Riv. Mat. Univ. Parma, Vol. **5** (2014), 39-63

Francesca Crispo and Paolo Maremonti

On the higher regularity of solutions to the p-Laplacean system in the subquadratic case

Abstract. We study the regularity properties of solutions to the non-homogeneous p-Laplacean system, $p \in (1,2)$, in a bounded domain Ω. Under suitable restrictions on the exponent p, we construct a $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ solution. Then we prove higher integrability results of the second-order derivatives of the solution. Finally, by means of semigroup properties of solutions to a special parabolic system, we prove a global pointwise bound for weak solutions under the only assumption $p \in \left(\frac{2n}{n+2}, 2\right)$.

Keywords. p-Laplacean, higher integrability, global regularity.

Mathematics Subject Classification (2010): 35J92, 35J55, 35B65.

1 - Introduction

This note deals with the existence and regularity issues for solutions of the p-Laplacean system

$$\begin{aligned} -\nabla \cdot \left((\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\, \Omega, \end{aligned} \tag{1.1}$$

with $p \in (1,2)$ and $\mu \geq 0$. Here we shall assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, whose boundary is C^2-smooth, and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \geq 1$, is a scalar or a vector field.

Received: April 5, 2013; accepted: May 13, 2013.
This work is performed under the auspices of GNFM-INdAM.

Our main purpose is to prove "global" (on the whole Ω) L^q-integrability properties ($q \in (1, \infty)$) of the second derivatives of solutions to system (1.1). Hence, by the Sobolev embedding theorems, we deduce "global" high regularity ($C^{1,\lambda}(\overline{\Omega})$). Our results are developed under two main assumptions. The former assumption concerns the bounded domain Ω. Indeed for unbounded domains the problem becomes more involved. In this connection it is enough to note that already in the case of the boundary value problem for the Laplacean in exterior domains, an estimate of the second order derivatives by means of a suitable norm of f does not hold for $q \in [\frac{n}{2}, \infty)$ (see [**21**]). The latter assumption is that we consider high regularity properties of solutions to the p-Laplacean operator and not to more general elliptic operators, whose structural properties have the p-Laplacean as a prototype. Indeed, a lot of results have been produced for more general quasilinear problems. We just recall a few of them. For instance we recall problems with quasiconvex functionals, initiated in [**12**] for $p \geq 2$ and achieved in the subquadratic non-singular case in [**7**], and, for both the singular and degenerate case in [**10**]. Moreover we mention problems with more general operators, including the p-Laplacean with coefficients, where borderline regularity conditions both for coefficients and for the data are taken into account. We refer for instance to the papers [**2**], [**11**], [**15**] and [**16**]. However, a case of more general quasilinear system is considered for global L^∞-estimates, see below.

The regularity problem for the p-Laplacean system and also for more general quasilinear systems has been extensively studied, especially as far as the interior Hölder continuity of the gradient is concerned. For a broader and deep discussion on the known results we refer to the recent survey article [**22**]. For results addressing the issue of integrability of the second derivatives of solutions of the homogeneous and non-homogeneous p-Laplacean system we mention, in particular, the pioneering paper [**1**] and the paper [**25**] for interior regularity, the paper [**5**], for up-to the boundary regularity in the three-dimensional case, and the paper [**6**], where results *close* to those we are going to present here are proved under some heavier restriction on p. Indeed in [**5**] and [**6**] the integrability exponent $q \geq 2$ of the second derivatives and the exponent p are connected by a relation of the kind $(2-p)\overline{C}(q) < 1$, for a suitable constant $\overline{C}$, with $q \to \infty$ iff $p \to 2$. In our paper at a first stage we obtain the same result by a different technique. The method employed enable us to show, in the non-singular case, the same higher integrability result for second derivatives by requiring only that $(2-p)\overline{C}(n+\varepsilon) < 1$. This means that we obtain the first $W^{2,q}$-regularity result, $q > n$, where the exponent p and the integrability exponent q are made independent. The drawback of the method employed to show this latter result is that it does not seem to work in the singular case[1]. Further, we prove global L^∞-estimates of solutions requiring $p > \dfrac{2n}{n+2}$. It is nice to point out two main facts. The first is that this result uses the Nash technique of the well known paper [**23**]. Actually the estimate of the L^∞-norm of the solutions is not deduced by means of Sobolev embedding theorems (which is not always available, see for example [**20**]), but this result is based on the semigroup properties of solutions to a special parabolic system, that we employ as adjoint of our steady problem (1.1) (for the connection between semigroup properties and Sobolev's inequality see also [**14**], chap. 6). The second fact is that with the same technique we are able to deal with a more general quasilinear system.

It is interesting to note that we do not prove that a weak solution has some integrability properties by applying regularity methods as the difference quotients technique. We prove the existence of a solution $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, that, by uniqueness, coincides with the unique weak solution, in the usual sense of Definition 3.1. The main tool is to appeal to an idea introduced by Prodi [**24**] in the context of non-stationary Navier-Stokes equations, where the existence of a solution was proved by the Galerkin method with eigenfunctions of the Stokes operator as basis functions. Clearly, here we replace this basis with the one given by eigenfunctions of the Laplace operator. In this regard, our technique seems to be new in the context of p-Laplacean system, even though it partially goes back to analogous topics in other PDEs systems.

Our main results are summarized in Theorems 1.1, 1.2 and 1.3 below. Let us introduce two main notations. We introduce the constant $\overline{C}(r)$, $r \geq 2$, defined as follows: for $v \in W_0^{1,2}(\Omega) \cap W^{2,r}(\Omega)$, one has

$$\|D^2 v\|_r \leq \overline{C}(r)\|\Delta v\|_r \tag{1.2}$$

with $\overline{C}(r)$ depending on r and Ω. The constant $\overline{C}(r)$ is connected with a Calderón-Zygmund type constant. Further we set

$$\overline{p} := 2 - \frac{1}{H}, \tag{1.3}$$

where H is the constant introduced in Lemma 2.1 in Sec. 2. We refer to this section for this and for further notations.

[1] In this connection, we would like to thank L. Diening, who has pointed out to the authors the impossibility of extending to the singular case the technique used in the case $\mu > 0$.

Theorem 1.1. *Let* $\mu \geq 0$ *and* $p \in (1,2)$ *if* Ω *is convex,* $p \in (\overline{p},2)$ *if* Ω *is not convex. Assume that* $f \in L^q(\Omega)$, *with* $q \in \Big[\dfrac{2n}{n(p-1)+2(2-p)},\infty\Big)$ *if* $n \geq 3$, *and* $q > 2$ *if* $n = 2$. *Then, if* $(2-p)\overline{C}(\widehat{q}) < 1$, *with* $\widehat{q} = \dfrac{nq(p-1)}{n-q(2-p)}$ *if* $q < n$, $\widehat{q} < n$ *if* $q = n$, *and* $\widehat{q} = q$ *if* $q > n$, *there exists a unique solution* $u \in W_0^{1,\widehat{q}}(\Omega) \cap W^{2,\widehat{q}}(\Omega)$ *of system* (1.1), *with*

$$\| u \|_{2,\widehat{q}} \leq C(\mu^{\frac{1}{2}} + \| f \|_q^{\frac{1}{p-1}}), \tag{1.4}$$

where C *is a constant independent of* μ *and* u.

Theorem 1.2. *Let* $\mu > 0$, $p \in (1,2)$ *if* Ω *is convex,* $p \in (\overline{p},2)$ *if* Ω *is not convex, and assume that, for some* $\varepsilon > 0$, $(2-p)\overline{C}(n+\varepsilon) < 1$. *Assume that* $f \in L^q(\Omega)$, *with* $q > n$. *There exists a unique solution* $u \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ *of system* (1.1), *with*

$$\| u \|_{2,q} \leq C(\mu) \| f \|_q^{\frac{1}{p-1}}, \tag{1.5}$$

where C *is a constant independent of* u.

By the Sobolev embedding theorem, if $q > n$ the previous theorems furnish the following $C^{1,\lambda}(\overline{\Omega})$ regularity result.

Corollary 1.1. *Let the assumptions of Theorem* 1.2 *be satisfied. Then the unique solution* u *of system* (1.1) *belongs to* $C^{1,\lambda}(\overline{\Omega})$ *for any* $\lambda \in [0, 1-\dfrac{n}{q}]$. *If* $\mu = 0$ *the same regularity holds, provided that* $(2-p)\overline{C}(q) < 1$.

We note that if Ω is a convex domain, then Theorem 1.1 for $\widehat{q} = 2$ holds for any p in the range $(1,2)$, without further restrictions from below, since for such a domain $\overline{C}(\widehat{q})$ can be taken equal to one (see Lemma 2.1).

Further, we observe that an approximation different from the one used in this paper can lead us to a result of L^p-integrability for the second derivatives, where the restriction on the exponent p does not depend anymore on the shape of the domain. Indeed we can get $u \in W^{2,p}(\Omega)$, for any $p \in \big(\max\{\frac{3}{2}, \frac{2n}{n+2}\}, 2\big)$ which could be a wider interval with respect to $(\overline{p},2)$. We do not develop these ideas here, and refer to **[9]** for the corresponding formulation in the parabolic setting.

Finally let us consider the quasilinear problem

$$\begin{aligned} -\nabla \cdot \big((\mu + A \cdot \nabla u \otimes \nabla u)^{\frac{p-2}{2}} \nabla u \big) = f, &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where $\mu \geq 0$ and $A(x) = A_{i\alpha j\beta}(x) \in L^\infty(\Omega)$ is a fourth-order tensor, satisfying the following ellipticity and boundedness conditions:

$$\theta|\xi|^2 \leq A_{i\alpha j\beta}(x)\xi_i^\alpha \xi_j^\beta \leq \Theta|\xi|^2, \ \forall \xi \in \mathbb{R}^{nN}, \text{ a.e. in } \Omega.$$

This kind of structure was considered, for instance, in [**16**] for elliptic equations and in [**2**] for parabolic systems, when $\mu = 0$. Clearly if $A_{i\alpha j\beta} = \delta_{i\alpha}\delta_{j\beta}$ and $\mu = 0$, we go back to the p-Laplacean.

