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A backward in time Harnack inequality for non-negative solutions to fully non-linear parabolic equations

Abstract. We consider fully non-linear parabolic equations of the form

$$Hu = F(D^2u(x,t), Du(x,t), x, t) - \partial_t u = 0$$

in bounded space-time domains $D \subset \mathbb{R}^{n+1}$, assuming only $F(0,0,x,t) = 0$ and a uniform parbolicity condition on F. For domains of the form $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz and $T > 0$, we establish a scale-invariant backward in time Harnack inequality for non-negative solutions vanishing on the lateral boundary. Our argument rests on the comparison principle, the Harnack inequality and local Hölder continuity estimates.

Keywords. Fully non-linear parabolic equations, Lipschitz domain, Harnack inequality, backward Harnack inequality.

Mathematics Subject Classification (2010): 35K55.

1 - Introduction

The study of the boundary behaviour of non-negative solutions to second order linear uniformly parabolic equations, in divergence and non-divergence form, in time-independent Lipschitz and NTA-cylinders as well as in more general time-dependent Lip(1,1/2)-domains and parabolic NTA-domains, has a long a rich history, see [**ACS**], [**ACS1**], [**FGS**], [**FS**], [**FSY**], [**G**], [**HLN**], [**N**], [**S**], [**SY**]. Key results in this theory, for instance in the context of domains $\Omega_T = \Omega \times (0,T)$ where

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$\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $T > 0$, include the backward in time Harnack inequality, and the Hölder continuity up to the boundary of quotients of non-negative solutions, vanishing on the lateral boundary. Important applications include regularity results for symmetry and free boundary type problems, e.g., see [**ACS**], [**ACS1**], [**HLN**]. However, the corresponding results concerning the boundary behavior for non-negative solution for various classes of non-linear generalizations of the heat equation, like for instance equations of p-parabolic type, porous medium equations and general fully non-linear parabolic equations, remain fundamentally undeveloped. Still there are some recent results for equations of p-parabolic type, see [**AGS**], [**KMN**], [**NPS**], and a recent result for general fully non-linear parabolic equations, see [**BG**]. The purpose of this remark is to establish a cornerstone in the theory, the backward in time Harnack inequality, for general uniformly parabolic fully non-linear parabolic equations in time-independent cylinders of the form $\Omega_T = \Omega \times (0, T)$ where Ω is a Lipschitz domain. So far, see [**BG**], this result has only been establish in cylinders $\Omega_T = \Omega \times (0, T)$ where Ω is a $C^{1,1}$-regular domain. In particular, we consider fully non-linear parabolic equations of the form

$$Hu = F(D^2u(x,t), Du(x,t), x, t) - \partial_t u = 0 \tag{1.1}$$

in $\mathbb{R}^{n+1}$ assuming that F satisfies $F(0, 0, x, t) = 0$. Let $\mathcal{M}_n$ denote the set of all real $n \times n$-matrices and given $M \in \mathcal{M}_n$ we let $\|M\| = \sup\limits_{|x|=1} |Mx|$. We say that the equation in (1.1) is uniformly parabolic in $\mathbb{R}^{n+1}$ if there exist λ, Λ, $0 < \lambda \leq \Lambda < \infty$ and $\eta > 0$, such that

$$n\lambda\|N\| - \eta|p - q| \leq F(M + N, p, x, t) - F(M, q, x, t) \leq \Lambda\|N\| + \eta|p - q| \tag{1.2}$$

whenever $M, N \in \mathcal{M}_n$, $p, q \in \mathbb{R}^n$, $(x, t) \in \mathbb{R}^{n+1}$. The fundamental theory for fully non-linear uniformly parabolic equations was developed in the papers of Lihe Wang, see [**W1**], [**W2**], but we also refer to [**CKS**]. Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set. We let $C(D)$ and $C^2(D)$ denote the set of functions defined in D which are continuous and have continuous partial derivatives up to order two, with respect to x and t, respectively in D. By a parabolic neighborhood of a point $(x_0, t_0) \in \mathbb{R}^{n+1}$ we mean the intersection of an Euclidean neighborhood U of (x_0, t_0) with $\mathbb{R}^n \times (-\infty, t_0]$. In the following a local extremum is to be understood with respect to parabolic neighborhoods.

