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Sharp asymptotics for the porous media equation in low dimensions via Gagliardo-Nirenberg inequalities

Abstract. We prove sharp asymptotic bounds for solutions to the porous media equation with homogeneous Dirichlet or Neumann boundary conditions on a bounded Euclidean domain, in dimension $N = 1$ and $N = 2$. This is achieved by making use of appropriate Gagliardo-Nirenberg inequalities only. The generality of the discussion allows to prove similar bounds for *weighted* porous media equations, provided one deals with weights for which suitable Gagliardo-Nirenberg inequalities hold true. Moreover, we show equivalence between such functional inequalities and the mentioned asymptotic bounds for solutions.

Keywords. Weighted porous media equation, smoothing effect, asymptotic behaviour, weighted Poincaré, Sobolev and Gagliardo-Nirenberg inequalities, nonlinear diffusion equations.

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Contents

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1 - Introduction

We consider weak solutions to the homogeneous Dirichlet and Neumann problems for the porous media equation posed on an Euclidean domain $\Omega \subset \mathbb{R}^N$, where $N = 1$ or $N = 2$. More precisely, the Dirichlet problem reads

$$\begin{cases} u_t = \Delta(u^m) & \text{in } \Omega \times (0,\infty)\,, \\ u = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u = u_0 & \text{on } \Omega \times \{0\}\,. \end{cases} \tag{1.1}$$

In problem (1.1) we might also consider the case $\Omega = \mathbb{R}^N$, in which of course no boundary condition is assumed. The Neumann problem reads (in this case we shall also assume that Ω is a regular bounded domain)

$$\begin{cases} u_t = \Delta(u^m) & \text{in } \Omega \times (0,\infty)\,, \\ \dfrac{\partial(u^m)}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u = u_0 & \text{on } \Omega \times \{0\}\,. \end{cases} \tag{1.2}$$

Notice that we define as usual, here and throughout the paper, $u^m := |u|^{m-1}u$. We refer the reader to [**25**] as the general reference concerning this equation.

Our goal will be to prove sharp bounds on $\|u(t)\|_\infty$ for short times and then to study the long-time behaviour of the solutions as well (Section 2). This has been done in dimension $N \geq 3$ in [**11**], improving on previous results given in [**2, 4**]. The method of proof used in such paper strongly depends on the validity of the Sobolev inequality on Ω, and fails to produce the correct short-time behaviour in lower dimension. In fact, consider for simplicity the Dirichlet problem with $|\Omega| < \infty$. By exploiting the methods of proof of [**11**] and using the Sobolev inequalities associated to the dimension of the space, namely

$$\|v\|_p \leq C(p)\,\|\nabla v\|_2 \quad \forall v \in W_0^{1,2}(\Omega)\,,\ \forall p \in (2,\infty)\,,\ \Omega \subset \mathbb{R}^2$$

and

$$\|v\|_\infty \leq C\,\|\nabla v\|_2 \quad \forall v \in W_0^{1,2}((a,b))\,,\ (a,b) \subset \mathbb{R}\,,$$

it can only be shown that the solution $u(t)$ corresponding to L^1 data satisfies the estimate $\|u(t)\|_\infty \leq C_\varepsilon\, t^{-\frac{1}{m}-\varepsilon}$ for any $\varepsilon > 0$ and $t \in (0,1]$ in the case $N = 2$, whereas it satisfies the estimate $\|u(t)\|_\infty \leq C\, t^{-\frac{1}{m}}$ for any $t \in (0,1]$ in the case $N = 1$. On the other hand, the correct time behaviour of solutions corresponding to such data is $t^{-\frac{1}{m}}$ when $N = 2$ and $t^{-\frac{1}{m+1}}$ when $N = 1$. Indeed this is exactly the short-time behaviour associated to the well-known Barenblatt solutions (see e.g. the monograph [**24**]).

Our first main result will recover the correct low-dimensional estimates by using a purely functional analytic approach involving, for the Dirichlet problem, suitable Gagliardo-Nirenberg inequalities which follow from the well-known Nash inequality

$$\|f\|_2 \le C(N)\,\|\nabla f\|_2^{\frac{N}{N+2}}\|f\|_1^{\frac{2}{N+2}} \quad \forall f \in C_c^\infty(\mathbb{R}^N)\,, \tag{1.3}$$

valid in any dimension $N \ge 1$. Such an approach is standard in the linear case, and it has been successfully adapted in some nonlinear setting in [**18**]. An entirely similar discussion holds for solutions to the Neumann problem too. For the detailed statements see Theorem 2.1, which deals with the case $N = 1$, and Theorem 2.2, which deals with the case $N = 2$. Notice that in higher dimensions our approach would yield nothing more than what we proved in [**11**], since for $N \ge 3$ the standard Sobolev inequality is equivalent to the Nash inequality corresponding to the same dimension (see, for instance, the results of [**3**]).

Having at our disposal a strategy of proof depending only on the validity of appropriate functional inequalities, we generalize the results of [**12**] and [**11**] to discuss the asymptotics of *weighted* porous media equations, as concerns both the Dirichlet and the Neumann case, under the sole assumption that a suitable weighted Gagliardo-Nirenberg inequality holds true (Section 3). We refer to [**15, 16, 9, 10, 19, 21, 7, 8, 22, 23, 14, 20, 11, 12, 13**] and references quoted therein, without any claim for completeness, as general references to weighted degenerate diffusion equations. Given two weights ρ_ν, ρ_μ which are positive and sufficiently smooth in Ω (but having possible degeneracies or singularities at the boundary), we shall consider weak solutions to the weighted Dirichlet problem

$$\begin{cases} \rho_\nu\, u_t = \operatorname{div}\Big(\rho_\mu\, \nabla(u^m)\Big) & \text{in } \Omega \times (0,\infty)\,, \\ u = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u = u_0 & \text{on } \Omega \times \{0\}\,, \end{cases} \tag{1.4}$$

and to the weighted Neumann problem (in this case under the additional hypothesis that $\rho_\nu \in L^1(\Omega)$)

$$\begin{cases} \rho_\nu\, u_t = \operatorname{div}\Big(\rho_\mu\, \nabla(u^m)\Big) & \text{in } \Omega \times (0,\infty)\,, \\ \rho_\mu\, \dfrac{\partial(u^m)}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u = u_0 & \text{on } \Omega \times \{0\}\,. \end{cases} \tag{1.5}$$

Notice that the case of unbounded Ω, in particular $\Omega = \mathbb{R}^N$, can be considered as well. Weak solutions to problems (1.4)-(1.5) are defined precisely in [**12**, Sect. 3], and are unique provided a suitable condition holds: we refer to such solution as to the *weak energy solution* to the problems considered. In fact, since non-uniqueness issues may occur, additional assumptions on the solution must be made: for the reader's convenience we recall briefly the situation in Section 3. In the latter section we shall also prove that suitable bounds on the L^∞ norm of the solutions are *equivalent* to certain *weighted* Gagliardo-Nirenberg inequalities which naturally generalize the Nash inequality (1.3). Actual examples of weights for which such weighted Gagliardo-Nirenberg inequalities do hold are given in Section 4. We remark that these examples are produced by choosing appropriate samples of the well-known Caffarelli-Kohn-Nirenberg inequalities [**6**].

As concerns the long-time asymptotics of solutions, we collect the relevant results in Sections 2.1 and 3.1 for the unweighted and weighted problems, respectively. The proofs of such estimates require less essential modifications w.r.t. the ones given in [**4, 11, 12**]. The resulting bounds will be particularly significant in the Neumann case, for which it is known (see [**2, 4, 11**]) that, as $t \to \infty$, solutions to (1.2) converge to the mean value $\overline{u}_0$ of the initial datum, which is conserved along the evolution. The time rate of such convergence is polynomial when $\overline{u}_0 = 0$, exponential otherwise, a fact which can be proved by using a Poincaré-type functional inequality. In fact, the sharp results given in [**11**] also hold when $N = 1, 2$ with an analogous proof, and can be readily adapted to the weighted case as well.

2 - The unweighted case

In this section we shall prove L^∞ regularizing estimates for solutions to the unweighted problems (1.1) and (1.2), by means of Gagliardo-Nirenberg inequalities which are consequences of the Nash inequality (1.3) for $N = 1$ and $N = 2$. Subsection 2.1 is then devoted to improving such estimates for $t \to \infty$.

In the proofs of the forthcoming results, we shall often make use of the following well-known properties of the solutions u to (1.1) or (1.2). First, the L^p norm of u is non-increasing:

$$\|u(t)\|_p \leq \|u_0\|_p \quad \forall t > 0\,, \ \forall p \in [1, \infty]\,. \tag{2.1}$$

Moreover, if u and v are solutions to (1.1) or (1.2) associated to the initial data u_0 and v_0 respectively, the L^1-contractivity principle

$$\|u(t) - v(t)\|_1 \leq \|u_0 - v_0\|_1 \quad \forall t > 0 \tag{2.2}$$

holds true.