Definition 1.1. Let $\mu \geq 0$ and $p \in (1, 2)$. Let $f \in W^{-1,p'}(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. A field $u\colon \Omega \to \mathbb{R}^N$ is a *weak solution* of system (1.6) if $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega (\mu + A \cdot \nabla u \otimes \nabla u)^{\frac{p-2}{2}} \nabla u \cdot \nabla \psi \, dx = (f, \psi), \quad \forall \psi \in W_0^{1,p}(\Omega). \tag{1.7}$$

Theorem 1.3. *Let $\mu \geq 0$ and $p \in \left(\dfrac{2n}{n+2}, 2\right)$. Let $f \in W^{-1,p'}(\Omega) \cap L^r(\Omega)$, for some $r > \dfrac{np}{p(n+2) - 2n}$. Then the unique weak solution u of system* (1.6) *belongs to $L^\infty(\Omega)$ and satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq cM^{\frac{2-p}{p}} \|f\|_r^{\gamma_1} \|f\|_{-1,p'}^{\gamma_2}, \tag{1.8}$$

with $M^{\frac{1}{p}} = c\left(\Theta^{\frac{2-p}{2p}} \|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}} |\Omega|^{\frac{1}{p}}\right)$, $\gamma_1 + \gamma_2 = \dfrac{1}{p-1}$ *and* $\gamma_1 = \dfrac{r(n-p)}{npr + pr - nr - np}$.

Our second result gives a global pointwise bound for weak solutions. Note that, since $\gamma_1 + \gamma_2 = \dfrac{1}{p-1}$, estimate (1.8) is completely in agreement with the dimensional balance of estimates (1.4) and (1.5) for the solution of the p-Laplacean.

2 - Notations and some preliminary results

By Ω we mean a bounded domain of $\mathbb{R}^n$, $n \geq 2$, whose boundary $\partial\Omega$ is assumed C^2-smooth. For $q \in [1, \infty]$, by $L^q(\Omega)$ we denote the Lebesgue space of functions defined on Ω. The norm of a function is indicated by $\|\cdot\|_q$, and, in the case $q = 2$ we put $\|\cdot\|_2 = \|\cdot\|$. $W^{m,q}(\Omega)$ denotes the usual Sobolev space of (m, q)-order of functions on Ω and $\|\cdot\|_{m,q}$ is its associated norm. Since it will be clear by the context, then, we do not introduce special notations of the functional spaces to distinguish the scalar and the vector spaces. Finally by c, C we denote numerical constants whose value is unessential for our aims. As well as, we can find in the same line $k > 1$ and $kc < c$. Relevant dependences are specified as, for instance, $C(\mu)$.

For the reader's convenience, below we recall some known results. Firstly, we recall the following inequality, for which we refer to [**17**] (chap. I, sec. 5) or [**18**] (chap. II, sec. 6), and to the more recent paper [**3**], where its extension to the L^q-norm, $q \in (1,2)$, is considered.

Lemma 2.1. *Assume that $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Then*

$$\|D^2 v\| \leq H\|\Delta v\|,$$

where the constant H depends on Ω. If Ω is a convex domain, the inequality holds with $H = 1$.

Note that, from the equivalence of the norms $\|v\|_{2,2}$ and $\|D^2 v\|$, the previous lemma ensures the validity of the following estimate $\|v\|_{2,2} \leq C(\Omega)\,\|\Delta v\|$.

We recall the classical "fixed point" theorem, for which we refer to [**19**], Lemma I.4.3.

Lemma 2.2. *Let P be a continuous function of $\mathbb{R}^m$, $m \geq 1$, into itself such that, for some $\rho > 0$, $P(\xi) \cdot \xi \geq 0$, for all $\xi \in \mathbb{R}^m$, with $|\xi| = \rho$. Then there exists a $\xi_0 \in \mathbb{R}^m$, with $|\xi_0| \leq \rho$, such that $P(\xi_0) = 0$.*

Further we recall the following regularity theorem, as given in the book [**13**], Theorem 7.3.

Lemma 2.3. *Let v be a $W^{2,2}$-solution of the linear system*

$$A_{i\alpha j\beta} D^2_{\alpha\beta} v_j = f_i,$$

with $A_{i\alpha j\beta} \in C(\overline{\Omega})$ satisfying the Legendre-Hadamard condition. If f belongs to $L^q(\Omega)$, for some $q \geq 2$, then $D^2 v \in L^q(\Omega)$, with

$$\|D^2 v\|_q \leq C(q,n,L,\omega)\,(\|f\|_q + \|D^2 v\|)\,,$$

where L is the constant of the Legendre-Hadamard condition and ω is the modulus of continuity of A.

We also give a useful inequality, referring, for instance, to [8].

Lemma 2.4. *Let $p \in (1,2)$ and $\mu > 0$. For any given real numbers $a, b \geq 0$, the following inequality holds true:*

$$\left| \frac{1}{(\mu + a)^{2-p}} - \frac{1}{(\mu + b)^{2-p}} \right| \leq \frac{2-p}{\mu^{3-p}}\,|a-b|. \tag{2.1}$$

Finally we give a result of existence and some semigroup properties for the solutions φ of the following integral equation

$$(2.2)\qquad \int_0^t(\varphi,\psi_\tau)d\tau - \nu\int_0^t(\nabla\varphi,\nabla\psi)d\tau - \int_0^t((\mu+J_\eta(A\cdot\nabla v\otimes\nabla v))^{\frac{p-2}{2}}\nabla\varphi,\nabla\psi)d\tau$$
$$= (\varphi(t),\psi(t)) - (\varphi_\circ,\psi(0))\,,\ \forall\psi\in C_0^\infty([0,T)\times\Omega),$$

where $\nu>0,\mu>0,p\in(1,2),J_\eta$ Friedrich's mollifier and $v\in W_0^{1,p}(\Omega)$ independent of t and satisfying

$$(2.3)\qquad \|(\mu+A\cdot\nabla v\otimes\nabla v)^{\frac12}\|_p^p\le M,$$

for a suitable constant $M>0$ depending on A. The proof of the following result can be immediately obtained following the proof of Lemma 2.4 in [**9**]. We omit the details.

Lemma 2.5. *Let* $p\in\left(\frac{2n}{n+2},2\right)$ *and let* $v\in W_0^{1,p}(\Omega)$ *satisfy* (2.3). *Then, for any* $\varphi_\circ(x)\in C_0^\infty(\Omega)$*, there exists a unique solution* $\varphi\in C(0,T;L^2(\Omega))\cap L^2(0,T;W_0^{1,2}(\Omega))$ *of* (2.2). *Moreover, for any* $r\in[1,2]$,

$$\|\varphi(t)\|_r\le\|\varphi_\circ\|_r,\quad \forall t\in[0,T)\,.$$

Further,

$$(2.4)\qquad \|\varphi(t)\|_r\le cM^{\frac{(2-p)\beta}{r'}}\|\varphi_\circ\|_1 t^{-p\beta(1-\frac1r)},\quad \forall t\in(0,T),$$

and

$$(2.5)\qquad \|\nabla\varphi(t)\|_p\le cM^{\frac{2-p}{2}(\beta+\frac1p)}\|\varphi_\circ\|_1 t^{-\frac{\beta p+1}{2}},\quad \forall t\in(0,T),$$

with

$$(2.6)\qquad \beta:=\frac{n}{p(n+2)-2n}$$

and r' *conjugate exponent of* r.

3 - The approximating system

Let us introduce the following auxiliary elliptic systems

$$(3.1)\qquad \begin{gathered}-\Delta u-(p-2)A(\mu,u)\cdot\nabla\nabla u=f\left(\mu+|\nabla u|^2\right)^{\frac{2-p}{2}},\quad \text{in }\Omega,\\ u=0,\quad \text{on }\partial\Omega,\end{gathered}$$

with $p\in(1,2)$ if Ω is convex and $p\in(\overline{p},2)$ if Ω is not convex, and

$$
\begin{aligned}
-\Delta U-(p-2)A(\mu,U)\cdot\nabla\nabla U &= f\Big(\mu+|J_\varepsilon(\nabla U)|^2\Big)^{\frac{2-p}{2}}, \quad \text{in } \Omega,\\
U &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
\tag{3.2}
$$

with $p\in(1,2)$ if Ω is convex and $p\in(\overline{p},2)$ if Ω is not convex,

where J_ε is the Friedrich's mollifier and by $A(\mu,v)$, $\mu>0$ and v vector field, we denote the fourth-order tensor

$$
A(\mu,v):=\frac{\nabla v\otimes\nabla v}{\mu+|\nabla v|^2}. \tag{3.3}
$$

Without loss of generality, throughout this section we assume $\mu\in(0,1]$. We set

$$
\overline{s}:=\frac{2n}{n(p-1)+2(2-p)}. \tag{3.4}
$$

Firstly we prove the following existence result for solutions of problem (3.2).

Proposition 3.1. *There exists a solution $U\in W^{1,2}_0(\Omega)\cap W^{2,2}(\Omega)$ of problem* (3.2), *with*

$$
\|U\|_{2,2}\le C\bigg(\mu^{\frac12}+\|f\|_{s_1}^{\frac{1}{p-1}}\bigg), \tag{3.5}
$$

where the constant C is independent of U, ε and μ, provided that $f\in L^{s_1}(\Omega)$ with $s_1=\overline{s}$ if $n\neq 2$, and $s_1>2$ if $n=2$.