Definition 1.1. Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set. A function $u \in C(D)$ is said to be a *viscosity supersolution* to (1.1) in D if, for given $\phi \in C^2(D)$, we have

$$F(D^2\phi(x_0, t_0), D\phi(x_0, t_0), x_0, t_0) - \partial_t\phi(x_0, t_0) \leq 0$$

whenever $u - \phi$ has a local minimum at (x_0, t_0). A function $u \in C(D)$ is said to be a *viscosity subsolution* to (1.1) in D if, for given $\phi \in C^2(D)$, we have

$$F(D^2\phi(x_0,t_0), D\phi(x_0,t_0), x_0, t_0) - \partial_t\phi(x_0,t_0) \geq 0$$

whenever $u - \phi$ has a local maximum at (x_0, t_0). If u is both a viscosity supersolution and a viscosity subsolution to (1.1) in D, then u is said to be a *viscosity solution* to (1.1) in D.

Points in Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ are denoted by $x = (x_1, \ldots, x_n, t)$. Given a set $E \subset \mathbb{R}^n$, let $\bar{E}, \partial E,$ diam E be the closure, boundary, and diameter of E. Let $\cdot$ denote the standard inner product on $\mathbb{R}^n$, $|x| = (x \cdot x)^{1/2}$, the Euclidean norm of x, and let dx be Lebesgue n-measure on $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. For $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$ we let $C_r(x,t) = B(x,r) \times (t - r^2, t + r^2)$. Furthermore, we let $d_p(x,t,y,s) = (|x-y|^2 + |t-s|)^{1/2}$ denote the parabolic distance between $(x,t), (y,s) \in \mathbb{R}^{n+1}$. Given $E, F \subset \mathbb{R}^n$, let $d(E,F)$ be the Euclidean distance from E to F. In case $E = \{y\}$, we write $d(y,F)$. Recall that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain if there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned} \Omega \cap B(x_i,r_i) &= \{y = (y', y_n) \in \mathbb{R}^n : y_n > \phi_i(y')\} \cap B(x_i, r_i), \\ \partial\Omega \cap B(x_i,r_i) &= \{y = (y', y_n) \in \mathbb{R}^n : y_n = \phi_i(y')\} \cap B(x_i, r_i), \end{aligned} \tag{1.3}$$

in an appropriate coordinate system and for a Lipschitz function $\phi_i : \mathbb{R}^{n-1} \to \mathbb{R}$. The Lipschitz constants of Ω are defined to be $M = \max_i |||\nabla\phi_i|||_\infty$, $r_0 := \min_i r_i$ and we will often refer to Ω as a Lipschitz domain with parameters M and r_0. If Ω is a Lipschitz domain with parameters M and r_0, then there exists, for any $x_0 \in \partial\Omega$, $0 < r < r_0$, a point $A_r(x_0) \in \Omega$, such that

$$M^{-1}r < d(x_0, A_r(x_0)) < r, \text{ and } d(A_r(x_0), \partial\Omega) \geq M^{-1}r.$$

We let $\Omega_T = \Omega \times (0,T)$, $T > 0$, and in following we let $A_r(x_0,t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T := \partial\Omega \times (0,T)$ and $0 < r < r_0$. We here prove the following theorem.