For notational simplicity, in the sequel we shall often denote inessential positive numerical constants (possibly changing from line to line) by C, without any further comment.

Theorem 2.1 (N = 1). *Assume that Ω is an interval. Let u be the solution to the Dirichlet problem* (1.1) *starting from an initial datum $u_0 \in L^1(\Omega) \cap L^{p_0}(\Omega)$, with $p_0 \geq 1$. Then the bound*

$$\|u(t)\|_\infty \leq C \, \frac{\|u_0\|_{p_0}^{\frac{2p_0}{2p_0+m-1}}}{t^{\frac{1}{2p_0+m-1}}} \quad \forall t > 0 \tag{2.3}$$

holds true.

Assume that Ω is a bounded interval. Let u be the solution to the Neumann problem (1.2) *starting from an initial datum $u_0 \in L^{p_0}(\Omega)$, with $p_0 \geq 1$. Then the bound*

$$\|u(t)\|_\infty \leq C \left(\frac{\|u_0\|_{p_0}^{\frac{2p_0}{2p_0+m-1}}}{t^{\frac{1}{2p_0+m-1}}} + \|u_0\|_{p_0} \right) \quad \forall t > 0 \tag{2.4}$$

holds true.

Proof. Let us consider first the Dirichlet problem (1.1). We start from the Nash inequality (1.3) corresponding to $N = 1$. Notice that this is a Gagliardo-Nirenberg inequality of the form considered in [3]: the parameter q associated to it, as defined in equation (3.1) of [3], can be readily calculated and is equal to -2. Hence Theorem 3.2 of such paper is applicable in our situation. As a consequence, the family of Gagliardo-Nirenberg inequalities

$$\|f\|_r \leq C \|\nabla f\|_2^{\vartheta} \|f\|_s^{1-\vartheta} \quad \forall f \in W_0^{1,2}(\Omega) \cap L^s(\Omega) \tag{2.5}$$

is valid provided $0 < s < r \leq \infty$ and ϑ are related by $\dfrac{\vartheta}{2} = \dfrac{1-\vartheta}{s} - \dfrac{1}{r}$. Notice that we adopt the symbol $\|\cdot\|_s$ even in the case $s \in (0,1)$ (here we do not need that such a functional is a norm). In particular one can choose $r = \infty$ and get that, for any $s \in (0, \infty)$, the inequality

$$\|f\|_\infty \leq C \|\nabla f\|_2^{\frac{2}{s+2}} \|f\|_s^{\frac{s}{s+2}} \quad \forall f \in W_0^{1,2}(\Omega) \cap L^s(\Omega) \tag{2.6}$$

holds true. Equivalently, (2.6) reads

$$\|\nabla f\|_2^2 \geq \frac{\|f\|_\infty^{s+2}}{C \, \|f\|_s^s} \quad \forall f \in W_0^{1,2}(\Omega) \cap L^s(\Omega)\,. \tag{2.7}$$

Now take $p_0 > 1$ and $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Fix $t > 0$. Multiplying (1.1) by u^{p_0-1} and formally integrating by parts in $\Omega \times (0,t)$ (this can be made rigorous by a standard approximation procedure) we get

$$\frac{4p_0(p_0-1)m}{(p_0+m-1)^2}\int_0^t\int_\Omega\left|\nabla\left(u(x,\tau)^{\frac{p_0+m-1}{2}}\right)\right|^2 \mathrm{d}x\,\mathrm{d}\tau \le \|u_0\|_{p_0}^{p_0}\,. \tag{2.8}$$

Using (2.7) applied to the function $f=u(\cdot,\tau)^{\frac{p_0+m-1}{2}}$ yields

$$\frac{4p_0(p_0-1)m}{C\,(p_0+m-1)^2}\int_0^t \frac{\left\|u(\tau)^{\frac{p_0+m-1}{2}}\right\|_\infty^{s+2}}{\left\|u(\tau)^{\frac{p_0+m-1}{2}}\right\|_s^{s}}\,\mathrm{d}\tau \le \|u_0\|_{p_0}^{p_0}. \tag{2.9}$$

Thanks to the monotonicity in time of any L^p norm (provided $s(p_0+m-1)/2\ge 1$ – recall (2.1)), from (2.9) we infer that there holds

$$t\,\frac{4p_0(p_0-1)m}{C\,(p_0+m-1)^2}\,\frac{\|u(t)\|_\infty^{\frac{(s+2)(p_0+m-1)}{2}}}{\|u_0\|_{\frac{s(p_0+m-1)}{2}}^{\frac{s(p_0+m-1)}{2}}} \le \|u_0\|_{p_0}^{p_0}. \tag{2.10}$$

Upon choosing $s=\dfrac{2p_0}{p_0+m-1}$ (2.10) reads

$$C\,t\,\frac{\|u(t)\|_\infty^{2p_0+m-1}}{\|u_0\|_{p_0}^{p_0}} \le \|u_0\|_{p_0}^{p_0}\,,$$

or equivalently

$$\|u(t)\|_\infty \le C\,\frac{\|u_0\|_{p_0}^{\frac{2p_0}{2p_0+m-1}}}{t^{\frac{1}{2p_0+m-1}}}\,, \tag{2.11}$$

which is exactly (2.3). However, recall that for the previous computations to work we needed to assume $p_0>1$ and $u_0\in L^1(\Omega)\cap L^\infty(\Omega)$. The first condition can be removed as follows. Write (2.11) in the time interval $(t/2,t)$, use the inequality $\|f\|_{p_0}^{p_0}\le\|f\|_\infty^{p_0-1}\|f\|_1$ and the fact that $\|u(t)\|_1$ is nonincreasing to get

$$\|u(t)\|_\infty \le C\left(\frac{2}{t}\right)^{\frac{\alpha}{2}}\|u(t/2)\|_\infty^{\alpha(p_0-1)}\|u_0\|_1^{\alpha}\,,\qquad \alpha=\frac{2}{2p_0+m-1}\,. \tag{2.12}$$

Iterating (2.12) k times one arrives at

$$\|u(t)\|_\infty \le \frac{C^{\sum_{h=0}^{k-1}[\alpha(p_0-1)]^h}\,2^{\frac{\alpha}{2}\sum_{h=0}^{k-1}(h+1)[\alpha(p_0-1)]^h}}{t^{\frac{\alpha}{2}\sum_{h=0}^{k-1}[\alpha(p_0-1)]^h}}\,\|u(t/2^k)\|_\infty^{[\alpha(p_0-1)]^k}\,\|u_0\|_1^{\alpha\sum_{h=0}^{k-1}[\alpha(p_0-1)]^h}\,. \tag{2.13}$$

Letting $k \to \infty$ in (2.13) one easily sees that the multiplicative constants above remain bounded, the exponents to which t^{-1} and $\|u_0\|_1$ are elevated tend to $1/(m+1)$ and $2/(m+1)$ respectively, while the one to which $\|u(t/2^k)\|_\infty \le \|u_0\|_\infty$ is elevated tends to zero. This gives (recall, up to relabelling constants)

$$\|u(t)\|_\infty \le C \frac{\|u_0\|_1^{\frac{2}{m+1}}}{t^{\frac{1}{m+1}}}, \tag{2.14}$$

which is (2.11) when $p_0 = 1$. Finally, we are left with removing the hypothesis $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. In order to do so (see also [**11**, pp. 2270]), take $u_0 \in L^1(\Omega) \cap L^{p_0}(\Omega)$, choose a sequence $\{u_{0n}\} \subset L^1(\Omega) \cap L^\infty(\Omega)$ which converges to u_0 in $L^1(\Omega) \cap L^{p_0}(\Omega)$ and exploit the fact that, for the corresponding sequence of solutions $\{u_n(\cdot, t)\}$, one has $u_n(t) \to u(t)$ in $L^1(\Omega)$ by the L^1 contraction principle (recall (2.2)) but also $u_n(t) \to u(t)$ in the weak* topology of $L^\infty(\Omega)$, for any fixed $t > 0$. Hence, formula (2.3) for the solution associated to an initial datum $u_0 \in L^1(\Omega) \cap L^{p_0}(\Omega)$ follows from the weak* lower semicontinuity of the L^∞ norm.