Proof. Let $\{a_j\}$ be the eigenfunctions of the Laplace operator Δ, and denote by $\{\lambda_j\}$ the corresponding (positive) eigenvalues:

$$
\begin{aligned}
-\Delta a_j &= \lambda_j a_j, \quad && \text{in } \Omega,\\
a_j &= 0, && \text{on } \partial\Omega.
\end{aligned}
$$

Recall that $\{a_j\}$ is a complete system in $W^{1,2}_0(\Omega)\cap W^{2,2}(\Omega)$, orthogonal in $W^{1,2}_0(\Omega)$. We construct the Galerkin approximations related to system (3.2), such that, for each $k\in\mathbb{N}$,

$$
U^k(x)=\sum_{j=1}^{k}c_{jk}a_j(x), \tag{3.6}
$$

and

$$
\begin{aligned}
Q_j(c_k):=-(\Delta U^k,a_j)-(p-2)\big(A(\mu,U^k)\cdot\nabla\nabla U^k,a_j\big)&\\
-\Big(f\big(\mu+|J_\varepsilon(\nabla U^k)|^2\big)^{\frac{2-p}{2}},a_j\Big)&=0,
\end{aligned}
\tag{3.7}
$$

for any $j = 1, \cdots, k$, with $c_k = (c_{1k}, \cdots, c_{kk})$. This is a system of k equations in the unknowns c_{jk}. Set

$$\begin{aligned} P_j(c_k) := -(\varDelta U^k, \lambda_j a_j) - (p-2)\big(A(\mu, U^k) \cdot \nabla\nabla U^k, \lambda_j a_j\big) \\ - \Big(f(\mu + |J_\varepsilon(\nabla U^k)|^2)^{\frac{2-p}{2}}, \lambda_j a_j\Big), \end{aligned} \tag{3.8}$$

$j = 1, \cdots, k$, and observe that

$$Q_j(c_k) = \frac{1}{\lambda_j} P_j(c_k). \tag{3.9}$$

Let us verify that $P(c_k) \cdot c_k \geq 0$ for suitable c_k. Note that, from our choice of the basis, we have

$$\begin{aligned} P(c_k) \cdot c_k = \|\varDelta U^k\|^2 + (p-2)\big(A(\mu, U^k) \cdot \nabla\nabla U^k, \varDelta U^k\big) \\ + \Big(f\big(\mu + |J_\varepsilon(\nabla U^k)|^2\big)^{\frac{2-p}{2}}, \varDelta U^k\Big). \end{aligned} \tag{3.10}$$

Let us estimate the L^2-scalar products on the right-hand side of (3.10). We start by considering the case of Ω convex. Then, from Lemma 2.1 we have

$$\|D^2 U^k\| \leq \|\varDelta U^k\|. \tag{3.11}$$

Therefore, since $|A(\mu, U^k)| \leq 1$, one readily gets

$$(2-p)\big(A(\mu, U^k) \cdot \nabla\nabla U^k, \varDelta U^k\big) \leq (2-p)\|\varDelta U^k\|^2. \tag{3.12}$$

For the last term in (3.10), we distinguish between $n = 2$ and $n \geq 3$. Let be $n \geq 3$. By applying Hölder's inequality with exponents $s_1 = \overline{s}, 2n/(n-2)(2-p)$ and 2, we have

$$\begin{aligned} \|f|J_\varepsilon(\nabla U^k)|^{2-p} \varDelta U^k\| \leq \|f\|_{s_1} \|J_\varepsilon(\nabla U^k)\|^{2-p}_{\frac{2n}{n-2}} \|\varDelta U^k\| \\ \leq \|f\|_{s_1} \|\nabla U^k\|^{2-p}_{\frac{2n}{n-2}} \|\varDelta U^k\|. \end{aligned} \tag{3.13}$$

If $n = 2$, we apply Hölder's inequality with exponents $r, \dfrac{2r}{r-2}$ and 2, with r such that $2 < r \leq s_1$ and $2(2-p)r/(r-2) > 1$ and we obtain

$$\begin{aligned} \|f|J_\varepsilon(\nabla U^k)|^{2-p} \varDelta U^k\| \leq \|f\|_r \|J_\varepsilon(\nabla U^k)\|^{2-p}_{\frac{(2-p)2r}{r-2}} \|\varDelta U^k\| \\ \leq C\|f\|_{s_1} \|\nabla U^k\|^{2-p}_{\frac{(2-p)2r}{r-2}} \|\varDelta U^k\|. \end{aligned} \tag{3.14}$$

In both cases, by the Sobolev embedding theorems and then by appealing to the estimate (3.11), we get

$$\|f|J_\varepsilon(\nabla U^k)|^{2-p} \varDelta U^k\| \leq C\|\varDelta U^k\|^{3-p} \|f\|_{s_1}. \tag{3.15}$$

Therefore, using Young's inequality in (3.15), (3.10) can be estimated as follows

$$\begin{aligned} P(c_k)\cdot c_k &\geq \|\varDelta U^k\|\Big((p-1)\|\varDelta U^k\| - C\,\|\,\varDelta U^k\,\|^{2-p}\,\|f\|_{s_1} - \mu^{\frac{2-p}{2}}\|f\|\Big)\\ &\geq \|\varDelta U^k\|\left(\frac{p-1}{2}\,\|\varDelta U^k\| - \frac{C}{2(p-1)}\,\|f\|_{s_1}^{\frac{1}{p-1}} - \mu^{\frac{2-p}{2}}\|f\|\right)\geq 0, \end{aligned}$$

provided that $\|\,\varDelta U^k\,\| \geq C\big(\|f\|_{s_1}^{\frac{1}{p-1}} + \mu^{\frac{2-p}{2}}\|f\|\big)$. Therefore, since $\|\,\varDelta U^k\,\| \geq \lambda_1\,|c_k|\, \equiv \lambda_1 R$, where λ_1 is the smallest eigenvalue of the Laplacean, then for sufficiently large R we have $P(c_k)\cdot c_k \geq 0$. By using Lemma 2.2, this proves the existence of a solution of the algebraic system $P_j(c_k) = 0$, for some $|c_k| \leq R$. Recalling (3.9) we also obtain a solution of the kind (3.6) for system (3.7). Now, we look to an estimate of $\|\varDelta U^k\|$ uniform with respect to $k\in\mathbb{N}$. Since c_k solves (3.8), that is $P(c_k) = 0$, from (3.10) we get

$$\|\varDelta U^k\|^2 + (p-2)\big(A(\mu, U^k)\cdot\nabla\nabla U^k, \varDelta U^k\big) + \big(f(\mu + |J_\varepsilon(\nabla\,U^k)|^2)^{\frac{2-p}{2}}, \varDelta U^k\big) = 0.$$

Hence, employing estimates (3.12) and (3.15), by using Young's inequality in (3.15), and observing that $s_1 > 2$, we obtain uniformly in k

$$\|\varDelta U^k\| \leq C\big(\mu^{\frac{2-p}{2}}\|f\| + \|f\|_{s_1}^{\frac{1}{p-1}}\big) \leq C\big(\mu^{\frac{1}{2}} + \|f\|_{s_1}^{\frac{1}{p-1}}\big). \tag{3.16}$$

The bound (3.16), thanks to the Rellich compactness theorem, implies the existence of a field U and of a subsequence, still denoted by $\{U^k\}$, such that

$$U^k \rightharpoonup U \text{ weakly in } W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega),$$
$$U^k \to U \text{ strongly in } W^{1,r}(\Omega),$$

for any $r\in[1,+\infty)$ if $n=2$, for any $r\in\big[1,\dfrac{2n}{n-2}\big)$, if $n\geq 3$, and, from the lower semi-continuity,

$$\|U\|_{2,2} \leq C(\mu^{\frac{1}{2}} + \|f\|_{s_1}^{\frac{1}{p-1}}).$$

Finally, we show that the limit U is actually a solution of system (3.2). Let φ be an arbitrary function in $C_0^\infty(\Omega)$. Then

$$\lim_k\,(\varDelta U^k - \varDelta U, \varphi)\, = 0,$$

thanks to the weak convergence. Moreover, by using the inequality (2.1)

$$|\,(\mu + |J_\varepsilon(\nabla\,U^k)|\,)^{2-p} - \,(\mu + |J_\varepsilon(\nabla\,U)|\,)^{2-p}\,| \leq \frac{2-p}{\mu^{p-1}}\,|J_\varepsilon(\nabla\,U^k) - \,J_\varepsilon(\nabla\,U)|\,,$$

and the strong convergence of the gradients in $L^2(\Omega)$, we get

$$\lim_k \left(f\left[(\mu+|J_\varepsilon(\nabla U^k)|^2)^{\frac{2-p}{2}}-(\mu+|J_\varepsilon(\nabla U)|^2)^{\frac{2-p}{2}}\right],\varphi\right)$$
$$\leq c\lim_k \|f\|\,\|J_\varepsilon(\nabla U^k)-J_\varepsilon(\nabla U)\|\|\varphi\|_\infty \leq c\lim_k \|f\|\|\nabla U^k-\nabla U\|\|\varphi\|_\infty=0\,.$$

Let us prove the convergence of the non linear term. We write

$$\begin{aligned}\big(A(\mu,U^k)\cdot\nabla\nabla U^k-A(\mu,U)\cdot\nabla\nabla U,\varphi\big)&=\big((A(\mu,U^k)-A(\mu,U))\cdot\nabla\nabla U^k,\varphi\big)\\&\quad+\big(A(\mu,U)\cdot(\nabla\nabla U^k-\nabla\nabla U),\varphi\big).\end{aligned}\tag{3.17}$$

We recall that L^r convergence of the gradients implies the almost everywhere convergence of a subsequence. Therefore $A(\mu,U^k)\to A(\mu,U)$, a.e. in Ω. Further $|A(\mu,U^k)|\leq 1$ point-wisely. Hence from Hölder's inequality, the Lebesgue dominated convergence theorem and the uniform bound (3.16), it follows that

$$\lim_k\big((A(\mu,U^k)-A(\mu,U))\cdot\nabla\nabla U^k,\varphi\big)$$
$$\leq\lim_k\|A(\mu,U^k)-A(\mu,U)\|\|\nabla\nabla U^k\|\|\varphi\|_\infty=0\,.$$