Theorem 1.2. *Let $F : \mathcal{M}_n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfy* (1.2) *for some (λ, Λ, η). Let $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T > 0$. Let u be a non-negative viscosity solution to* (1.1) *in Ω_T vanishing continuously on S_T. Let $0 < \delta \ll \sqrt{T}$ be a fixed constant, let $(x_0, t_0) \in S_T$, $\delta^2 \leq t_0 \leq T - \delta^2$, and assume that $r < \min\{r_0/2, \sqrt{(T - t_0 - \delta^2)/4}, \sqrt{(t_0 - \delta^2)/4}\}$.*

Then, there exists $c = c(\lambda, \Lambda, \eta, M, \operatorname{diam}(\Omega), T, \delta)$*,* $1 \leq c < \infty$*, such that*

$$u(x,t) \leq cu(A_r(x_0,t_0))$$

whenever $(x,t) \in \Omega_T \cap C_{r/4}(x_0,t_0)$*.*

Note that Theorem 1.2 is proved in [**BG**] under the additional assumption that Ω is a $C^{1,1}$-domain. The latter implies that Ω satisfies a uniform inner and outer ball condition based on which one can prove, using a barrier argument, see [**BG**], that any non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ introduced in Definition 2.1 below, decays linearly at the lateral boundary. This results in a comparison principle for non-negative functions in the class $\mathcal{S}(\lambda, \Lambda, \eta)$, vanishing on the lateral boundary, based on which the authors in [**BG**] are able to conclude Theorem 1.2 in the case Ω is a $C^{1,1}$-domain. Since Theorem 1.2 is stated under much weaker geometric assumptions compared to [**BG**] a different route of proof is needed and we claim that Theorem 1.2 can be proved using only fundamental principles like comparison principles, the Harnack inequality and Hölder decay estimates at the lateral boundary. In particular, our proof relies on the following simple decay estimate at the bottom of cylinders. Let $(x_0,t_0) \in \mathbb{R}^{n+1}$, $r > 0$, and let $C_r^+(x_0,t_0) = B(x_0,r) \times (t_0, t_0 + r^2)$. Let F be as in the statement of Theorem 1.2. Then there exist constants $c = c(\lambda, \Lambda, \eta)$, $1 \leq c < \infty$, and $\alpha = \alpha(\lambda, \Lambda, \eta)$, $0 < \alpha < 1$, such that the following is true. Assume that u is a viscosity solution to (1.1) in $C_{2r}^+(x_0,t_0)$, that u is continuous on the closure of $C_{2r}^+(x_0,t_0)$ and that $u = 0$ on $\partial_p C_{2r}^+(x_0,t_0) \cap \{(x,t) : t = t_0\}$. Then,

$$|u(x,t)| \leq c\left(\frac{|x| + |t|^{1/2}}{r}\right)^{\alpha} \sup_{C_{2r}^+(x_0,t_0)} |u| \tag{1.4}$$

whenever $(x,t) \in C_r^+(x_0,t_0)$. Note that this estimate follows from elementary barrier type arguments and we refer to section 2.4 in [**W2**] for a proof of (1.4). Furthermore, we claim that Theorem 1.2 is simply a consequence of the comparison principle, the solvability of the Dirichlet problem in cylinders of the form $C_r(x,t)$, the Harnack inequality, Hölder decay estimates at the lateral boundary, and the estimate in (1.4). In fact, we also claim that Theorem 1.2 remains true also under the weaker assumption that $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 in the sense of [**JK**], and that Theorem 1.2 probably also extends to the setting of the time-dependent Lip(1,1/2)-domains considered in [**N**] and in the more general setting of parabolic NTA-domains considered in [**HLN**]. However, for the sake of simplicity and brevity we here stay with the formulation in Theorem 1.2.