We now consider the Neumann problem (1.2). For simplicity we shall assume that $|\Omega| = 1$: this constraint can be easily removed by a standard scaling argument identical to the one given in [**11**, pp. 2270]. To begin with, we claim that the inequality

$$\|f\|_\infty \le C(\|\nabla f\|_2 + \|f\|_1)^{\frac{2}{s+2}} \|f\|_s^{\frac{s}{s+2}} \quad \forall f \in W^{1,2}(\Omega) \tag{2.15}$$

holds true for all $s \in (0, \infty)$. In fact, this can be proved starting from (2.6) (with $\Omega = \mathbb{R}$) as follows. Given $f \in W^{1,2}(\Omega)$, consider the extension operator $P : W^{1,2}(\Omega) \longrightarrow W^{1,2}(\mathbb{R})$. As it is well-known, there exists a positive constant C such that $\|f\|_{L^s(\Omega)} \le \|Pf\|_{L^s(\mathbb{R})} \le C\|f\|_{L^s(\Omega)}$ for all given $s \in (0, \infty]$ and $\|Pf\|_{W^{1,2}(\mathbb{R})} \le C\|f\|_{W^{1,2}(\Omega)}$. Applying (2.6) (with $\Omega = \mathbb{R}$) to Pf and using the above properties of the operator P, we get:

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|_{W^{1,2}(\Omega)}^{\frac{2}{s+2}} \|f\|_{L^s(\Omega)}^{\frac{s}{s+2}}.$$

In order to prove (2.15), just notice that the usual norm on $W^{1,2}(\Omega)$ is equivalent to the norm given by $\|\nabla f\|_2 + \|f\|_1$, since Ω is bounded: this is an immediate consequence of the validity of the Poincaré inequality in $W^{1,2}(\Omega)$. It is convenient to rewrite (2.15) (upon relabelling C) as

$$\|\nabla f\|_2^2 \ge \frac{\|f\|_\infty^{s+2}}{C\,\|f\|_s^s} - \|f\|_1^2 \quad \forall f \in W^{1,2}(\Omega). \tag{2.16}$$

Let $t > 0$. Inequality (2.8) can be proved to hold exactly as in the Dirichlet case. Given $p_0 \in (1, \infty) \cap [m-1, \infty)$ and $u_0 \in L^\infty(\Omega)$, we can apply (2.16) to $f = u(\cdot, \tau)^{\frac{p_0+m-1}{2}}$

in (2.8) to get

$$\|u_0\|_{p_0}^{p_0} \geq \frac{4p_0(p_0-1)m}{(p_0+m-1)^2}\left(\int_0^t \frac{\left\|u(\tau)^{\frac{p_0+m-1}{2}}\right\|_\infty^{s+2}}{C\left\|u(\tau)^{\frac{p_0+m-1}{2}}\right\|_s^s}\,\mathrm{d}\tau - \int_0^t \left\|u(\tau)^{\frac{p_0+m-1}{2}}\right\|_1^2 \mathrm{d}\tau\right)$$

$$\geq \frac{4p_0(p_0-1)m}{(p_0+m-1)^2}\left(\int_0^t \frac{\|u(\tau)\|_\infty^{\frac{(p_0+m-1)(s+2)}{2}}}{C\|u(\tau)\|_{\frac{s(p_0+m-1)}{2}}^{\frac{s(p_0+m-1)}{2}}}\,\mathrm{d}\tau - \int_0^t \|u(\tau)\|_{\frac{p_0+m-1}{2}}^{p_0+m-1}\mathrm{d}\tau\right)$$

$$\geq \frac{4p_0(p_0-1)m}{(p_0+m-1)^2}\left(t\,\frac{\|u(t)\|_\infty^{\frac{(p_0+m-1)(s+2)}{2}}}{C\|u_0\|_{\frac{s(p_0+m-1)}{2}}^{\frac{s(p_0+m-1)}{2}}} - \int_0^t \|u(\tau)\|_{p_0}^{p_0+m-1}\mathrm{d}\tau\right)$$

$$\geq t\,\frac{4p_0(p_0-1)m}{(p_0+m-1)^2}\left(\frac{\|u(t)\|_\infty^{\frac{(p_0+m-1)(s+2)}{2}}}{C\|u_0\|_{\frac{s(p_0+m-1)}{2}}^{\frac{s(p_0+m-1)}{2}}} - \|u_0\|_{p_0}^{p_0+m-1}\right),$$

where we also exploited (2.1) and used the fact that $|\Omega| = 1$ and $p_0 \geq m-1$ to bound $\|u(\tau)\|_{(p_0+m-1)/2}$ with $\|u(\tau)\|_{p_0}$. If we set $s = 2p_0/(p_0+m-1)$ the previous inequality becomes

$$\|u_0\|_{p_0}^{p_0} \geq t\,\frac{4p_0(p_0-1)m}{(p_0+m-1)^2}\left(\frac{\|u(t)\|_\infty^{2p_0+m-1}}{C\|u_0\|_{p_0}^{p_0}} - \|u_0\|_{p_0}^{p_0+m-1}\right),$$

or equivalently (upon relabelling C)

$$\|u(t)\|_\infty \leq C\left(\frac{\|u_0\|_{p_0}^{2p_0}}{t} + \|u_0\|_{p_0}^{2p_0+m-1}\right)^{\frac{1}{2p_0+m-1}} \leq C\left(\frac{\|u_0\|_{p_0}^{\frac{2p_0}{2p_0+m-1}}}{t^{\frac{1}{2p_0+m-1}}} + \|u_0\|_{p_0}\right),$$

which is our claim. To extend such result to general values of p_0 (other than $p_0 \in (1,\infty)\cap[m-1,\infty)$) one can proceed similarly to the Dirichlet case, when we proved the validity of (2.3) down to $p_0 = 1$. We omit details and we limit ourselves to pointing out that, with respect to the Dirichlet case, there are slight technical difficulties that can be dealt with following closely the procedure given in [**11**, pp. 2268-2269].

Finally, the methods to remove the hypothesis $u_0 \in L^\infty(\Omega)$ are identical to the ones we exploited for the Dirichlet problem. □

Theorem 2.2 (N = 2). *Assume that Ω is a domain of $\mathbb{R}^2$. Let u be the solution to the Dirichlet problem* (1.1) *corresponding to an initial datum $u_0 \in L^1(\Omega) \cap L^{p_0}(\Omega)$,* *with* $p_0 \geq 1$. *Then the bound*

$$\|u(t)\|_\infty \leq C\,\frac{\|u_0\|_{p_0}^{\frac{p_0}{p_0+m-1}}}{t^{\frac{1}{p_0+m-1}}} \quad \forall t > 0 \tag{2.17}$$

holds true.

Assume that Ω is a bounded regular domain of $\mathbb{R}^2$. *Let u be the solution to the Neumann problem* (1.2) *corresponding to an initial datum* $u_0 \in L^{p_0}(\Omega)$, *with* $p_0 \geq 1$. *Then the bound*

$$\|u(t)\|_\infty \leq C\left(\frac{\|u_0\|_{p_0}^{\frac{p_0}{p_0+m-1}}}{t^{\frac{1}{p_0+m-1}}} + \|u_0\|_{p_0}\right) \quad \forall t > 0 \tag{2.18}$$

holds true.

Proof. Let us consider first the Dirichlet problem (1.1). Again, we start from the Nash inequality (1.3). In the case $N = 2$, thanks to Theorem 3.4 of [**3**] (here the parameter q defined in equation (3.1) of [**3**] is equal to ∞), (1.1) implies the validity of the following family of Gagliardo-Nirenberg inequalities:

$$\|f\|_r \leq C_r\,\|\nabla f\|_2^{1-\frac{s}{r}}\|f\|_s^{\frac{s}{r}} \quad \forall f \in W_0^{1,2}(\Omega) \cap L^s(\Omega) \tag{2.19}$$

for all s, r such that $0 < s < r$, where C_r is a positive constant depending continuously on r. Since in the sequel the parameter r will be constrained to lie inside a bounded interval, we can assume $C_r \leq C$ for a given C independent of r.

Take $p_0 > 1$ and $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Fix $t > 0$. In order to obtain (2.17), we shall proceed by a Moser iteration technique (see also [**1**]). Given two increasing sequences of positive numbers $\{p_k\}$ and $\{t_k\}$, to be defined later, let us multiply by u^{p_k-1} the equation (1.1) satisfied by u and formally integrate by parts in $\Omega \times (t_k, t_{k+1})$. We obtain:

$$\frac{4p_k(p_k-1)m}{(p_k+m-1)^2}\int_{t_k}^{t_{k+1}}\int_\Omega \left|\nabla\left(u(x,\tau)^{\frac{p_k+m-1}{2}}\right)\right|^2 \mathrm{d}x\,\mathrm{d}\tau \leq \|u(t_k)\|_{p_k}^{p_k}. \tag{2.20}$$

Writing (2.19) in the form

$$\frac{\|f\|_r^{\frac{2r}{r-s}}}{C\,\|f\|_s^{\frac{2s}{r-s}}} \leq \|\nabla f\|_2^2 \quad \forall f \in W_0^{1,2}(\Omega) \cap L^s(\Omega) \tag{2.21}$$

and applying it to $f = u(\cdot,\tau)^{\frac{p_0+m-1}{2}}$ in order to bound the l.h.s. of (2.20), one gets

$$\frac{4p_k(p_k-1)m}{(p_k+m-1)^2 C}\int_{t_k}^{t_{k+1}} \frac{\left\|u(\tau)^{\frac{p_k+m-1}{2}}\right\|_r^{\frac{2r}{r-s}}}{\left\|u(\tau)^{\frac{p_k+m-1}{2}}\right\|_s^{\frac{2s}{r-s}}}\,\mathrm{d}\tau \leq \|u(t_k)\|_{p_k}^{p_k}. \tag{2.22}$$