As far as the second integral on the right-hand side of (3.17) is concerned, it tends to zero thanks to the weak convergence of U^k to U in $W^{2,2}(\Omega)$ and since $A(\mu,U)\cdot\varphi\in L^2(\Omega)$. We have therefore shown that

$$(\varDelta U+(p-2)A(\mu,U)\cdot\nabla\nabla U+f(\mu+|J_\varepsilon(\nabla U)|)^{2-p},\varphi)=0\,,$$

for all $\varphi\in C_0^\infty(\Omega)$, which ensures that U satisfies (3.2) a.e. in Ω. Now, we conclude the proof considering Ω not convex domain. This case follows simply replacing estimate (3.11) with the estimate of Lemma 2.1

$$\|D^2U^k\|\leq H\|\varDelta U^k\|\tag{3.18}$$

and consequently modifying (3.12). Indeed, estimate (3.18) implies

$$(2-p)\big(A(\mu,U^k)\cdot\nabla\nabla U^k,\varDelta U^k\big)\leq H(2-p)\|\varDelta U^k\|^2,\tag{3.19}$$

hence, under our assumption $p\in\left(2-\dfrac{1}{H},2\right)$ we can repeat all the considerations leading to the result in the case of Ω convex. The proof is accomplished. □

Let us consider the problem

$$\begin{aligned}-\varDelta w-(p-2)A(\mu,U)\cdot\nabla\nabla w&=f\left(\mu+|J_\varepsilon(\nabla U)|^2\right)^{\frac{2-p}{2}},\ \text{ in }\Omega,\\ w&=0,\quad\text{ on }\partial\Omega,\end{aligned}\tag{3.20}$$

with $p\in(1,2)$ if Ω is convex and $p\in(\overline{p},2)$ if Ω is not convex, where U is the solution of system (3.2), constructed in Proposition 3.1, and $A(\mu, U)$ is defined in (3.3).

We set

$$\widehat{q}\begin{cases} = \dfrac{nq(p-1)}{n-q(2-p)} & \text{if } q \in [\overline{s},\, n) \text{ and } n \neq 2\,, \\ < n & \text{if } q = n \neq 2\,, \\ = q & \text{if } q > n\,. \end{cases} \tag{3.21}$$

We note that $\widehat{q} \leq q$. Further we define

$$r(\widehat{q}) := \begin{cases} \dfrac{\widehat{q}q}{\widehat{q}-q}(2-p) & \text{if } q \in [\overline{s},\, n]\,, \\ \infty & \text{if } q > n\,. \end{cases} \tag{3.22}$$

Lemma 3.1. *Let $f \in L^q(\Omega)$, $q \geq \overline{s}$ if $n \neq 2$, $q > 2$ if $n = 2$, and let $\widehat{q}$ be given by* (3.21)*. Then*

i) if $(2-p)\overline{C}(\widehat{q}) < 1$ there exists a unique solution to problem (3.20)*, with*

$$\| w \|_{2,\widehat{q}} \leq C \left(\mu^{\frac{2-p}{2}} \|f\|_q + \|f\|_q \|J_\varepsilon(\nabla\, U)\|^{2-p}_{r(\widehat{q})} \right), \tag{3.23}$$

where $r(\widehat{q})$ is given by (3.22) *and the constant C is independent of U, ε and μ;*

ii) if, for some $\alpha \in [0,1]$, $U \in C^{1,\alpha}(\overline{\Omega})$, then there exists a unique solution to problem (3.20) *which satisfies estimate* (3.23)*, with a constant C depending on μ.*

Proof. Let us consider the regularized problem

$$\begin{gathered} -\Delta w^\eta - (p-2)A_\eta(\mu, U)\cdot \nabla\nabla w^\eta = f\left(\mu + |J_\varepsilon(\nabla U)|^2\right)^{\frac{2-p}{2}}, \quad \text{in } \Omega, \\ w^\eta = 0, \text{ on } \partial\Omega, \end{gathered} \tag{3.24}$$

where

$$A_\eta(\mu, U) = \frac{\nabla J_\eta(U) \otimes \nabla J_\eta(U)}{\mu + |J_\eta(\nabla U)|^2}\,.$$

This is a linear elliptic problem with $C^\infty(\mathbb{R}^n)$ coefficients. Observing that $f(\mu + |J_\varepsilon(\nabla U)|^2)^{\frac{2-p}{2}} \in L^2(\Omega)$, we can employ the Galerkin method, following the same arguments used in Proposition 3.1, to claim the existence of $w^\eta \in W^{2,2}(\Omega)$. In particular, as $\widehat{q} \geq 2$, we get

$$\|D^2 w^\eta\| \leq c\|f(\mu + |J_\varepsilon(\nabla U)|^2)^{\frac{2-p}{2}}\| \leq c\|f(\mu + |J_\varepsilon(\nabla U)|^2)^{\frac{2-p}{2}}\|_{\widehat{q}}\,. \tag{3.25}$$

By using Hölder's inequality the $L^{\widehat{q}}$-norm of the right-hand side is estimated as

$$\|f\big(\mu+|J_\varepsilon(\nabla U)|^2\big)^{\frac{2-p}{2}}\|_{\widehat{q}} \le c\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}} + c\|f\|_q\|J_\varepsilon(\nabla U)\|^{2-p}_{r(\widehat{q})}, \tag{3.26}$$

with $q>\overline{s}$ and $r(\widehat{q})$ given by (3.22). Therefore, observing again that $\widehat{q}\ge 2$ for any $q\ge\overline{s}$ if $n\ge 3$, and $\widehat{q}=q>2$ if $n=2$, Lemma 2.3 for the linear elliptic system (3.24) ensures that $w^\eta\in W^{2,\widehat{q}}(\Omega)$ with

$$\begin{aligned}\|D^2w^\eta\|_{\widehat{q}} &\le C(\eta,\mu)\bigg(\|f\Big(\mu+|J_\varepsilon(\nabla U)|^2\Big)^{\frac{2-p}{2}}\|_{\widehat{q}}+\|D^2w^\eta\|\bigg)\\ &\le C(\eta,\mu)\bigg(\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}}+\|f\|_q\|J_\varepsilon(\nabla U)\|^{2-p}_{r(\widehat{q})}\bigg).\end{aligned}\tag{3.27}$$

If $(2-p)\overline{C}(\widehat{q})<1$, we can make estimate (3.27) uniform both in η and μ. Indeed let us multiply equation (3.24) by $\varphi\in C_0^\infty(\Omega)$ and integrate over Ω. We get

$$\begin{aligned}|-(\varDelta w^\eta,\varphi)| &\le (2-p)|(A_\eta(\mu,U)\cdot\nabla\nabla w^\eta,\varphi)| + \Big|\Big(f\big(\mu+|J_\varepsilon(\nabla U)|^2\big)^{\frac{2-p}{2}},\varphi\Big)\Big|\\ &\le (2-p)\|\nabla\nabla w^\eta\|_{\widehat{q}}\|\varphi\|_{\widehat{q}'} + \|f\big(\mu+|J_\varepsilon(\nabla U)|^2\big)^{\frac{2-p}{2}}\|_{\widehat{q}}\|\varphi\|_{\widehat{q}'},\end{aligned}$$

whence we obtain

$$\|\varDelta w^\eta\|_{\widehat{q}} = \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\varphi\|_{\widehat{q}'}=1}} |(\varDelta w^\eta,\varphi)| \le (2-p)\|\nabla\nabla w^\eta\|_{\widehat{q}} + \|f\big(\mu+|J_\varepsilon(\nabla U)|^2\big)^{\frac{2-p}{2}}\|_{\widehat{q}}.$$

By using estimates (1.2) and (3.26) we have

$$\|\varDelta w^\eta\|_{\widehat{q}} \le (2-p)C(\widehat{q})\|\varDelta w^\eta\|_{\widehat{q}} + c\,\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}} + c\,\|f\|_q\|J_\varepsilon(\nabla U)\|^{2-p}_{r(\widehat{q})}\,.$$

Hence, the assumption $1-(2-p)C(\widehat{q})>0$ gives

$$\|\varDelta w^\eta\|_{\widehat{q}} \le c\,\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}} + c\,\|f\|_q\|J_\varepsilon(\nabla U)\|^{2-p}_{r(\widehat{q})}\,,$$

uniformly in $\eta>0$, non-singular in $\mu>0$. Applying again inequality (1.2) we get

$$\|D^2w^\eta\|_{\widehat{q}} \le c\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}} + c\|f\|_q\|J_\varepsilon(\nabla U)\|^{2-p}_{r(\widehat{q})},\ \text{ for all } \eta>0 \text{ and } \mu>0. \tag{3.28}$$

Let us remove the assumption $(2-p)\overline{C}(\widehat{q})<1$ and let $U\in C^{1,\alpha}(\overline{\Omega})$. Then, setting

$$A_\eta(\mu,U) = (A_\eta(\mu,U)-A(\mu,U)) + A(\mu,U)\,,$$

one easily shows that the boundedness of the modulus of continuity of A_η is uniform in $\eta > 0$, but does not hold uniformly in $\mu > 0$. Therefore (3.27) becomes

$$\|D^2 w^\eta\|_{\widehat{q}} \le C(\mu)\Big(\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}} + \|f\|_q \|J_\varepsilon(\nabla U)\|^{2-p}_{r(\widehat{q})}\Big), \text{ for all } \eta > 0. \tag{3.29}$$

