{S}}_\sigma} \left(1 - \left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma} \right)^{-\frac{m}{2}} \right) \right] dx \\
&= \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \left[\frac{1}{2-m} + \frac{2}{2-m} \frac{u(x,0)}{\bar{\mathcal{S}}_\sigma} \left(\frac{1 - \left(\frac{u(x,0)}{\bar{\mathcal{S}}_\sigma} \right)^{-\frac{m}{2}}}{m} \right) \right] dx \\
&\le \frac{1}{2-m} \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} dx \\
&\quad + \frac{2}{2-m} \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} u(x,0) \frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - u(x,0)^{-\frac{m}{2}}}{m} dx = J_3' + J_3''.
\end{aligned}
$$

$$
J_3' = \frac{1}{2-m} \bar{\mathcal{S}}_\sigma^{1-\frac{m}{2}} \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} dx \le \frac{\gamma}{2-m} \mathcal{S}_\sigma^{1-\frac{m}{2}} \rho^{N\frac{m}{2}} \quad \text{where} \quad \gamma = 2^N.
$$

$$
\begin{aligned}
J_3'' &= \frac{2}{2-m} \int\limits_{K_{(1+\sigma)\rho}\cap[u>\bar{\mathcal{S}}_\sigma]} u(x,0) \frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - u(x,0)^{-\frac{m}{2}}}{m} dx \\
&\le \frac{2}{2-m} \frac{\bar{\mathcal{S}}_\sigma^{-\frac{m}{2}} - M^{-\frac{m}{2}}}{m} \mathcal{S}_\sigma = \frac{2}{2-m} \frac{M^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{m M^{\frac{m}{2}}} \rho^{N\frac{m}{2}} \mathcal{S}_\sigma^{1-\frac{m}{2}}.
\end{aligned}
$$

As in the interval $(0, M]$ the function

$$
f(s) = \frac{M^{\frac{m}{2}} - s^{\frac{m}{2}}}{m M^{\frac{m}{2}}}
$$

is convex, we can apply Jensen's inequality and conclude that

$$
\begin{aligned}
J_3'' &\le \frac{2}{2-m}\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}} \fint_{K_{(1+\sigma)\rho}} \frac{M^{\frac{m}{2}} - u(x,s_1)^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\,dx \\
&\le \gamma \Lambda_{\frac{m}{2},1}\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}}.
\end{aligned}
$$

Therefore,

$$
J_3 \le \gamma(1+\Lambda_{\frac{m}{2},1})\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}}.
$$

As for J_4 we have

$$
\begin{aligned}
J_4 &= \frac{\gamma}{\sigma^2\rho^2}\iint_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \frac{u^{\frac{3}{2}m}}{\bar{\mathcal{S}}_\sigma^m}\left(\frac{u^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{mu^{\frac{m}{2}}}\right)^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2}\frac{1}{\bar{\mathcal{S}}_\sigma^m}\iint_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} u^{\frac{3}{2}m}\left(\frac{M^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 dx\,d\tau \\
&\le \frac{\gamma}{\sigma^2\rho^2}\frac{1}{\bar{\mathcal{S}}_\sigma^m}\left(\frac{M^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 t\sup_{0<\tau<t}\int_{K_{(1+\sigma)\rho}} u^{\frac{3}{2}m}(x,\tau)\,dx \\
&\le \frac{\gamma}{\sigma^2\rho^2}\frac{\rho^{Nm}}{\mathcal{S}_\sigma^m}\left(\frac{M^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 t\left(\sup_{0<\tau<t}\int_{K_{(1+\sigma)\rho}} u(x,\tau)\,dx\right)^{\frac{3}{2}m}\rho^{N(1-\frac{3}{2}m)} \\
&= \frac{\gamma}{\sigma^2\rho^2}\mathcal{S}_\sigma^{\frac{m}{2}}\left(\frac{M^{\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{\frac{m}{2}}}{mM^{\frac{m}{2}}}\right)^2 t\,\rho^{N(1-\frac{m}{2})} \\
&\le \frac{\gamma}{\sigma^2\rho^2}\Lambda_{\frac{m}{2},1}^2\mathcal{S}_\sigma^{\frac{m}{2}}\,t\,\rho^{N(1-\frac{m}{2})},
\end{aligned}
$$