Let us set $t_k = (1 - 2^{-k})\, t$, so that $t_0 = 0$, $t_{k+1} - t_k = t/2^{k+1}$ and $\lim_{k\to\infty} t_k = t$. Exploiting (2.1), we have that (2.22) implies

$$t\,\frac{4p_k(p_k-1)m}{(p_k+m-1)^2\,2^{k+1}\,C}\,\frac{\|u(t_{k+1})\|_{\frac{r}{2}(p_k+m-1)}^{\frac{r}{r-s}(p_k+m-1)}}{\|u(t_k)\|_{\frac{s}{2}(p_k+m-1)}^{\frac{s}{r-s}(p_k+m-1)}} \le \|u(t_k)\|_{p_k}^{p_k}. \tag{2.23}$$

If we choose $s = s_k = \dfrac{2p_k}{p_k+m-1}$, so that $\dfrac{s}{2}(p_k+m-1) = p_k$, from (2.23) we deduce that, upon relabelling C (notice that by construction $\{s_k\}$ is a bounded sequence), the bound

$$\|u(t_{k+1})\|_{\frac{r}{2}(p_k+m-1)}^{\frac{r}{r-s_k}(p_k+m-1)} \le \frac{2^k\,C}{t}\,\|u(t_k)\|_{p_k}^{p_k+\frac{s_k}{r-s_k}(p_k+m-1)} \tag{2.24}$$

holds. From now on it will be convenient to set $r = r_k = s_k + 2$. By that, $\{r_k\}$ is also a bounded sequence. Moreover, we require $\{p_k\}$ to satisfy the recursive relation $p_{k+1} = 2p_k + m - 1$. We can therefore rewrite (2.24) as

$$\|u(t_{k+1})\|_{p_{k+1}} \le \frac{2^{\frac{k}{p_{k+1}}}\,C^{\frac{1}{p_{k+1}}}}{t^{\frac{1}{p_{k+1}}}}\,\|u(t_k)\|_{p_k}^{\frac{2p_k}{p_{k+1}}}. \tag{2.25}$$

Iterating (2.25) and using the fact that $p_{k+1} = (p_0+m-1)2^{k+1} + 1 - m$, we get:

$$\begin{aligned}
\|u(t_{k+1})\|_{p_{k+1}} &\le 2^{\frac{1}{p_{k+1}}\sum_{h=0}^{k} 2^h(k-h)} \left(\frac{C}{t}\right)^{\frac{1}{p_{k+1}}\sum_{h=0}^{k} 2^h} \|u_0\|_{p_0}^{\frac{2^{k+1}p_0}{p_{k+1}}} \\
&\le 2^{\frac{\sum_{h=0}^{k} 2^h(k-h)}{(p_0+m-1)2^{k+1}+1-m}} \left(\frac{C}{t}\right)^{\frac{2^{k+1}-1}{(p_0+m-1)2^{k+1}+1-m}} \|u_0\|_{p_0}^{\frac{2^{k+1}p_0}{(p_0+m-1)2^{k+1}+1-m}} \\
&\le C\,\frac{\|u_0\|_{p_0}^{\frac{2^{k+1}p_0}{(p_0+m-1)2^{k+1}+1-m}}}{t^{\frac{2^{k+1}-1}{(p_0+m-1)2^{k+1}+1-m}}}\,,
\end{aligned}$$

where C is again a suitable positive constant which can be taken to be independent of k. Noticing that $p_k \uparrow \infty$ and $t_k \uparrow t$ as $k \to +\infty$ and exploiting the monotonicity of the L^p norms (2.1), in the limit $k \to \infty$ we obtain

$$\|u(t)\|_\infty = \lim_{k\to+\infty} \|u(t)\|_{p_{k+1}} \le \liminf_{k\to+\infty} \|u(t_{k+1})\|_{p_{k+1}} \le C\,\frac{\|u_0\|_{p_0}^{\frac{p_0}{p_0+m-1}}}{t^{\frac{1}{p_0+m-1}}}\,, \tag{2.26}$$

that is (2.17). However, recall that the proof above is valid only for p_0 strictly larger than 1. Nonetheless, in order to extend (2.26) to the case $p_0 = 1$ as well, we can proceed exactly as in the proof of the bound (2.3), namely by using interpolation inequalities and exploiting (2.26) recursively. Also the relaxation of the assumption $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$ works just as in the mentioned proof.

Let us now consider the Neumann problem. As in the corresponding one-dimensional context we shall assume, with no loss of generality, that $|\Omega| = 1$. First of all we claim that the inequality

$$\|f\|_r \leq C_r(\|\nabla f\|_2 + \|f\|_1)^{1-\frac{s}{r}} \|f\|_s^{\frac{s}{r}} \quad \forall f \in W^{1,2}(\Omega) \tag{2.27}$$

holds true for all s, r such that $0 < s < r$. This fact can be proved similarly to the case $N = 1$, that is applying (2.19) (with $\Omega = \mathbb{R}^2$) to the extended function Pf and exploiting the properties of the extension operator.

Afterwards we start an iterative procedure which is similar to the one outlined in the Dirichlet case. Fixed $t > 0$, given $p_0 \in (1, \infty) \cap [m-1, \infty)$, $u_0 \in L^\infty(\Omega)$ and two increasing sequences $\{p_k\}$ (to be defined later) and $\{t_k\} = \{(1 - 2^{-k})\,t\}$, multiply equation (1.1) by $u^{p_k - 1}$ and formally integrate by parts to obtain again (2.20). Writing (2.27) in the form (let us relabel C)

$$\|\nabla f\|_2^2 \geq \frac{\|f\|_r^{\frac{2r}{r-s}}}{C\,\|f\|_s^{\frac{2s}{r-s}}} - \|f\|_1^2 \quad \forall f \in W^{1,2}(\Omega) \tag{2.28}$$

and applying it to the function $f = u(\cdot, \tau)^{\frac{p_k+m-1}{2}}$ in (2.20), we get:

$$\frac{4p_k(p_k-1)m}{(p_k+m-1)^2} \int_{t_k}^{t_{k+1}} \left(\frac{\left\| u(\tau)^{\frac{p_k+m-1}{2}} \right\|_r^{\frac{2r}{r-s}}}{C \left\| u(\tau)^{\frac{p_k+m-1}{2}} \right\|_s^{\frac{2s}{r-s}}} - \left\| u(\tau)^{\frac{p_k+m-1}{2}} \right\|_1^2 \right) \mathrm{d}\tau \leq \|u(t_k)\|_{p_k}^{p_k},$$

that is

$$\frac{4p_k(p_k-1)m}{(p_k+m-1)^2} \int_{t_k}^{t_{k+1}} \left(\frac{\|u(\tau)\|_{\frac{r}{2}(p_k+m-1)}^{\frac{r}{r-s}(p_k+m-1)}}{C\|u(\tau)\|_{\frac{s}{2}(p_k+m-1)}^{\frac{s}{r-s}(p_k+m-1)}} - \|u(\tau)\|_{\frac{p_k+m-1}{2}}^{p_k+m-1} \right) \mathrm{d}\tau \leq \|u(t_k)\|_{p_k}^{p_k}. \tag{2.29}$$

Since $(p_k + m - 1)/2 \leq p_k$ (recall that $p_0 \geq m - 1$ and $\{p_k\}$ is increasing) and (2.1) holds, (2.29) implies

$$t\,\frac{4p_k(p_k-1)m}{(p_k+m-1)^2\,2^{k+1}} \left(\frac{\|u(t_{k+1})\|_{\frac{r}{2}(p_k+m-1)}^{\frac{r}{r-s}(p_k+m-1)}}{C\|u(t_k)\|_{\frac{s}{2}(p_k+m-1)}^{\frac{s}{r-s}(p_k+m-1)}} - \|u(t_k)\|_{p_k}^{p_k+m-1} \right) \leq \|u(t_k)\|_{p_k}^{p_k}. \tag{2.30}$$

Choosing $s = s_k = \dfrac{2p_k}{p_k + m - 1}$, $r = r_k = s_k + 2$ and letting $p_{k+1} = 2p_k + m - 1$ we have that, for another positive constant C, there holds

$$\begin{aligned}&\|u(t_{k+1})\|_{\frac{r_k}{2}(p_k+m-1)}^{\frac{r_k}{r_k-s_k}(p_k+m-1)}\\&\le C\left(\frac{2^k}{t}\,\|u(t_k)\|_{p_k}^{p_k+\frac{s_k}{r_k-s_k}(p_k+m-1)}+\|u(t_k)\|_{p_k}^{p_k+\frac{s_k}{r_k-s_k}(p_k+m-1)+m-1}\right),\end{aligned}$$

which reads

$$\|u(t_{k+1})\|_{p_{k+1}}^{p_{k+1}}\le C\left(\frac{2^k}{t}\,\|u(t_k)\|_{p_k}^{2p_k}+\|u(t_k)\|_{p_k}^{2p_k+m-1}\right). \tag{2.31}$$