Since both the bounds (3.28) and (3.29) are uniform in η, from the family of solutions $\{w^\eta\}$, we can extract a subsequence, still denoted by $\{w^\eta\}$, whose weak limit w, as $\eta \to 0$, belongs to $W^{2,\widehat{q}}(\Omega)$. The convergence is also strong in $W^{1,r}(\Omega)$, for any r if $q \ge n$, and for any $r \in \Big(1, \dfrac{n\widehat{q}}{n-\widehat{q}}\Big)$, if $\widehat{q} \in [2, n)$. Let us show that w is a solution of system (3.20). Let φ be an arbitrary function in $C_0^\infty(\Omega)$. Then

$$\lim_{\eta\to 0}\,(\Delta w^\eta - \Delta w, \varphi) = 0,$$

from the weak convergence. We write

$$\begin{aligned}\big(A_\eta(\mu,U)\cdot\nabla\nabla w^\eta - A(\mu,U)\cdot\nabla\nabla w, \varphi\big) =& \big((A_\eta(\mu,U) - A(\mu,U))\cdot\nabla\nabla w^\eta, \varphi\big)\\ &+ (A(\mu,U)\cdot(\nabla\nabla w^\eta - \nabla\nabla w), \varphi).\end{aligned} \tag{3.30}$$

We recall that the L^r-convergence of $J_\eta(\nabla U)$ to ∇U, implies the almost everywhere convergence of a subsequence. Therefore, along a subsequence, $A_\eta(\mu,U) - A(\mu,U) \to 0$, a.e. in Ω. Further $|A_\eta(\mu,U) - A(\mu,U)| \le 2$, point-wisely. Hence from Hölder's inequality, the Lebesgue dominated convergence theorem and the η-uniform bounds (3.29) or (3.28), it follows that

$$\lim_\eta \big((A_\eta(\mu,U) - A(\mu,U))\cdot\nabla\nabla w^\eta, \varphi\big) \le \lim_\eta \|A_\eta(\mu,U) - A(\mu,U)\|_{\widehat{q}'}\|\nabla\nabla w^\eta\|_{\widehat{q}}\|\varphi\|_\infty = 0.$$

As far as the second integral on the right-hand side of (3.30) is concerned, it tends to zero thanks to the weak convergence of w^η to w in $W^{2,\widehat{q}}(\Omega)$ and since $A(\mu,U)\cdot\varphi \in L^{\widehat{q}'}(\Omega)$. Recalling that w^η solves (3.20), we have shown that

$$(\Delta w + (p-2)A(\mu,U)\cdot\nabla\nabla w + f(\mu + |J_\varepsilon(\nabla U)|)^{2-p}, \varphi) = 0\,,$$

for all $\varphi \in C_0^\infty(\Omega)$, which ensures that w satisfies (3.20) a.e. in Ω. Finally, we prove the uniqueness. Let us consider another solution of (3.20) and denote by $\widehat{w}$ their difference. We have

$$\begin{aligned} -\Delta\widehat{w} - (p-2)A(\mu,U)\cdot\nabla\nabla\widehat{w} &= 0, \quad &&\text{in } \Omega\,,\\ \widehat{w} &= 0, &&\text{on } \partial\Omega\,.\end{aligned}$$

Since $\widehat{q} \ge 2$, multiplying the above equation by $\Delta\widehat{w}$ and integrating on Ω, then employing Lemma 2.1, one readily recognizes that, under our assumptions on p, one has $\|\,\Delta\widehat{w}\,\| = 0$. Hence $\widehat{w} \equiv 0$ holds. The lemma is completely proved. $\square$

Proposition 3.2. *Let U be the solution of system* (3.2), *constructed in Proposition* 3.1. *Let $f \in L^q(\Omega)$, $q \geq \overline{s}$ if $n \neq 2$, $q > 2$ if $n = 2$. If $(2-p)\overline{C}(\widehat{q}) < 1$, then $U \in W^{2,\widehat{q}}(\Omega)$, where $\widehat{q}$ is given in* (3.21), *with*

$$\| U \|_{2,\widehat{q}} \leq C\Big(\mu^{\frac{1}{2}} + \| f \|_q^{\frac{1}{p-1}} \Big), \tag{3.31}$$

where the constant C is independent of U, ε and μ.

Proof. From Lemma 3.1-*i*), the unique solution w of problem (3.20) belongs to $W^{2,\widehat{q}}(\Omega)$. In the sequel we prove that $w = U$, which completes the proof. By taking the difference of (3.2) and (3.20), side by side, and by setting $V = U - w$, we get

$$\begin{aligned} -\Delta V - (p-2)A(\mu, U)\cdot \nabla\nabla V = 0, &\quad \text{in } \Omega\,,\\ V = 0\,, &\quad \text{on } \partial\Omega\,. \end{aligned}$$

Multiply the above equation by ΔV and integrate in Ω. By appealing to arguments already used in Lemma 3.1 to prove the uniqueness of w, one readily recognizes that, under our assumptions on p, the field $V \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ satisfies $\|\Delta V\| = 0$. Hence $V = 0$ holds. Therefore, $U \in W^{2,\widehat{q}}(\Omega)$ too. Next, we obtain an estimate of the $W^{2,\widehat{q}}$-norm, uniform in ε. Firstly we observe that, by standard embedding theorems, $U \in W^{2,\widehat{q}}(\Omega)$ implies $\nabla U \in L^{r(\widehat{q})}(\Omega)$, with $r(\widehat{q})$ given in (3.22). From estimate (3.23) we have

$$\|U\|_{2,\widehat{q}} \leq C\left(\mu^{\frac{2-p}{2}} \|f\|_{\widehat{q}} + \|f\|_q \|J_\varepsilon(\nabla U)\|_{r(\widehat{q})}^{2-p} \right).$$

Therefore, using the properties of the mollifier of a $L^{r(\widehat{q})}(\Omega)$-function, estimate (3.5) and Young's inequality, we get

$$\begin{aligned} \|U\|_{2,\widehat{q}} &\leq C\Big(\mu^{\frac{2-p}{2}} \|f\|_{\widehat{q}} + \|f\|_q \|\nabla U\|_{r(\widehat{q})}^{2-p} \Big) \leq C\mu^{\frac{2-p}{2}}\|f\|_{\widehat{q}} + \frac{C}{2}\|f\|_q^{\frac{1}{p-1}} + \frac{1}{2}\|U\|_{2,\widehat{q}}\\ &\leq C\Big(\mu^{\frac{1}{2}} + \|f\|_q^{\frac{1}{p-1}} \Big) + \frac{1}{2}\|U\|_{2,\widehat{q}}, \end{aligned}$$

which gives the result. □

From Proposition 3.2 and by a Sobolev embedding, we get

Corollary 3.1. *Let $f \in L^q(\Omega)$, $q > n \geq 2$. If, for some $\varepsilon > 0$, $(2-p)\overline{C}(n+\varepsilon) < 1$, then $U \in C^{1,\alpha}(\overline{\Omega})$, with $\alpha = 1 - \dfrac{n}{n+\varepsilon}$, and*

$$| U |_{1,\alpha} \leq C\Big(\mu^{\frac{1}{2}} + \|f\|_q^{\frac{1}{p-1}} \Big), \tag{3.32}$$

where the constant C is independent of U, ε and μ.

Corollary 3.2. *Let* $f \in L^q(\Omega)$, $q > n \geq 2$. *If, for some* $\varepsilon > 0$, $(2-p)\overline{C}(n+\varepsilon) < 1$, *then* $U \in W^{2,q}(\Omega)$, *with*

$$\| U \|_{2,q} \leq C(\mu)\left(\mu^{\frac{1}{2}} + \| f \|_q^{\frac{1}{p-1}} \right), \tag{3.33}$$

where the constant C is independent of U and ε.

Proof. From Corollary 3.1 the solution U belongs to $C^{1,\alpha}(\overline{\Omega})$, with $\alpha = 1 - \dfrac{n}{n+\varepsilon}$. Therefore we can apply Lemma 3.1-*ii*) and obtain that $w \in W^{2,q}(\Omega)$ and satisfies (3.23) with $C = C(\mu)$. Finally, as in the proof of Proposition 3.2, one shows that $U = w$, so that U satisfies (3.33). $\square$

Proposition 3.3. *Let* $f \in L^q(\Omega)$, $q \geq \overline{s}$ *if* $n \neq 2$, $q > 2$ *if* $n = 2$. *Then*

i) if $(2-p)\overline{C}(\widehat{q}) < 1$, *then exists a solution* $u \in W_0^{1,\widehat{q}}(\Omega) \cap W^{2,\widehat{q}}(\Omega)$ *of system* (3.1), $\widehat{q}$ *given in* (3.21), *with*

$$\| u \|_{2,\widehat{q}} \leq C\left(\mu^{\frac{1}{2}} + \| f \|_q^{\frac{1}{p-1}} \right),$$

where the constant C is independent of u and μ;

ii) let $q > n$. *If, for some* $\varepsilon > 0$, $(2-p)\overline{C}(n+\varepsilon) < 1$, *then exists a solution* $u \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ *of system* (3.1), *with*

$$\| u \|_{2,q} \leq C(\mu)\left(\mu^{\frac{1}{2}} + \| f \|_q^{\frac{1}{p-1}} \right),$$

where the constant C is independent of u.