2 - Preliminaries

In the following we assume that $F : \mathcal{M}_n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies (1.2) for some (λ, Λ, η). We will use the notation $C_r^+(x,t) = B(x,r) \times (t, t+r^2)$, $C_r^-(x,t) = B(x,r) \times (t-r^2, t)$, $C_{r_1,r_2}^-(x,t) = B(x,r_1) \times (t-r_2^2, t)$, for $(x,t) \in \mathbb{R}^{n+1}$ and $r, r_1, r_2 > 0$. We will work in cylindrical domains $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, i.e., a bounded, connected and open set, and $T > 0$. The parabolic boundary of the cylinder Ω_T, $\partial_p \Omega_T$, is defined as

$$\partial_p \Omega_T = S_T \cup (\bar{\Omega} \times \{0\}), \quad S_T = \partial\Omega \times [0,T].$$

We let, for $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$,

$$\Delta(x,t,r) = S_T \cap C_r(x,t). \tag{2.1}$$

2.1 - *Pucci extremal operators*

Following standard notation we let, given $\lambda, \Lambda, 0 < \lambda \le \Lambda < \infty$, $\mathcal{P}^+_{\lambda,\Lambda}$, $\mathcal{P}^-_{\lambda,\Lambda}$, denote the maximal and minimal Pucci extremal operators corresponding to λ, Λ, i.e., for every $M \in \mathcal{M}_n$ we have

$$\mathcal{P}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i, \ \mathcal{P}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i, \tag{2.2}$$

where $e_i = e_i(M)$, $i = 1, ..., n$, denote the eigenvalues of M. In addition we let

$$\begin{aligned} \mathcal{H}^+_{\lambda,\Lambda,\eta} u &= \mathcal{P}^+_{\lambda,\Lambda}(D^2 u) + \eta |Du| - \partial_t u, \\ \mathcal{H}^-_{\lambda,\Lambda,\eta} u &= \mathcal{P}^-_{\lambda,\Lambda}(D^2 u) - \eta |Du| - \partial_t u. \end{aligned} \tag{2.3}$$

Note that $\mathcal{H}^+_{\lambda,\Lambda,\eta}$, $\mathcal{H}^-_{\lambda,\Lambda,\eta}$, are uniformly parabolic in the sense stated above. Furthermore, we also note that if u be a viscosity solution to (1.1), then u satisfies the differential inequality

$$\mathcal{H}^+_{\lambda,\Lambda,\eta} u \ge 0 \ge \mathcal{H}^-_{\lambda,\Lambda,\eta} u \tag{2.4}$$

in the viscosity sense. Following **[W1]** we introduce the following standard notation.

Definition 2.1. Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set. We let $\mathcal{S}(\lambda, \Lambda, \eta) = \mathcal{S}(\lambda, \Lambda, \eta, D)$ denote the set of all functions u which are continuous in D and which simultaneously are viscosity subsolutions to the equation $\mathcal{H}^+_{\lambda,\Lambda,\eta} u = 0$ and viscosity supersolutions to the equation $\mathcal{H}^-_{\lambda,\Lambda,\eta} u = 0$.

Remark 2.2. Note that the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in $\mathbb{R}^{n+1}$ is invariant under orthogonal transformations in the space variables and under translation in the t-variable. This is in contrast to viscosity solutions to (1.1) which are not necessarily invariant under these transformations.

Remark 2.3. Note that if u is a viscosity solutions to (1.1), then $u \in \mathcal{S}(\lambda, \Lambda, \eta)$ in $\mathbb{R}^{n+1}$.

Remark 2.4. Note that the equations $\mathcal{H}^+_{\lambda,\Lambda} u = 0$ and $\mathcal{H}^-_{\lambda,\Lambda} u = 0$ are special cases of the general structure in (1.1).