In order to handle (2.31) we assume the additional constraint $\|u_0\|_\infty\le 1$, so that $\|u(t_k)\|_{p_k}\le 1$ (recall (2.1) and that $|\Omega|=1$), which in turn implies $\|u(t_k)\|_{p_k}^{2p_k+m-1}\le\|u(t_k)\|_{p_k}^{2p_k}$. This allows to deduce from (2.31) the following inequality:

$$\|u(t_{k+1})\|_{p_{k+1}}\le 2^{\frac{k}{p_{k+1}}}\,C^{\frac{1}{p_{k+1}}}\left(t^{-1}+1\right)^{\frac{1}{p_{k+1}}}\|u(t_k)\|_{p_k}^{\frac{2p_k}{p_{k+1}}}. \tag{2.32}$$

Notice that (2.32) is the same as (2.25) provided one replaces t^{-1} with $t^{-1}+1$. So, proceeding exactly as in the proof of the Dirichlet estimate (2.17), we get

$$\|u(t)\|_\infty\le C\left(t^{-1}+1\right)^{\frac{1}{p_0+m-1}}\|u_0\|_{p_0}^{\frac{p_0}{p_0+m-1}}. \tag{2.33}$$

However, recall that (2.33) has been obtained under the hypothesis $\|u_0\|_\infty\le 1$. In order to deduce from it an estimate that works for any (bounded) initial datum, it is enough to apply (2.33) itself to the rescaled solution $\widehat{u}(\cdot,t)=\|u_0\|_\infty^{-1}u(\cdot,\|u_0\|_\infty^{1-m}t)$, which starts from the initial datum $u_0/\|u_0\|_\infty$, whose $L^\infty(\Omega)$ norm is equal to 1:

$$\|\widehat{u}(t)\|_\infty\le C\left(t^{-1}+1\right)^{\frac{1}{p_0+m-1}}\|\widehat{u}_0\|_{p_0}^{\frac{p_0}{p_0+m-1}}. \tag{2.34}$$

Evaluating (2.34) at time $t\|u_0\|_\infty^{m-1}$ in place of t yields

$$\|u(t)\|_\infty\le C\left(t^{-1}+\|u_0\|_\infty^{m-1}\right)^{\frac{1}{p_0+m-1}}\|u_0\|_{p_0}^{\frac{p_0}{p_0+m-1}}, \tag{2.35}$$

which holds for any $u_0\in L^\infty(\Omega)$. Of course (2.35) is not a regularizing estimate any longer, since on its r.h.s. there appears $\|u_0\|_\infty$. Our technique to remove the dependence on $\|u_0\|_\infty$ is not particularly complicated, but it involves long and tedious computations. The idea is to use (2.35) in the time interval $(t/2^{n+1},t/2^n)$, that is

$$\begin{aligned}\|u(t/2^n)\|_\infty&\le C\left((t/2^{n+1})^{-1}+\|u(t/2^{n+1})\|_\infty^{m-1}\right)^{\frac{1}{p_0+m-1}}\|u(t/2^{n+1})\|_{p_0}^{\frac{p_0}{p_0+m-1}}\\&\le C(n)\left(t^{-1}+\|u(t/2^{n+1})\|_\infty^{m-1}\right)^{\frac{1}{p_0+m-1}}\|u_0\|_{p_0}^{\frac{p_0}{p_0+m-1}},\end{aligned}$$

and apply it recursively. This allows to reduce the dependence of (2.35) on $\|u_0\|_\infty$, in the sense that it will appear as $\|u_0\|_\infty^\theta$, where $\theta = \theta(n)$ gets smaller as n gets larger. Then it can be shown that starting from data whose L^{p_0} norm is *larger* than 1, one can obtain a true regularity estimate similar to (2.33), but with different powers. Combining such regularity estimate with the improved (2.35) (through the recursive procedure sketched above), one infers that (2.18) holds for *any* $u_0 \in L^\infty(\Omega)$ (provided $p_0 \in (1,\infty) \cap [m-1,\infty)$). We omit the technical details and refer the reader to [**11**, pp. 2267-2268], where thorough computations are given in an similar context.

Finally, as concerns the methods to remove the hypotheses $u_0 \in L^\infty(\Omega)$ and $p_0 \in (1,\infty) \cap [m-1,\infty)$, the same comments of the one-dimensional Neumann case apply.

2.1 - *Long-time estimates*

The aim of this section is to improve the results of Theorems 2.1 and 2.2 for $t \to \infty$. In particular, we shall see that, at least when $|\Omega| < \infty$, the rates of decay of solutions to the Dirichlet problem (1.1) given by (2.3) or (2.17) can be sharpened. However, the most important improvements will be made for the Neumann problem (from which we therefore start this section): indeed, either (2.4) or (2.18) give no information for t large. As shown first in [**2**], and later improved in [**4, 11**], as $t \to \infty$ solutions to (1.2) converge to the mean value $\overline{u}_0$ of the initial datum, which is conserved along the evolution. The time rate of such convergence is polynomial when $\overline{u}_0 = 0$, exponential otherwise. In [**2**], as concerns the case $\overline{u}_0 \neq 0$, this was proved for $N = 1$ only, whereas a similar result in higher dimensions was shown to hold for data which are strictly bounded away from zero. The latter assumption was later removed in [**4**], while sharp rates of convergence have been recently established in [**11**]. Moreover, the results given in [**4, 11**] also hold when $N = 1, 2$ with an identical proof, and read as follows.

Theorem 2.3. *Let u be the solution to the Neumann problem* (1.2) *corresponding to an initial datum $u_0 \in L^1(\Omega)$, with $\overline{u}_0 \neq 0$. There exists a positive constant C such that the following estimate holds:*

$$\|u(t) - \overline{u}_0\|_\infty \leq C\, e^{-\frac{m}{C_P^2}|\overline{u}_0|^{m-1}\, t} \quad \forall t \geq 1\,, \tag{2.36}$$

where C_P is the best constant appearing in the Poincaré inequality

$$\|f - \overline{f}\|_2 \leq C_P \|\nabla f\|_2 \quad \forall f \in W^{1,2}(\Omega)\,,\ \overline{f} = \frac{\int_\Omega f(x)\,\mathrm{d}x}{|\Omega|}\,. \tag{2.37}$$

Proof. One just replicates the proof of [**11**, Th. 4.3]. Indeed, in order to do that, one only needs the validity of the Sobolev inequality $\|f-\overline{f}\|_q \le C\|\nabla f\|_2$ for some $q>2$, which here readily follows from (1.3) and (2.37). □

Notice that (2.36) is sharp in any dimension, as shown in [**11**, Prop 4.5]: there always exist initial data for which a matching lower bound holds. As for solutions corresponding to data such that $\overline{u}_0 = 0$ we have the following result.

Theorem 2.4. *Let u be the solution to the Neumann problem* (1.2) *corresponding to an initial datum $u_0 \in L^{p_0}(\Omega)$, with $p_0 \ge 1$ and $\overline{u}_0 = 0$. Assume that $N = 1$. Then for some positive constants C_1, C_2 the following estimate holds:*

$$\|u(t)\|_\infty \le C_1\, t^{-\frac{1}{2p_0+m-1}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0}^{1-m}\right)^{\frac{2p_0}{(m-1)(2p_0+m-1)}}} \quad \forall t>0\,. \tag{2.38}$$

Assume instead that $N=2$. Then for another positive constants C_1, C_2 the following estimate holds:

$$\|u(t)\|_\infty \le C_1\, t^{-\frac{1}{p_0+m-1}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0}^{1-m}\right)^{\frac{p_0}{(m-1)(p_0+m-1)}}} \quad \forall t>0\,. \tag{2.39}$$

In particular, in both cases the absolute bound

$$\|u(t)\|_\infty \le C_3\, t^{-\frac{1}{m-1}} \quad \forall t>0 \tag{2.40}$$

holds true for some $C_3 > 0$ (independent of u_0).

Proof. The proof proceeds along the lines of the one given in [**11**, Th. 4.1], exploiting both the Poincaré inequality (2.37) and the regularizing estimates (2.4) when $N=1$ and (2.18) when $N=2$. □

The absolute bound (2.40) is sharp, in the sense that in [**2**] the authors also prove, for solutions corresponding to $u_0 \in L^\infty(\Omega)$ with $\overline{u}_0 = 0$, a *lower* bound of order $t^{-1/(m-1)}$.

The results of Theorems 2.1 and 2.2, for t large, can be improved for the Dirichlet problem too provided $|\Omega|<\infty$. Notice that the bounds contained in the following result also recover the short-time behaviour of the solutions.