Proof. The result readily follows from Proposition 3.1, Proposition 3.2 and Corollary 3.2. Indeed, denoting by $\{U^\varepsilon\}$ the sequence of solutions of (3.2), for any $\varepsilon > 0$, we may pass to the limit, up to a subsequence, thanks to the ε-uniform estimate (3.31). Now, let us prove that the limit, say u, is a solution. For its linear character, the convergence of the Laplacian term is immediate. As far as the convergence of the non linear term $A(\mu, U^\varepsilon)\nabla\nabla U^\varepsilon$ we can follow the same arguments employed in Proposition 3.1 for the convergence of the Galerkin approximation. Finally, for the term $f(\mu + |J_\varepsilon(\nabla U^\varepsilon)|^2)^{\frac{2-p}{2}}$ we observe that, by means of Lemma 2.4,

$$\begin{aligned}
&|(\mu + |J_\varepsilon(\nabla U^\varepsilon)|^2)^{\frac{2-p}{2}} - (\mu + |\nabla u|^2)^{\frac{2-p}{2}}| \\
&\qquad \leq |(\mu + |J_\varepsilon(\nabla U^\varepsilon)|^2)^{\frac{2-p}{2}} - (\mu + |J_\varepsilon(\nabla u)|^2)^{\frac{2-p}{2}}| \\
&\qquad\quad + |(\mu + |J_\varepsilon(\nabla u)|^2)^{\frac{2-p}{2}} - (\mu + |\nabla u|^2)^{\frac{2-p}{2}}| \\
&\qquad \leq \frac{2(2-p)}{\mu^{3-p}}\Big(|J_\varepsilon(\nabla U^\varepsilon) - J_\varepsilon(\nabla u)|^{2-p} + |J_\varepsilon(\nabla u) - \nabla u| \Big),
\end{aligned}$$

whence that the limit u is solution easily follows. $\square$

Definition 3.1. Let $\mu \geq 0$ and $p \in (1,2)$. Let $f \in W^{-1,p'}(\Omega)$, $\dfrac{1}{p}+\dfrac{1}{p'}=1$. A field $u\colon \Omega \to \mathbb{R}^N$ is a *weak solution* of system (1.1) if $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega (\mu+|\nabla u|^2)^{\frac{p-2}{2}} \nabla u \cdot \nabla \psi \, dx = (f,\psi)\,, \quad \forall \psi \in W_0^{1,p}(\Omega). \tag{3.34}$$

The existence and uniqueness of a weak solution is a well established result, which traces back to [**19**] (chap.II).

Theorem 3.1. *Let $\mu > 0$ and $p \in (1,2)$ if Ω is convex, $p \in (\overline{p},2)$ if Ω is not convex. Assume that $f \in L^q(\Omega)$, $q \geq \overline{s}$ if $n \neq 2$, $q > 2$ if $n=2$. Then*

i) if $(2-p)\overline{C}(\widehat{q}) < 1$, there exists a unique solution $u \in W_0^{1,\widehat{q}}(\Omega) \cap W^{2,\widehat{q}}(\Omega)$, of system (1.1)*, $q = q(s)$ given by* (3.21)*, with*

$$\| u \|_{2,\widehat{q}} \leq C\left(\mu^{\frac{1}{2}} + \|f\|_q^{\frac{1}{p-1}} \right), \tag{3.35}$$

where the constant C is independent of u and μ;

ii) if $q > n$ and, for some $\varepsilon > 0$, $(2-p)\overline{C}(n+\varepsilon) < 1$, there exists a unique solution $u \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$, of system (1.1)*, with*

$$\| u \|_{2,q} \leq C(\mu)\left(\mu^{\frac{1}{2}} + \|f\|_q^{\frac{1}{p-1}} \right), \tag{3.36}$$

where the constant C is independent of u.

Proof. Let $u \in W_0^{1,\widehat{q}}(\Omega) \cap W^{2,\widehat{q}}(\Omega)$ be the solution of (3.1), obtained in Proposition 3.3. Dividing both sides of equation $(3.2)_1$ by $(\mu+|\nabla u|)^{2-p}$, it is immediate to verify that u is also a weak solutions to (1.1), in the sense of Definition 3.1. So the result follows from the uniqueness of weak solutions of (1.1). $\square$

4 - Proof of Theorem 1.1 and Theorem 1.2

Theorem 1.1 for $\mu > 0$ is Theorem 3.1-*i*) , while Theorem 1.2 is Theorem 3.1-*ii*). Hence we have just to prove Theorem 1.1 for $\mu = 0$. In doing this, the starting point is the existence and regularity result given in Theorem 3.1-*i*). Then we show the convergence of the sequence of approximating solutions $\{u^\mu\}$, as μ goes to zero, to a weak solution of problem (1.1) with $\mu = 0$. From the existence and uniqueness of a weak solution of the p-Laplacean system, the result follows.

In order to prove the convergence of the nonlinear term, we will not exploit any strong convergence at disposal, for instance the strong convergence in $W_0^{1,p}(\Omega)$, but we will use just a weaker one, and follow the classical monotonicity method, as in [**19**]. This will enable us to employ the same arguments in the proof of Theorem 1.3, where a strong convergence in $W_0^{1,p}(\Omega)$ is not known.

Proof of Theorem 1.1. For $\mu > 0$, let $\{u^\mu\}$ be the sequence of solutions of (1.1), obtained in Theorem 3.1-*i*). This sequence is uniformly bounded in $W_0^{1,\widehat{q}}(\Omega) \cap W^{2,\widehat{q}}(\Omega)$ and satisfies estimate (3.35). Therefore, there exists a field $u \in W_0^{1,\widehat{q}}(\Omega) \cap W^{2,\widehat{q}}(\Omega)$ and a subsequence, still denoted by $\{u^\mu\}$, such that $u^\mu \rightharpoonup u$ weakly in $W^{2,\widehat{q}}(\Omega)$, and, by Rellich's theorem, strongly in $W_0^{1,p}(\Omega)$. It remains to show that u is solution of problem (1.1) for $\mu = 0$.

Let us introduce the operators from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ defined as

$$L_\mu(\psi) := -\nabla \cdot \Big((\mu + |\nabla \psi|^2)^{\frac{p-2}{2}} \nabla \psi \Big), \tag{4.1}$$

and

$$L(\psi) := -\nabla \cdot \Big(|\nabla \psi|^{\frac{p-2}{2}} \nabla \psi \Big). \tag{4.2}$$

They are both monotonous and emicontinuous operators[2]. It is standard to prove that the weak solution $u \in W_0^{1,p}(\Omega)$ of (1.1) satisfies the estimate

$$\|u\|_{1,p} \le c\Big(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}} |\Omega|^{\frac{1}{p}} \Big), \tag{4.3}$$

with c independent of μ (see, for instance, [**4**], sec. 4). From (4.3) it follows that $L_\mu(u^\mu)$ is bounded in $W^{-1,p'}(\Omega)$, uniformly in μ. Hence, along a subsequence, in the limit as μ tends to zero, we have

$$L_\mu(u^\mu) \rightharpoonup \chi \text{ weakly in } W^{-1,p'}(\Omega). \tag{4.4}$$

Let us show that $\chi = L(u)$. Firstly we observe that

$$|\,(\mu + |\nabla\psi|^2)^{\frac{p-2}{2}}\,\nabla\psi - |\nabla\psi|^{p-2}\nabla\psi\,|^{p'} \to 0, \quad \text{a.e. in } \Omega,$$

and

$$|\,(\mu + |\nabla\psi|^2)^{\frac{p-2}{2}}\,\nabla\psi - |\nabla\psi|^{p-2}\nabla\psi\,|^{p'} \le 2^{p'} |\nabla\psi|^p.$$

[2] For the sake of brevity for the corresponding definitions we refer, for instance, to [**19**] Ch. II, Sec.1.2.

The Lebesgue dominated convergence theorem ensures that

$$(\mu + |\nabla\psi|^2)^{\frac{p-2}{2}}\nabla\psi \to |\nabla\psi|^{p-2}\nabla\psi \text{ strongly in } L^{p'}(\Omega). \tag{4.5}$$

Set

$$X_\mu := (L_\mu(u^\mu) - L_\mu(\psi), u^\mu - \psi), \ \forall\psi \in W_0^{1,p}(\Omega).$$

From the monotonicity of L_μ it follows that

$$X_\mu \geq 0, \ \forall\psi \in W_0^{1,p}(\Omega).$$

By using that u^μ is a solution of (1.1), we write X_μ as follows

$$\begin{aligned} X_\mu &= (L_\mu(u^\mu), u^\mu) - (L_\mu(u^\mu), \psi) - (L_\mu(\psi), u^\mu - \psi) \\ &= (f, u^\mu) - (L_\mu(u^\mu), \psi) - (L_\mu(\psi), u^\mu) + (A_\mu(\psi), \psi). \end{aligned}$$

Let us pass to the limit. Observe that

$$(f, u^\mu) \to (f, u)$$

from the weak convergence of u^μ in $W_0^{1,p}(\Omega)$,

$$(L_\mu(u^\mu), \psi) \to (\chi, \psi),$$

thanks to (4.4),

$$(L_\mu(\psi), \psi) \longrightarrow (L(\psi), \psi),$$

thanks to (4.5). Moreover, since

$$(L_\mu(\psi), u^\mu) - (L(\psi), u) = (L_\mu(\psi) - L(\psi), u^\mu) + (L(\psi), u^\mu - u),$$

from the strong convergence (4.5), the uniform bound (4.3) and the weak convergence of ∇u^μ to ∇u in $L^p(\Omega)$, we get

$$\lim_{\mu\to 0}(L_\mu(\psi), u^\mu) = (L(\psi), u).$$

Hence

$$0 \leq \lim X_\mu \leq (f, u) - (\chi, \psi) - (L(\psi), u - \psi).$$

On the other hand, it is easy to see that the limit u satisfies

$$(\chi, u) = (f, u).$$

Therefore we obtain

$$(\chi - L(\psi), u - \psi) \geq 0, \quad \forall\psi \in W_0^{1,p}(\Omega).$$

Taking $\psi = u - \lambda v$, for $\lambda > 0$ and for some $v \in W_0^{1,p}(\Omega)$, and then letting λ tend to zero the thesis follows. $\square$

5 - The global pointwise bound of a weak solution: proof of Theorem 1.3

For the study of global pointwise bound of a weak solution of (1.6), we introduce the following family of approximating systems. Let $p \in (1,2)$, $\mu > 0$, $\nu > 0$ and let us consider the elliptic system

$$\begin{aligned} -\nu \Delta w - \nabla \cdot \left((\mu + A \cdot \nabla w \otimes \nabla w)^{\frac{p-2}{2}} \nabla w \right) &= f, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

Definition 5.1. Let $f \in W^{-1,2}(\Omega)$. A field $w\colon \Omega \to \mathbb{R}^N$ is said a *weak solution* of system (5.1) if $w \in W_0^{1,2}(\Omega)$ satisfies

$$\int_\Omega (\nu \nabla w + (\mu + A \cdot \nabla w \otimes \nabla w)^{\frac{(p-2)}{2}} \nabla w) \cdot \nabla \psi \, dx = (f, \psi), \forall \psi \in C_0^\infty(\Omega). \tag{5.2}$$

Note that for this system and for system (1.6) the existence and uniqueness of a weak solution can be proved by using the methods of monotone operators (see [**19**], chap. II, sec. 2). We further note, for later use, that the test functions in (5.2) can be replaced by test functions in $W_0^{1,2}(\Omega)$, by density arguments.