2.2 - *Fundamental principles*

Lemma 2.5. *Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set. Let u^- be a viscosity subsolution to* (1.1) *in D and let u^+ be a viscosity supersolution to* (1.1) *in D. If $u^- \leq u^+$ on ∂D then $u^- \leq u^+$ in $\bar{D}$.*

Proof. See [**GGIS**] or Lemma 2.5 and Corollary 2.6 in [**BG**]. □

Lemma 2.6. *Consider r, $0 < r < \infty$, and assume that u be a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in $C^-_r(x_0, t_0)$. Then, given $0 < h_1 < h_2 < 1$, $0 < \beta < 1$, there exists a constant $c = c(\lambda, \Lambda, \eta, h_1, h_2, \beta)$, $1 \leq c < \infty$, such that*

$$u(x,t) \leq c u(x_0, t_0) \quad \textit{whenever } (x,t) \in C^-_{\beta r, h_2 r}(x_0, t_0) \setminus C^-_{\beta r, h_1 r}(x_0, t_0).$$

Proof. See Theorem 4.18 in [**W1**]. □

Recall, assuming that $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, with parameters M, r_0, that in the introduction we introduced the point of reference $A_r(x_0, t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$. In the following we will also use the notation

$$A^+_r(x_0, t_0) = (A_r(x_0), t_0 + 2r^2), \quad A^-_r(x_0, t_0) = (A_r(x_0), t_0 - 2r^2), \tag{2.5}$$

whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$. The following two lemmas follows immediately from Lemma 2.6 and for proofs we refer to [**N**].

Lemma 2.7. *Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T > 0$. Let $(x_0, t_0) \in S_T$ and*

$$r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}.$$

Let u be a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in Ω_T. Then, there exist $c = c(\lambda, \Lambda, \eta, M)$, $1 \leq c < \infty$, and $\gamma = \gamma(\lambda, \Lambda, \eta, M) > 0$, such that for every $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$,

$$u(x, t) d_p(x, t, S_T)^\gamma \leq c r^\gamma u(A_r^+(x_0, t_0)). \tag{2.6}$$

Lemma 2.8. *Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T > 0$. Let $(x_0, t_0) \in S_T$ and*

$$r < \min\left\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\right\}.$$

Let u be a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in Ω_T. Then, there exist $c = c(\lambda, \Lambda, \eta, M)$, $1 \leq c < \infty$, and $\gamma = \gamma(\lambda, \Lambda, \eta, M) > 0$, such that

$$u(A_r^-(x_0, t_0)) \leq c \left(\frac{r}{d_p(x, t, S_T)}\right)^\gamma u(x, t),$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$.

3 - Gaussian type decay estimates in thin cylinders

Lemma 3.1. *Let $(x_0, t_0) \in \mathbb{R}^{n+1}$, $r > 0$. Then there exist constants $c = c(\lambda, \Lambda, \eta)$ and $\alpha = \alpha(\lambda, \Lambda, \eta)$ such that the following is true. Assume that u be a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in $C_{2r}^+(x_0, t_0)$, that u is continuous on the closure of $C_{2r}^+(x_0, t_0)$ and that $u = 0$ on $\partial_p C_{2r}^+(x_0, t_0) \cap \{(x, t) : t = t_0\}$. Then,*

$$|u(x, t) - u(y, s)| \leq c \left(\frac{d_p(x, t, y, s)}{r}\right)^\alpha \sup_{C_{2r}^+(x_0, t_0)} |u|$$

whenever $(x, t), (y, s) \in C_r^+(x_0, t_0)$.

Proof. First, making the transformation

$$\tilde{u}(x, t) := \frac{u\big(rx + x_0, r^2 t + t_0\big)}{\|u\|_\infty},$$

we see that $\tilde{u}$ is a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in $C_2^+(0, 0)$, that $\tilde{u}$ is continuous on the closure of $C_2^+(0, 0)$, $\tilde{u} = 0$ on $\partial_p C_2^+(0, 0) \cap \{(x, t) : t = 0\}$, and

$$\sup_{C_2^+(0,0)} |\tilde{u}| = 1. \tag{3.1}$$

Now, applying the interior Hölder estimates derived in [**W2**], and the Hölder estimate at the bottom of cylinders proved in Theorem 2.11 in **[W2]**, we see that there exist constants $c = c(\lambda, \Lambda, \eta)$ and $\alpha = \alpha(\lambda, \Lambda, \eta)$ such that

$$|\tilde{u}(x,t) - \tilde{u}(y,s)| \leq c\big(d_p(x,t,y,s)\big)^{\alpha} \text{ whenever } (x,t),(y,s) \in C_1^+(0,0).$$