Theorem 2.5. *Let u be the solution to the Dirichlet problem* (1.1) *corresponding to an initial datum $u_0 \in L^{p_0}(\Omega)$, with $p_0 \ge 1$. Assume that $|\Omega|<\infty$. If*

$N = 1$, *for some positive constants* C_1, C_2 *there holds*

$$\|u(t)\|_\infty \le C_1\, t^{-\frac{1}{2p_0+m-1}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0}^{1-m}\right)^{\frac{2p_0}{(m-1)(2p_0+m-1)}}} \quad \forall t > 0\,. \tag{2.41}$$

If instead $N = 2$, *for another positive constants* C_1, C_2 *there holds*

$$\|u(t)\|_\infty \le C_1\, t^{-\frac{1}{p_0+m-1}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0}^{1-m}\right)^{\frac{p_0}{(m-1)(p_0+m-1)}}} \quad \forall t > 0\,. \tag{2.42}$$

In particular, in both cases the absolute bound

$$\|u(t)\|_\infty \le C_3\, t^{-\frac{1}{m-1}} \quad \forall t > 0 \tag{2.43}$$

holds true for some $C_3 > 0$ (*independent of* u_0).

Proof. The proof is not very different from the one of the Neumann case when $\overline{u}_0 = 0$, but it is simpler. Hence, we sketch it. Exploiting the fact that $|\Omega| < \infty$ and the consequent validity of the gap inequality $\|f\|_2 \le C_G \|\nabla f\|_2$ for all $f \in W_0^{1,2}(\Omega)$, from (1.1) one easily gets (let $p_0 > 1$)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_{p_0}^{p_0} &= -\left(\frac{2}{p_0+m-1}\right)^2 m p_0 (p_0-1) \left\|\nabla\left(u(s)^{\frac{p_0+m-1}{2}}\right)\right\|_2^2 \\
&\le -\left(\frac{2}{p_0+m-1}\right)^2 \frac{m p_0 (p_0-1)}{C_G^2} \|u(s)\|_{p_0+m-1}^{p_0+m-1} \\
&\le -\left(\frac{2}{p_0+m-1}\right)^2 \frac{m p_0 (p_0-1)}{C_G^2\, |\Omega|^{\frac{m-1}{p_0}}} \|u(s)\|_{p_0}^{p_0+m-1} \\
&= -\,C\left(\|u(s)\|_{p_0}^{p_0}\right)^{\frac{p_0+m-1}{p_0}}.
\end{aligned}$$

The claims follow by integrating the above differential inequality in the time interval $(t/2, t)$ and using the regularizing estimates (2.3) (when $N = 1$) or (2.17) (when $N = 2$) in the time interval $(0, t/2)$. The limiting case $p_0 = 1$ can be recovered in a standard way. We omit details and again refer the reader to [**11**, Th. 4.1]. □

The sharpness of estimates (2.41)-(2.43) is a consequence of the fact that, for t small, the well-known Barenblatt solutions behave like the r.h.s. of (2.41) or (2.42), whereas for t large the time behaviour predicted by (2.43) coincides with the one of separable solutions to (1.1) (see the monographs [**24, 25**]).

3 - The weighted case

We shall briefly consider here the weighted problems (1.4) and (1.5), for which suitable analogues of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5 can be established providing that the weights ρ_ν, ρ_μ involved satisfy certain functional requirements.

In the sequel, it is understood that the measures ν and μ are defined as $\mathrm{d}\nu = \rho_\nu(x)\mathrm{d}x$ and $\mathrm{d}\mu = \rho_\mu(x)\mathrm{d}x$, respectively.

For suitable concepts of *weak solution* to (1.4) and (1.5) we refer the reader to [**12**, Sect. 3]. Here we limit ourselves to commenting that the standard running assumptions on the weights will be inner positivity and regularity. More precisely, we shall ask:

$$\rho_\nu \in C^{3,\alpha}_{\mathrm{loc}}(\Omega)\,, \ \ \rho_\mu \in C^{2,\alpha}_{\mathrm{loc}}(\Omega), \quad \rho_\nu(x), \rho_\mu(x) > 0 \ \ \forall x \in \Omega\,,$$

so that possible degeneracies or singularities of $\rho_\nu(x), \rho_\mu(x)$ on $\partial\Omega$ are allowed.

As it is well-known, non-uniqueness issues can occur for the problems (1.4), (1.5). Referring to [**12**] again for details, we point out that in the Dirichlet case uniqueness is restored, for example, by requiring that

$$u^m \in L^{\frac{m+1}{m}}((0,T); V_0^{\frac{m+1}{m}}(\Omega;\nu,\mu))\,, \ \ \nabla(u^m) \in L^2((0,T);[L^2(\Omega;\mu)]^N) \quad \forall T > 0\,, \tag{3.1}$$

where $V_0^p(\Omega;\nu,\mu)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\| f\|_{p,2;\nu,\mu} := \| f\|_{p;\nu} + \|\nabla f\|_{2;\mu}\,, \tag{3.2}$$

where $\|\cdot\|_{p;\nu}$ and $\|\cdot\|_{2;\mu}$ are the norms of the weighted Lebesgue spaces $L^p(\Omega;\nu)$ and $L^2(\Omega;\mu)$, respectively. In the Neumann case, uniqueness is restored by imposing that

$$u \in L^{m+1}((0,T); L^{m+1}(\Omega;\nu)) \tag{3.3}$$

(however, unlike the Dirichlet case, stronger requirements are asked directly in the definition of weak solution). Existence results under such conditions are also proved in [**12**]. Both in the Dirichlet and in the Neumann case we shall refer to the solution satisfying the above conditions as the (unique) *weak energy solution* of the problem considered.

As concerns regularizing and long-time estimates, the methods of proof of the results of Section 2 can be readily extended to the weighted case provided appropriate Gagliardo-Nirenberg (and, for long times, Poincaré) inequalities hold true in the weighted Sobolev spaces $W_0^{1,2}(\Omega;\nu,\mu) := V_0^2(\Omega;\nu,\mu)$ or $W^{1,2}(\Omega;\nu,\mu)$, the latter being the space of functions f for which $\| f\|_{2,2;\nu,\mu} < \infty$. Notice that the weighted analogues of formulas (2.1) and (2.2) hold true in the present context as well. They are used in the proofs given in [**11**], which we generalize below.

The following is an extension of the results valid, in the unweighted context, for $N = 1$.

Theorem 3.1 [One-dimensional-like cases]. *Assume that Ω is a domain of $\mathbb{R}^N$. Suppose that, for some $s > 0$ and $q \le -2$, the functional inequality*

$$\|f\|_\infty \le C\,\|\nabla f\|_{2;\mu}^{\frac{-q}{s-q}}\|f\|_{s;\nu}^{\frac{s}{s-q}} \quad \forall f \in W_0^{1,2}(\Omega;\nu,\mu) \cap L^s(\Omega;\nu) \tag{3.4}$$

is valid. Let u be the solution to the Dirichlet problem (1.4) *satisfying condition* (3.1) *and starting from an initial datum $u_0 \in L^1(\Omega;\nu) \cap L^{p_0}(\Omega;\nu)$, with $p_0 \ge 1$. Then the bound*

$$\|u(t)\|_\infty \le C\,\frac{\|u_0\|_{p_0;\nu}^{\frac{(q-2)p_0}{(q-2)p_0+q(m-1)}}}{t^{\frac{q}{(q-2)p_0+q(m-1)}}} \quad \forall t > 0 \tag{3.5}$$

holds true.

Suppose instead that, for some $s > 0$ and $q \le -2$, the functional inequality

$$\|f\|_\infty \le C\Big(\|\nabla f\|_{2;\mu} + \|f\|_{1;\nu}\Big)^{\frac{-q}{s-q}}\|f\|_{s;\nu}^{\frac{s}{s-q}} \quad \forall f \in W^{1,2}(\Omega;\nu,\mu) \tag{3.6}$$

is valid and that $\nu(\Omega) < \infty$. Let u be the solution to the Neumann problem (1.5) *satisfying condition* (3.3) *and starting from an initial datum $u_0 \in L^{p_0}(\Omega;\nu)$, with $p_0 \ge 1$. Then the bound*

$$\|u(t)\|_\infty \le C\left(\frac{\|u_0\|_{p_0;\nu}^{\frac{(q-2)p_0}{(q-2)p_0+q(m-1)}}}{t^{\frac{q}{(q-2)p_0+q(m-1)}}} + \|u_0\|_{p_0;\nu}\right) \quad \forall t > 0 \tag{3.7}$$

holds true.