The following estimate can be proved in a standard way by using the properties of $A(x)$

$$\|w\|_{1,p} \le c \left(\Theta^{\frac{2-p}{2p}} \|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}} |\Omega|^{\frac{1}{p}} \right), \tag{5.3}$$

with c independent of μ and ν. In virtue of (5.3), the constant M introduced in (2.3) can be specified as

$$M^{\frac{1}{p}} = c \left(\Theta^{\frac{2-p}{2p}} \|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}} |\Omega|^{\frac{1}{p}} \right). \tag{5.4}$$

Lemma 5.1. *Let* $p \in \left(\frac{2n}{n+2}, 2 \right)$ *and* $\mu > 0$. *Let* $f \in W^{-1,p'}(\Omega) \cap L^r(\Omega)$, *for some* $r > \dfrac{np}{p(n+2) - 2n}$. *Then the weak solution* u *of system* (1.6) *belongs to* $L^\infty(\Omega)$ *and satisfies the estimate*

$$\|u\|_\infty \le c M^{\frac{(2-p)}{p}} \|f\|_r^{\frac{r(p\beta - 1)}{r + p\beta(r-2)}} \|f\|_{-1,p'}^{\frac{2(r - p\beta)}{r + p\beta(r-2)}}, \tag{5.5}$$

with M *given in* (5.4) *and* β *given in* (2.6).

Proof. Let w^ν be the unique weak solution of the approximating system (5.1). In the sequel, to easier the notation, we drop the superscript ν. Let us consider the parabolic system (2.2), with $v \equiv w$ and with the test functions ψ in the completion space $L^2(0,T;W_0^{1,2}(\Omega))$, $\psi_t \in L^2(0,T;L^2(\Omega))$. Then w satisfies the assumption (2.3) with M given by (5.4). Let us choose $\psi(\tau,x)=h(\tau)w(x)$ as test function in (2.2), where h belongs to $C_0^\infty[0,T)$ and satisfies $h(\tau)=1$, if $\tau\in[0,t]$, $h(\tau)=0$, if $\tau>2t$. Therefore we get

$$(5.6)\qquad \begin{aligned}(w,\varphi_\circ) &= (w,\varphi^\eta(t)) + \nu\int_0^t (\nabla\varphi^\eta(\tau),\nabla w)d\tau\\ &\quad+\int_0^t\Big((\mu+J_\eta(A\cdot\nabla w\otimes\nabla w))^{\frac{(p-2)}{2}}\nabla\varphi^\eta(\tau),\nabla w\Big)d\tau.\end{aligned}$$

On the other hand, by density, we can use test functions in (5.2) belonging to $W_0^{1,2}(\Omega)$. Therefore, employing the regularity of φ^η, $\varphi^\eta\in C(0,T;L^2(\Omega))\cap L^2(0,T;W_0^{1,2}(\Omega))$, we can use φ^η as test function in (5.2) and obtain, from (5.6),

$$(5.7)\qquad \begin{aligned}(w,\varphi_\circ) &= (w,\varphi^\eta(t)) + \int_0^t (f,\varphi^\eta(\tau))d\tau\\ &\quad+\int_0^t((\mu+J_\eta(A\cdot\nabla w\otimes\nabla w))^{\frac{(p-2)}{2}}-(\mu+A\cdot\nabla w\otimes\nabla w)^{\frac{(p-2)}{2}})(\nabla\varphi^\eta(\tau),\nabla w)d\tau\\ &= (w,\varphi^\eta(t)) + \int_0^t (f,\varphi^\eta(\tau))d\tau + I_\eta,\quad \forall t>0.\end{aligned}$$

One easily verifies that the integral I_η goes to zero, as η goes to zero, along a subsequence, by using the boundedness of $\nabla v\in L^p(\Omega)$ and of $\nabla\varphi^\eta\in L^2(0,T\times\Omega)$ (for details we refer to Lemma 2.5 in [**9**]). By using (2.4) in Lemma 2.5, and the Sobolev embedding theorem with embedding exponent $\dfrac{np}{n-p}$, we have

$$(5.8)\qquad |(w,\varphi^\eta(t))|\le \|\varphi^\eta(t)\|_{\frac{np}{np-n+p}}\|w\|_{\frac{np}{n-p}}\le c\|\varphi_\circ\|_1 M^{\frac{(2-p)(np-n+p)}{p(np-2n+2p)}}M^{\frac{1}{p}}t^{-\frac{np-n+p}{np-2n+2p}},$$

for all $t>0$, uniformly in $\eta>0$. Moreover, for any $0<\bar t<t$

$$(5.9)\qquad |\int_0^t (f,\varphi^\eta(\tau))d\tau|\le|\int_0^{\bar t}(f,\varphi^\eta(\tau))d\tau|+|\int_{\bar t}^t(f,\varphi^\eta(\tau))d\tau|.$$

By applying Hölder's inequality with $r'<\dfrac{np}{2n-2p}$ and $\dfrac{1}{r}+\dfrac{1}{r}=1$, then employing estimate (2.4) in Lemma 2.5, we can bound the first integral as follows

$$\begin{aligned}|\int_0^{\bar t}(f,\varphi^\eta(\tau))\,d\tau| &\le \|f\|_r\int_0^{\bar t}\|\varphi^\eta(\tau)\|_{r'}\,d\tau\le cM^{\frac{(2-p)\beta}{r}}\|f\|_r\|\varphi_\circ\|_1\bar t^{\,1-\frac{p\beta}{r}}\\ &\le c\,|\Omega|^{\frac{1}{r}}M^{\frac{(2-p)\beta}{r}}\|f\|_r\|\varphi_\circ\|_{\bar r'}\bar t^{\,1-\frac{p\beta}{r}},\end{aligned}$$

where $\frac{1}{\bar r}+\frac{1}{\bar r'}=1$. Similarly, by applying Hölder's inequality, and then using estimate (2.5) in Lemma 2.5, we can bound the second integral as follows

$$\begin{aligned}|\int_{\bar t}^{t}(f,\varphi^\eta(\tau))\,d\tau| &\le c\,\|f\|_{-1,p'}\int_{\bar t}^{t}\|\nabla\varphi^\eta(\tau)\|_p\,d\tau\\ &\le c\,M^{\frac{2-p}{2}(\beta+\frac1p)}\|f\|_{-1,p'}\|\varphi_\circ\|_1(\bar t^{\,\frac{1-\beta p}{2}}-t^{\frac{1-\beta p}{2}})\\ &\le c\,|\Omega|^{\frac{1}{\bar r}}M^{\frac{2-p}{2}(\beta+\frac1p)}\|f\|_{-1,p'}\|\varphi_\circ\|_{\bar r'}\bar t^{\,\frac{1-\beta p}{2}}.\end{aligned}$$

From the above estimates and (5.7)-(5.9), letting $t\to\infty$ we get

$$\frac{|(w,\varphi_\circ)|}{\|\varphi_\circ\|_{\bar r'}}\le c\,|\Omega|^{\frac{1}{\bar r}}\Big(M^{\frac{(2-p)\beta}{r}}\|f\|_r\bar t^{\,1-\frac{p\beta}{r}}+M^{\frac{2-p}{2}(\beta+\frac1p)}\|f\|_{-1,p'}\,\bar t^{\,\frac{1-\beta p}{2}}\Big),$$

for all $\varphi_\circ\in C_0^\infty(\Omega)$. Hence the last estimate implies

$$\|w\|_{\bar r}\le\ c\,|\Omega|^{\frac{1}{\bar r}}\Big(\|f\|_rM^{\frac{(2-p)\beta}{r}}\bar t^{\,1-\frac{p\beta}{r}}+\|f\|_{-1,p'}M^{\frac{2-p}{2}(\beta+\frac1p)}\,\bar t^{\,\frac{1-\beta p}{2}}\Big).$$

Since the right-hand side is uniform with respect to $\bar r$, letting $\bar r\to\infty$, we obtain

$$\|w\|_\infty\le c\Big(M^{\frac{(2-p)\beta}{r}}\|f\|_r\bar t^{\,1-\frac{p\beta}{r}}+M^{\frac{2-p}{2}(\beta+\frac1p)}\|f\|_{-1,p'}\,\bar t^{\,\frac{1-\beta p}{2}}\Big).$$

Choosing $\bar t=\Big(\|f\|_r^{-1}\|f\|_{-1,p'}\Big)^\gamma M^{\frac{2-p}{p}}$, with $\gamma=\dfrac{2r}{r+p\beta(r-2)}$, we obtain

$$\|w\|_\infty\le cM^{\frac{(2-p)}{p}}\|f\|_r^{1-\gamma(1-\frac{p\beta}{r})}\|f\|_{-1,p'}^{\gamma(1-\frac{p\beta}{r})}=cM^{\frac{(2-p)}{p}}\|f\|_r^{\frac{r(p\beta-1)}{r+p\beta(r-2)}}\|f\|_{-1,p'}^{\frac{2(r-p\beta)}{r+p\beta(r-2)}}. \tag{5.10}$$

Let us give back the superscript ν to the solution w^ν. From (5.3) and the above arguments, the sequence of solutions $\{w^\nu\}$ is bounded in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, uniformly in $\nu>0$, and satisfies estimate (5.10). Therefore, by Rellich's theorem, passing to a suitable subsequence, we have

$$\begin{gathered}w^\nu\to u\quad\text{strongly in } L^p(\Omega)\,,\ \text{as}\ \ \nu\to0,\\ \nabla w^\nu\rightharpoonup\nabla u\quad\text{weakly in } L^p(\Omega)\,,\ \text{as}\ \ \nu\to0,\\ w^\nu\to u\quad\text{almost everywhere in } \Omega\,,\ \text{as}\ \ \nu\to0\,.\end{gathered}$$

Hence

$$
\begin{aligned}
|u(x)| &\leq |u(x)-w^\nu(x)|+|w^\nu(x)| \leq |u(x)-w^\nu(x)|+\|w^\nu\|_\infty \\
&\leq |u(x)-w^\nu(x)|+cM^{\frac{(2-p)}{p}}\|f\|_r^{\frac{r(p\beta-1)}{r+p\beta(r-2)}}\|f\|_{-1,p'}^{\frac{2(r-p\beta)}{r+p\beta(r-2)}},
\end{aligned}
\tag{5.11}
$$

for each $\nu>0$, almost everywhere in Ω. Passing to the limit in (5.11) and using the expression (2.6) of β, we get (5.5).