Scaling back we get the conclusion stated in the lemma. □

Lemma 3.2 (Gaussian decay estimates). *There exists a $\hat{K} \gg 1$, $\hat{K} = \hat{K}(\lambda, \Lambda, \eta)$ such that the following is true whenever $(x_0,t_0) \in \mathbb{R}^{n+1}$, $r > 0$, $K \geq \hat{K}$. Let u be a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in $C^-_{Kr,2r}(x_0,t_0)$ vanishing continuously on $\partial_p C^-_{Kr,2r}(x_0,t_0) \cap \big(B(x_0,(K-2)r) \times \{t_0 - 4r^2\}\big)$. Then, there exists a constant $c = c(\lambda, \Lambda, \eta)$, $1 \leq c < \infty$, such that*

$$\sup_{C_r^-(x_0,t_0)} u \leq c \exp(-K/c) \sup_{C^-_{Kr,2r}(x_0,t_0)} u.$$

Proof. In the following we let $\hat{K} \gg 1$ be a constant to be chosen, we consider $K \geq \hat{K}$, and we note that we can without loss of generality assume that $(x_0,t_0) = (0,0)$ and $r = 1$. Let $\tilde{x}_0 \in \mathbb{R}^n$ and $R > 2$ be such that $B(\tilde{x}_0, 2R) \subset B(0, 2K)$. Furthermore, assume that v is a non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ in $C^+_{2R}(\tilde{x}_0, -4)$, that v is continuous on the closure of $C^+_{2R}(\tilde{x}_0, -4)$ and that $v = 0$ on $\partial_p C^+_{2R}(\tilde{x}_0, -4) \cap \{(x,t) : t = -4\}$. Then, using Lemma 3.1 we have that

$$|v(x,t) - v(y,s)| \leq c\left(\frac{d_p(x,t,y,s)}{R}\right)^{\alpha} \sup_{C^+_{2R}(\tilde{x}_0,-4)} |v| \tag{3.2}$$

whenever $(x,t),(y,s) \in C^+_R(\tilde{x}_0,-4)$. In particular, using (3.2) with $(y,s) \in B(\tilde{x}_0, 2) \times \{t = -4\}$, we have that

$$|v(x,t)| \leq cR^{-\alpha} \sup_{C^+_{2R}(\tilde{x}_0,-4)} |v| \tag{3.3}$$

whenever $(x,t) \in C^+_2(\tilde{x}_0,-4)$. Based on (3.3) we in the following let θ, $0 < \theta < 1$, be a degree of freedom to be fixed and we let R be such that $cR^{-\alpha} = \theta$. In particular, θ fixes R. Using this R we let suppose $\hat{K} = 2(R+1) + R$. Hence, $\hat{K} = \hat{K}(\lambda, \Lambda, \eta)$. Let now $x_1 \in \partial B(0, K - 2(R+1))$ and using the function u we introduce an auxiliary function v as follows. We let v be a solution to $\mathcal{H}^+_{\lambda,\Lambda,\eta} v = 0$ in $C^+_{2R}(x_1,-4)$, continuous on the closure of $C^+_{2R}(x_1,-4)$, $v = u$ on $\partial_p C^+_{2R,2}(x_1,-4)$ and $v(x,t) = u(x,0)$ whenever $(x,t) \in \partial_p\big(C^+_{2R}(x_1,-4) \setminus C^+_{2R,2}(x_1,-4)\big)$. Then, by the comparison principle, see Lemma 2.5, we have that