Conversely, assume that $\nu(\Omega) < \infty$. Suppose that the bound (3.5) *holds, with $p_0 = m$, for any solution u to the Dirichlet problem* (1.4) *starting from $u_0 \in L^m(\Omega;\nu)$. Then the functional inequality* (3.4) *is valid. If instead the bound* (3.7) *holds, with $p_0 = m$, for any solution u to the Neumann problem* (1.5) *starting from $u_0 \in L^m(\Omega;\nu)$, then the functional inequality* (3.6) *is valid.*

Proof. Estimates (3.5) and (3.7) can be proved using exactly the same arguments given in the proof of Theorem 2.1. As for the converse one can proceed, both in the Dirichlet and the Neumann case, along the lines of the proof of [**11**, Th. 5.3] (one puts formally $\sigma = q/2$ there). Notice that the finiteness of the measure is used there to ensure the validity of the crucial inequality (5.7) of that paper (which is proved in [**12**, Ths. 4.3, 5.6]). □

Remark 3.2. The results of Theorem 2.1 here correspond to the case $N=1$, $\rho_\nu=\rho_\mu=1$ and $q=-2$.

Similarly, one can generalize the results valid, in the unweighted context, for $N=2$.

Theorem 3.3 [Two-dimensional-like cases]. *Assume that Ω is a domain of $\mathbb{R}^N$. Suppose that, for all $0<s<r$, the functional inequality*

$$\|f\|_{r;\nu} \le C_r \|\nabla f\|_{2;\mu}^{1-\frac{s}{r}} \|f\|_{s;\nu}^{\frac{s}{r}} \quad \forall f \in W_0^{1,2}(\Omega;\nu,\mu) \cap L^s(\Omega;\nu) \tag{3.8}$$

is valid for a positive constant C_r depending continuously on r. Let u be the solution to the Dirichlet problem (1.4) *satisfying condition* (3.1) *and starting from an initial datum $u_0 \in L^1(\Omega;\nu) \cap L^{p_0}(\Omega;\nu)$, with $p_0 \ge 1$. Then the bound* (2.17) *holds true upon replacing $\|u_0\|_{p_0}$ with $\|u_0\|_{p_0;\nu}$.*

Suppose instead that, for all $0<s<r$, the functional inequality

$$\|f\|_{r;\nu} \le C_r \Big(\|\nabla f\|_{2;\mu} + \|f\|_{1;\nu} \Big)^{1-\frac{s}{r}} \|f\|_{s;\nu}^{\frac{s}{r}} \quad \forall f \in W^{1,2}(\Omega;\nu,\mu) \tag{3.9}$$

is valid for a positive constant C_r depending continuously on r, and assume that $\nu(\Omega)<\infty$. Let u be the solution to the Neumann problem (1.5) *satisfying condition* (3.3) *and starting from an initial datum $u_0 \in L^{p_0}(\Omega;\nu)$, with $p_0 \ge 1$. Then the bound* (2.18) *holds true upon replacing $\|u_0\|_{p_0}$ with $\|u_0\|_{p_0;\nu}$.*

Conversely, assume that $\nu(\Omega)<\infty$. Suppose that the bound (2.17) *holds, with $p_0=m$ and $\|u_0\|_m$ replaced by $\|u_0\|_{m;\nu}$, for any solution u to the Dirichlet problem* (1.4) *starting from $u_0 \in L^m(\Omega;\nu)$. Then the functional inequality* (3.8) *is valid. If instead the bound* (2.18) *holds, with $p_0=m$ and $\|u_0\|_m$ replaced by $\|u_0\|_{m;\nu}$, for any solution u to the Neumann problem* (1.5) *starting from $u_0 \in L^m(\Omega;\nu)$, then the functional inequality* (3.9) *is valid.*

Proof. The regularizing estimates can be proved using the same arguments given in the proof of Theorem 2.2. As for the converse one can proceed, both in the Dirichlet and the Neumann case, again along the lines of the proof of [**11**, Th. 5.3] (one puts formally $\sigma=\infty$ there). □

3.1 - *Long-time estimates*

Similarly to what we did in Section 2.1, here we aim at improving the results of Theorems 3.1 and 3.3 for $t\to\infty$. As for the Neumann problem (1.5) we shall see below that, under the *additional* hypothesis that the weights ρ_ν, ρ_μ satisfy the

Poincaré inequality

$$\|f-\overline{f}\|_{2;\nu} \leq C_P \|\nabla f\|_{2;\mu} \quad \forall f \in W^{1,2}(\Omega;\nu,\mu)\,,\; \overline{f} = \frac{\int_\Omega f(x)\,\rho_\nu(x)\mathrm{d}x}{\nu(\Omega)}\,, \tag{3.10}$$

the long-time estimates of Section 2.1 work in the weighted context too.

Theorem 3.4. *Let u be the solution to the Neumann problem* (1.5) *satisfying condition* (3.3) *and starting from an initial datum $u_0 \in L^1(\Omega;\nu)$, with $\overline{u}_0 \neq 0$. Suppose that either of the Gagliardo-Nirenberg inequalities* (3.6) *or* (3.9) *is valid. Moreover, assume that the weights ρ_ν, ρ_μ satisfy the weighted Poincaré inequality* (3.10)*, with best constant C_P. Then there exists a positive constant $C>0$ such that* (2.36) *holds true.*

Proof. It suffices to notice that the validity of either of the Gagliardo-Nirenberg inequalities (3.6) or (3.9) and of the Poincaré inequality (3.10), which together imply the validity of the Sobolev inequality $\|f-\overline{f}\|_\infty \leq C\|\nabla f\|_{2;\mu}$, is exactly what one needs in order to reproduce the proof of [**11**, Th. 4.3]. □

Theorem 3.5. *Let u be the solution to the Neumann problem* (1.5) *satisfying condition* (3.3) *and starting from an initial datum $u_0 \in L^{p_0}(\Omega;\nu)$, with $p_0 \geq 1$ and $\overline{u}_0 = 0$. Assume that the weights ρ_ν, ρ_μ satisfy the weighted Poincaré inequality* (3.10)*. If the family of Gagliardo-Nirenberg inequalities* (3.6) *is valid for some $q \leq -2$, then for some positive constants C_1, C_2 the following estimate holds:*

$$\|u(t)\|_\infty \leq C_1\, t^{-\frac{q}{(q-2)p_0+q(m-1)}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0;\nu}^{1-m}\right)^{\frac{(q-2)p_0}{(m-1)[(q-2)p_0+q(m-1)]}}} \quad \forall t>0\,. \tag{3.11}$$

If instead the family of Gagliardo-Nirenberg inequalities (3.9) *is valid, then for another positive constants C_1, C_2 the following estimate holds:*

$$\|u(t)\|_\infty \leq C_1\, t^{-\frac{1}{p_0+m-1}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0;\nu}^{1-m}\right)^{\frac{p_0}{(m-1)(p_0+m-1)}}} \quad \forall t>0\,. \tag{3.12}$$

In particular, in both cases the absolute bound

$$\|u(t)\|_\infty \leq C_3\, t^{-\frac{1}{m-1}} \quad \forall t>0 \tag{3.13}$$

holds true for some $C_3 > 0$ (independent of u_0).

Proof. Just notice that the validity of the regularizing estimates of Theorems 3.1, 3.3 and of the Poincaré inequality (3.10) allow to reproduce the proofs of [**4**, Cor. 1.3] and [**11**, Th. 4.1]. □

Regarding the Dirichlet problem (1.4), the analogue of Theorem 2.5 can be established upon requiring $\nu(\Omega) < \infty$.

Theorem 3.6. *Let u be the solution to the Dirichlet problem* (1.4) *satisfying condition* (3.1) *and starting from an initial datum $u_0 \in L^{p_0}(\Omega;\nu)$, with $p_0 \geq 1$. Assume that $\nu(\Omega) < \infty$. If the family of Gagliardo-Nirenberg inequalities* (3.4) *is valid for some $q \leq -2$, then for some positive constants C_1, C_2 there holds*

$$\|u(t)\|_\infty \leq C_1\, t^{-\frac{q}{(q-2)p_0+q(m-1)}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0;\nu}^{1-m}\right)^{\frac{(q-2)p_0}{(m-1)[(q-2)p_0+q(m-1)]}}} \quad \forall t > 0\,. \tag{3.14}$$

If instead the family of Gagliardo-Nirenberg inequalities (3.8) *is valid, then for other positive constants C_1, C_2 there holds*

$$\|u(t)\|_\infty \leq C_1\, t^{-\frac{1}{p_0+m-1}} \frac{1}{\left(C_2\, t + \|u_0\|_{p_0;\nu}^{1-m}\right)^{\frac{p_0}{(m-1)(p_0+m-1)}}} \quad \forall t > 0\,. \tag{3.15}$$

In particular, in both cases the absolute bound

$$\|u(t)\|_\infty \leq C_3\, t^{-\frac{1}{m-1}} \quad \forall t > 0 \tag{3.16}$$

holds true for some $C_3 > 0$ (independent of u_0).