We want to show that u is a weak solution of (1.1) with $\mu>0$. To this aim, we again appeal to the monotonicity trick, with some suitable changes. Set

$$
L_\mu(\psi)=-\nabla\cdot\left((\mu+A\cdot\nabla\psi\otimes\nabla\psi)^{\frac{p-2}{2}}\nabla\psi\right),
$$

$$
L(\psi)=-\nabla\cdot\left(A\cdot\nabla\psi\otimes\nabla\psi)^{\frac{p-2}{2}}\nabla\psi\right),
$$

and

$$
X_\nu := (L_\mu(w^\nu)-L_\mu(\psi),w^\nu-\psi)\geq 0,\ \forall\psi\in W_0^{1,p}(\Omega).
$$

By using (5.3) it is easy to see that $L_\mu(w^\nu)$ is bounded, uniformly in ν, in $W^{-1,p'}(\Omega)$. Hence

$$
L_\mu(w^\nu)\rightharpoonup\widetilde{\chi}\ \text{ weakly in }\ W^{-1,p'}(\Omega).
$$

Let us show that $\widetilde{\chi}=L_\mu(u)$. By using that w^ν is a solution of (5.1), we write X_ν as follows

$$
\begin{aligned}
X_\nu &= (L_\mu(w^\nu),w^\nu)-(L_\mu(w^\nu),\psi)-(L_\mu(\psi),w^\nu-\psi)\\
&=(f,w^\nu)-\nu\|\nabla w^\nu\|^2-(L_\mu(w^\nu),\psi)-(L_\mu(\psi),w^\nu-\psi).
\end{aligned}
$$

Passing to the lim sup and observing that

$$
\limsup(-\nu\|\nabla w^\nu\|^2)=-\liminf(\nu\|\nabla w^\nu\|^2)\leq 0,
$$

we get

$$
\begin{aligned}
0 &\leq \limsup X_\nu \leq (f,u)+\limsup(-\nu\|\nabla w^\nu\|^2)-(\widetilde{\chi},\psi)-(L_\mu(\psi),u-\psi)\\
&\leq (f,u)-(\widetilde{\chi},\psi)-(L_\mu(\psi),u-\psi).
\end{aligned}
$$

On the other hand, the limit u satisfies

$$
(\widetilde{\chi},u)=(f,u).
\tag{5.12}
$$

In this regard we note that, for any $\psi\in C_0^\infty(\Omega)$,

$$
\nu|(\Delta w^\nu,\psi)|=\nu|(\nabla w^\nu,\nabla\psi)|\leq\nu\|\nabla w^\nu\|_p\|\nabla\psi\|_{p'},
$$

and therefore tends to zero, thanks to (5.3). Hence

$$
\begin{aligned}
0 &= \lim_{\nu\to 0}\left[-\nu(\Delta w^\nu,\psi)+\langle L_\mu(w^\nu),\psi\rangle-\langle f,\psi\rangle\right]\\
&=(\widetilde{\chi},\psi)-(f,\psi),\ \forall\psi\in C_0^\infty(\Omega),
\end{aligned}
$$

which implies (5.12), by using a density argument. Summarizing we have found

$$(\widetilde{\chi} - L_\mu(\psi), u - \psi) \geq 0, \ \forall \psi \in W_0^{1,p}(\Omega)\,.$$

Taking $\psi = u - \lambda v$, for $\lambda > 0$ and for some $v \in W_0^{1,p}(\Omega)$, and then letting λ tend to zero the thesis follows.

Proof of Theorem 1.3. Let us consider, for all $\mu > 0$, the sequence $\{u^\mu\}$ of solutions of (1.6). From Lemma 5.1, $\{u^\mu\}$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, uniformly in $\mu > 0$, and satisfies estimates (5.5) and (4.3). Therefore, by Rellich's theorem, in the limit as μ tends to zero, the sequence $\{u^\mu\}$ converges weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and, almost everywhere along a subsequence to a function u. Denoting the subsequence by the same symbol $\{u^\mu\}$, for any $\mu > 0$ we have

$$|u(x)| \leq |u(x) - u^\mu(x)| + |u^\mu(x)| \leq |u(x) - u^\mu(x)| + \|u^\mu\|_\infty, \quad \text{a.e. in } x \in \Omega.$$

Passing to the limit, recalling (5.5) and using the expressions of β and M we get (1.8). The fact that the weak limit u a solution of (1.6) follows by using the same argument lines as in the proof of Theorem 1.1. $\square$

References

[1] E. Acerbi and N. Fusco, *Regularity for minimizers of nonquadratic functionals: the case* $1 < p < 2$, J. Math. Anal. Appl. **140** (1989), 115-135.

[2] E. Acerbi and G. Mingione, *Gradient estimates for a class of parabolic systems*, Duke Math. J. **136** (2007), 285-320.

[3] V. Adolfsson, *L^p-integrability of the second order derivatives of Green potentials in convex domains*, Pacific J. Math. **159** (1993), 201-225.

[4] H. Beirão da Veiga, *Navier-Stokes equations with shear thinning viscosity. Regularity up to the boundary*, J. Math. Fluid Mech. **11** (2009), 258-273.

[5] H. Beirão da Veiga and F. Crispo, *On the global regularity for nonlinear systems of the p-Laplacian type*, Discrete Contin. Dyn. Syst. Ser. S **6** (2013), 1173-1191.

[6] H. Beirão da Veiga and F. Crispo, *On the global $W^{2,q}$ regularity for nonlinear N-systems of the p-Laplacian type in n space variables*, Nonlinear Anal. **75** (2012), 4346-4354.

[7] M. Carozza, N. Fusco and G. Mingione, *Partial regularity of minimizers of quasiconvex integrals with subquadratic growth*, Ann. Mat. Pura Appl. (IV) **175** (1998), 141-164.

[8] F. Crispo and C. R. Grisanti, *On the existence, uniqueness and $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ regularity for a class of shear-thinning fluids*, J. Math. Fluid Mech. **10** (2008), 455-487.

[9] F. CRISPO and P. MAREMONTI, *Higher regularity of solutions to the singular p-Laplacean parabolic system, Adv.* Differential Equations **18** (2013), 849-894.

[10] F. DUZAAR and G. MINGIONE, *Regularity for degenerate elliptic problems via p-harmonic approximation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 735-766.

[11] F. DUZAAR and G. MINGIONE, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal. **259** (2010), 2961-2998.

[12] L. C. EVANS, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Rational Mech. Anal. **95** (1986), 227-252.

[13] M. GIAQUINTA and L. MARTINAZZI, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Appunti. Sc. Norm. Super. Pisa (N. S.), Edizioni della Normale, Pisa 2005.

[14] M.-H. GIGA, Y. GIGA and J. SAAL, *Nonlinear partial differential equations. Asymptotic behavior of solutions and self-similar solutions*, Progr. Nonlinear Differential Equations Appl., **79**, Birkhäuser, Boston 2010.

[15] T. KILPELÄINEN and J. MALÝ, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137-161.

[16] J. KINNUNEN and S. ZHOU, *A local estimate for nonlinear equations with discontinuous coefficients*, Comm. Partial Differential Equations **24** (1999), 2043-2068.

[17] O. A. LADYZHENSKAYA, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York-London-Paris 1969.

[18] O. A. LADYZHENSKAYA, *The boundary value problems of mathematical physics*, Applied Mathematical Sciences, **49**, Springer-Verlag, New York 1985.

[19] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris 1969.

[20] P. MAREMONTI, *Some interpolation inequalities involving Stokes operator and first order derivatives*, Ann. Mat. Pura Appl. (IV) **175** (1998), 59-91.

[21] P. MAREMONTI and V. A. SOLONNIKOV, *Estimates of the solution of the Dirichlet problem for the Laplace operator in exterior domains*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **146** (1985), 92-101. Translated in J. Soviet Math. **40** (1988), 72-79.

[22] G. MINGIONE, *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math. **51** (2006), 355-426.

[23] J. NASH, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931-954.

[24] G. PRODI, *Teoremi di tipo locale per il sistema di Navier-Stokes e stabilità delle soluzioni stazionarie*, Rend. Sem. Mat. Univ. Padova **32** (1962), 374-397.

[25] P. TOLKSDORF, *Everywhere-regularity for some quasilinear systems with a lack of ellipticity*, Ann. Mat. Pura Appl. (IV) **134** (1983), 241-266.

FRANCESCA CRISPO, PAOLO MAREMONTI
Second University of Naples
Via Vivaldi 43
Caserta, 81100, Italy
e-mail: francesca.crispo@unina2.it
e-mail: paolo.maremonti@unina2.it