$$\sup_{C^+_{2R}(x_1,-4)} |v| \leq \sup_{C^-_{K,2}(0,0)} u. \tag{3.4}$$

Furthermore, using (3.3), (3.4), and the definition of θ, we see that

$$|v(x,t)| \leq \theta \sup_{C^-_{K,2}(0,0)} u \tag{3.5}$$

whenever $(x,t) \in C^+_2(x_1,-4)$. Note that $v \equiv u$ on $C^+_{2R,2}(x_1,-4)$ and hence

$$|u(x,t)| \leq \theta \sup_{C^-_{K,2}(0,0)} u \tag{3.6}$$

whenever $(x,t) \in C^+_2(x_1,-4)$. Since x_1 is arbitrary in this argument we see that

$$|u(x,t)| \leq \theta \sup_{C^-_{K,2}(0,0)} u, \tag{3.7}$$

whenever $(x,t) \in C^-_{K-2(R+1),2}(0,0)$. Repeating this argument we can conclude, by induction, that

$$|u(x,t)| \leq \theta^j \sup_{C^-_{K,2}(0,0)} u \quad \text{whenever} \quad (x,t) \in C^-_{K-2j(R+1),2}(0,0), \tag{3.8}$$

as long as $K-2j(R+1)>0$. Let j_0 be the largest j such that $K-2j(R+1) \geq R$ and note that by construction of $\hat{K}$ we know that $j_0 \geq 1$. In particular, $j_0 \approx ((K-R)/(R+1))$ and we deduce that

$$|u(x,t)| \leq ce^{-K/c} \sup_{C^-_{K,2}(0,0)} u \tag{3.9}$$

whenever $(x,t) \in C^-_1(0,0)$. Hence the proof of the lemma is complete. □

4 - Proof of Theorem 1.2

Lemma 4.1. *Let $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T>0$. Let $(x_0,t_0) \in S_T$ and let $r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let u be a non-negative function in the class $\mathcal{S}(\lambda,\Lambda,\eta)$ in $\Omega_T \cap C_{2r}(x_0,t_0)$ vanishing continuously on $\Delta(x_0,t_0,2r)$. There exist c, $1 \leq c < \infty$, $c = c(\lambda,\Lambda,\eta,M)$, $\theta = \theta(\lambda,\Lambda,\eta,M)$, $0<\theta<1$, such that*

$$\sup_{\Omega_T \cap C_{r/c}(x_0,t_0)} u \leq \theta \sup_{\Omega_T \cap C_r(x_0,t_0)} u.$$

Proof. This is essentially a special case of Theorem 2.5 in [**W2**], see also Lemma 2.6 in [**W2**]. A proof can also be found as the proof of Lemma 3.1 in [**BG**]. □

Lemma 4.2. *Let $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T>0$. Let $(x_0,t_0) \in S_T$ and let $r <$*

$\min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. *Let u be a non-negative function in the class $\mathcal{S}(\lambda,\Lambda,\eta)$ in Ω_T vanishing continuously on $\Delta(x_0,t_0,2r)$. Then, there exists a constant $c=c(\lambda,\Lambda,\eta,M)$, $1\le c<\infty$, such that*

$$u(x,t)\le cu(A_r^+(x_0,t_0))$$

whenever $(x,t)\in\Omega_T\cap C_{r/4}(x_0,t_0)$.

Proof. Using the Harnack inequality and Lemma 4.1 we see that Lemma 4.2 follows by standard arguments along the lines of [**S**]. □

Lemma 4.3. *Let $\Omega_T=\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T>0$. Let u be a non-negative function in the class $\mathcal{S}(\lambda,\Lambda,\eta)$ in Ω_T which vanishes continuously on S_T. Let $0<\delta\ll\sqrt{T}$ be given. Then, there exists a constant $c=c(\lambda,\Lambda,\eta,M,\operatorname{diam}(\Omega),T,\delta)$, $1\le c<\infty$, such that*

$$\sup_{(x,t)\in\Omega^\delta\times(\delta^2,T)} u(x,t)\le c\inf_{(x,t)\in\Omega^\delta\times(\delta^2,T)} u(x,t),$$

where $\Omega^\delta=\{x\in\Omega: d(x,\partial\Omega)>\delta\}$.