Proof. Upon noticing that the Gagliardo-Nirenberg inequalities plus the finiteness of the measure ensure the validity of the L^2-gap inequality $\|f\|_{2;\nu} \leq C_G \|\nabla f\|_{2;\mu}$ $\forall f \in W_0^{1,2}(\Omega;\nu,\mu)$, the same arguments of the proof of Theorem 2.5 apply. □

4 - Examples of weighted Gagliardo-Nirenberg inequalities

In what follows, we show explicit examples of weights for which one-dimensional-like (Section 4.1) or two-dimensional-like (Section 4.2) Gagliardo-Nirenberg inequalities hold. We shall make use of the well-known Caffarelli-Kohn-Nirenberg inequalities [**6**], and for simplicity we restrict ourselves to examples on real intervals, which can anyway be constructed so that also what we called two-dimensional-like weighted Gagliardo-Nirenberg inequalities hold.

4.1 - *One-dimensional-like inequalities*

Below we list choices of explicit weights (on real intervals) for which the 1-dimensional-like inequalities (3.4) and (3.6) are valid, so that the conclusions of Theorem 3.1 hold true.

The Dirichlet case

Let $q \leq -2$, $\beta \in (-1,1)$ and s,r be such that $0 < s < r$. The inequalities

$$\left(\int_{\mathbb{R}} \frac{|f(x)|^r}{|x|^{\frac{q}{2}(1-\beta)+1}}\,\mathrm{d}x\right)^{\frac{1}{r}} \leq C\left(\int_{\mathbb{R}} |f'(x)|^2|x|^\beta\,\mathrm{d}x\right)^{-\frac{q(r-s)}{2r(s-q)}}\left(\int_{\mathbb{R}} \frac{|f(x)|^s}{|x|^{\frac{q}{2}(1-\beta)+1}}\,\mathrm{d}x\right)^{\frac{r-q}{r(s-q)}}, \tag{4.1}$$

valid e.g. for any $f \in C_c^\infty(\mathbb{R})$, are special cases of the aforementioned Caffarelli-Kohn-Nirenberg inequalities, at least for $s \geq 1$. If instead $s \in (0,1)$ their validity follows e.g. using once again the results of [3]. Let us set $\rho_\nu(x) = |x|^{-\frac{q}{2}(1-\beta)-1}$ and $\rho_\mu(x) = |x|^\beta$. Since identical inequalities hold trivially also for any $f \in C_c^\infty((0,\infty))$, (4.1) implies that

$$\|f\|_{r;\nu} \leq C\,\|\nabla f\|_{2;\mu}^{\vartheta}\|f\|_{s;\nu}^{1-\vartheta} \quad \forall f \in W_0^{1,2}((0,\infty);\nu,\mu) \cap L^s((0,\infty);\nu)\,, \tag{4.2}$$

having set $\vartheta = -\dfrac{q(r-s)}{r(s-q)}$. By the results of [3] we infer the validity of the functional inequality (3.4). Hence (3.5) holds true for the solution u to the Dirichlet problem (1.4) associated to the weights $\rho_\nu(x), \rho_\mu(x)$ given above. Moreover, since $\rho_\nu(x) = |x|^{-\frac{q}{2}(1-\beta)-1}$ is locally integrable in $[0,\infty)$, also estimate (3.14) holds if we restrict, for instance, to the interval $(0,1)$.

The Neumann case

Consider again (4.1) but restricted to the space X of functions belonging to $C^\infty([0,1+\varepsilon])$ and vanishing at $1+\varepsilon$, $\varepsilon > 0$ being a fixed parameter. As above let $q \leq -2$, $\beta \in (-1,1)$ and s,r be such that $0 < s < r$. Set $\rho_\nu(x) = |x|^{-\frac{q}{2}(1-\beta)-1}$ and $\rho_\mu(x) = |x|^\beta$. It is easy to prove, for example considering appropriate cut-off functions, that X is dense in the space Y of functions in $W^{1,2}((0,1+\varepsilon);\nu,\mu)$ vanishing at $1+\varepsilon$. This ensures that (4.2) is valid for any $f \in Y$. Let P be the extension operator $P : W^{1,2}((0,1);\nu,\mu) \to Y$. Following considerations analogous to the ones we made along the proof of Theorem 2.1 (after formula (2.15)), we conclude that there holds (ϑ is as above)

$$\|f\|_{r;\nu} \leq C\,\|f\|_{2,2;\nu,\mu}^{\vartheta}\|f\|_{s;\nu}^{1-\vartheta} \quad \forall f \in W^{1,2}((0,1);\nu,\mu)\,. \tag{4.3}$$

It is now necessary to show that

$$\|f\|_{r;\nu} \leq C\Big(\|\nabla f\|_{2;\mu} + \|f\|_{1;\nu}\Big)^{\vartheta}\|f\|_{s;\nu}^{1-\vartheta} \quad \forall f \in W^{1,2}((0,1);\nu,\mu)\,.$$

To this end, notice that the Poincaré inequality (3.10) here does hold. In fact, this can be proved through a standard argument by contradiction, using the compactness of the embedding of $W^{1,2}((0,1);\nu,\mu)$ into $L^2((0,1);\nu)$ (for these special choices of weights), which is shown for instance in [**17**, Sect. 7.10, ii)]. In this regard, see also the examples listed in [**11**, Sect. 5.1], where we used the results of [**5**].

Hence the functional inequality (3.6) holds as a consequence of the results of [**3**]. Thanks to the validity of the Poincaré inequality as well, we have that for the choices of weights above the solution u to the Neumann problem (1.5) satisfies estimates (3.7), and (2.36) or (3.11).

4.2 - *Two-dimensional-like inequalities*

Below we list choices of explicit weights (on real intervals) for which the 2-dimensional-like inequalities (3.8) and (3.9) are valid, so that the conclusions of Theorem 3.3 hold true.

The Dirichlet case

Let $\alpha > -1$ and s, r be such that $0 < s < r$. The validity of the inequalities

$$\left(\int_{\mathbb{R}} |f(x)|^r\, |x|^\alpha\, \mathrm{d}x\right)^{\frac{1}{r}} \le C_r \left(\int_{\mathbb{R}} |f'(x)|^2 |x|\, \mathrm{d}x\right)^{\frac{1}{2}\left(1-\frac{s}{r}\right)} \left(\int_{\mathbb{R}} |f(x)|^s\, |x|^\alpha\, \mathrm{d}x\right)^{\frac{1}{r}} \tag{4.4}$$

for any $f \in C_c^\infty(\mathbb{R})$ is again a consequence of the Caffarelli-Kohn-Nirenberg inequalities and the results of [**3**]. Notice that this corresponds to choosing $q = \infty$ in (4.2). Let us set $\rho_\nu(x) = |x|^\alpha$ and $\rho_\mu(x) = |x|$. Reasoning as in Section 4.1, we deduce that (4.4) yields the validity of

$$\|f\|_{r;\nu} \le C_r\, \|\nabla f\|_{2;\mu}^{1-\frac{s}{r}} \|f\|_{s;\nu}^{\frac{s}{r}} \quad \forall f \in W_0^{1,2}((0,\infty);\nu,\mu) \cap L^s((0,\infty);\nu)\,, \tag{4.5}$$

that is (3.8). Hence (2.17) holds true (with $\|u_0\|_{p_0}$ replaced by $\|u_0\|_{p_0;\nu}$) for the solution u to the Dirichlet problem (1.4) associated to the weights $\rho_\nu(x), \rho_\mu(x)$ given above. Moreover, since $\rho_\nu(x) = |x|^\alpha$, for $\alpha > -1$, is locally integrable in $[0,\infty)$, also estimate (3.15) holds if we restrict, for instance, to the interval $(0,1)$.

The Neumann case

Let $\alpha > -1$ and s, r be such that $0 < s < r$. Set, as above, $\rho_\nu(x) = |x|^\alpha$ and $\rho_\mu(x) = |x|$. Proceeding exactly as in Section 4.1, one can show that there holds

$$\|f\|_{r;\nu} \le C_r\, \|f\|_{2,2;\nu,\mu}^{1-\frac{s}{r}} \|f\|_{s;\nu}^{\frac{s}{r}} \quad \forall f \in W^{1,2}((0,1);\nu,\mu)\,.$$

To prove that also

$$\|f\|_{r;\nu} \leq C_r \Big(\|\nabla f\|_{2;\mu} + \|f\|_{1,\nu} \Big)^{1-\frac{s}{r}} \|f\|_{s;\nu}^{\frac{s}{r}} \qquad \forall f \in W^{1,2}((0,1);\nu,\mu)$$

is valid, we use again the fact that the Poincaré inequality (3.10) holds for the present choices of weights. However this cannot be shown through compactness, since for such weights the compact embedding of $W^{1,2}((0,1);\nu,\mu)$ into $L^2((0,1);\nu)$ fails (see again [**17**, Sect. 7.10]). Nevertheless we can use [**5**, Th. 1.4], which gives the required Poincaré inequality (actually more – see also the examples of [**11**, Sect. 5.1]).

Hence, for the choices of weights above, the solution u to the Neumann problem (1.5) satisfies estimates (2.18) (with $\|u_0\|_{p_0}$ replaced by $\|u_0\|_{p_0;\nu}$), and (2.36) or (3.12).

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