where we have taken into account Jensen's inequality once more. Hence, we have

$$
\begin{aligned}
&\frac{C_o}{4}\iint_{Q\cap[u>\bar{\mathcal{S}}_\sigma]} \zeta^2 u^{\frac{m}{2}-2}|Du|^2 dx\,d\tau \\
&\le \gamma(\Lambda_{\frac{m}{2},1}+1)\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}} + \frac{\gamma}{\sigma^2\rho^2}\Lambda_{\frac{m}{2},1}^2\mathcal{S}_\sigma^{\frac{m}{2}}\,t\,\rho^{N(1-\frac{m}{2})}.
\end{aligned}
\tag{B.2}
$$

The lemma follows by combining estimates (B.1) and (B.2).

The use of $\left(\dfrac{u^{-\frac{m}{2}} - \bar{\mathcal{S}}_\sigma^{-\frac{m}{2}}}{m}\right)_+\zeta^2$ as test function can be justified using $u+\varepsilon$ instead of u, and then letting $\varepsilon \to 0$. □

Corollary B.1. *Let u be a non-negative, local, weak solution to the singular equations* (4.5)-(4.6), *in E_T. There exists a positive constant γ depending only on the data $\{N, C_o, C_1\}$, such that for all cylinders $K_{4\rho}(y)\times[s,t]\subset E_T$, and all $\sigma\in(0,1)$,*

$$\frac{1}{\rho}\int_s^t\int_{K_\rho(y)}|\boldsymbol{A}(x,\tau,u,Du)|dx\,d\tau\le\frac{\gamma}{\sigma}(\Lambda^2_{\frac{m}{2},1}+\Lambda^2_{\frac{m}{2},2})^{\frac12}\left(\frac{t-s}{\rho^\lambda}\right)\mathcal{S}^m_\sigma$$
$$+\gamma(1+\Lambda_{\frac{m}{2},1})^{\frac12}\left(\frac{t-s}{\rho^\lambda}\right)^{\frac12}\mathcal{S}_\sigma^{\frac{m+1}{2}}$$

Proof. Assume $(y,s)=(0,0)$, and let $Q=K_\rho\times(0,t]$. By the structure conditions of $\boldsymbol{A}$

$$\frac{1}{\rho}\int_0^t\int_{K_\rho}|\boldsymbol{A}(x,\tau,u,Du)|dx\,d\tau\le\frac{C_1}{\rho}\iint_Q u^{m-1}|Du|dx\,d\tau$$
$$\le\frac{C_1}{\rho}\left(\iint_Q u^{\frac{m}{2}-2}|Du|^2dx\,d\tau\right)^{\frac12}\left(\iint_Q u^{\frac32 m}dx\,d\tau\right)^{\frac12}$$
$$\le\gamma\frac{C_1}{\rho}\left[(1+\Lambda_{\frac{m}{2},1})\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}}+\frac{1}{\sigma^2\rho^2}(\Lambda^2_{\frac{m}{2},1}+\Lambda^2_{\frac{m}{2},2})\,\mathcal{S}_\sigma^{\frac{m}{2}}\,t\,\rho^{N(1-\frac{m}{2})}\right]^{\frac12}$$
$$\times\left[t\sup_{0<\tau<t}\int_{K_{(1+\sigma)\rho}}u^{\frac32 m}(x,\tau)dx\right]^{\frac12}$$
$$\le\gamma\frac{C_1}{\rho}\left[(1+\Lambda_{\frac{m}{2},1})\rho^{N\frac{m}{2}}\mathcal{S}_\sigma^{1-\frac{m}{2}}+\frac{1}{\sigma^2\rho^2}(\Lambda^2_{\frac{m}{2},1}+\Lambda^2_{\frac{m}{2},2})\,\mathcal{S}_\sigma^{\frac{m}{2}}\,t\,\rho^{N(1-\frac{m}{2})}\right]^{\frac12}$$
$$\times\left[t\,\mathcal{S}_\sigma^{\frac32 m}\rho^{N(1-\frac32 m)}\right]^{\frac12}.$$

By simple computations, we conclude. □

Proof of Proposition 4.1

We conclude as in the proof of Proposition 2.1, relying on Corollary B.1, instead of Corollary 6.1. □

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