Proof. The lemma follows from Lemma 4.2, the Harnack inequality and Lemma 4.2, see Theorem 1.3 in [**FGS**], or Lemma 2.7 in [**N**], or Theorem 3.4 in [**BG**], for details. □

Lemma 4.4. *Let $\Omega_T=\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T>0$. Let $K\gg 1$ be given and consider $(x_0,t_0)\in S_T$. Furthermore, assume that $r<\min\{r_0/(2K), \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let $\gamma=\gamma(\lambda,\Lambda,\eta,M)\in(0,1)$ be as in Lemma 2.7 and Lemma 2.8. Assume that u is a non-negative function in the class $\mathcal{S}(\lambda,\Lambda,\eta)$ in Ω_T, vanishing continuously on $S_T\cap C^-_{Kr,2r}(x_0,t_0)$, and that*

$$\sup_{\Omega_T\cap C_r^-(x_0,t_0)} u\ge(2K)^{-\gamma}\sup_{\Omega_T\cap C^-_{Kr,2r}(x_0,t_0)} u.$$

Then, provided $K=K(\lambda,\Lambda,\eta,M)$ is chosen large enough,

$$\sup_{\Omega_T\cap\left(B(x_0,Kr)\times\{t_0-4r^2\}\right)} u\ge\frac{1}{10}\sup_{\Omega_T\cap C_r^-(x_0,t_0)} u. \tag{4.1}$$

Proof. We may without loss of generality assume that $r=1$. Obviously (4.1) is true if

$$\sup_{\Omega_T\cap C^-_{K,2}(x_0,t_0)} u=\sup_{\Omega_T\cap C^-_K(x_0,t_0)\cap\{(x,t):t=t_0-4\}} u. \tag{4.2}$$

Hence we in the following assume that (4.2) does not hold. Let $\tilde{u}$ satisfy $\mathcal{H}^+_{\lambda,\Lambda,\eta}\tilde{u} = 0$ in $C^-_{K,2}(x_0, t_0)$ with boundary data u on $(\partial_p C^-_{K,2}(x_0, t_0)) \cap \Omega_T$ and boundary data 0 on $(\partial_p C^-_{K,2}(x_0, t_0)) \cap (\mathbb{R}^{n+1} \setminus \Omega_T)$. Then $\tilde{u}$ has continuous boundary data on $\partial_p C^-_{K,2}(x_0, t_0)$ and, by the comparison principle $u \leq \tilde{u}$ in $(C^-_{K,2}(x_0, t_0)) \cap \overline{\Omega_T}$. In particular,

$$
\begin{aligned}
\sup_{\Omega_T \cap C^-_1(x_0,t_0)} \tilde{u} \geq & \sup_{\Omega_T \cap C^-_1(x_0,t_0)} u \\
\geq & (2K)^{-\gamma} \sup_{\Omega_T \cap C^-_{K,2}(x_0,t_0)} u = (2K)^{-\gamma} \sup_{\Omega_T \cap C^-_{K,2}(x_0,t_0)} \tilde{u}.
\end{aligned} \tag{4.3}
$$

Next, let $\phi \in C_0^\infty(B(x_0, K))$ be a function such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $B(x_0, K-1)$. Let h satisfy $\mathcal{H}^+_{\lambda,\Lambda,\eta} h = 0$ in $C^-_{K,2}(x_0, t_0)$ with the boundary data

$$
h(x,t) = \begin{cases} 0, & (x,t) \in \partial_p C^-_{K,2}(x_0, t_0) \setminus \Omega_T, \\ (1-\phi(x))\tilde{u}(x,t), & (x,t) \in B(x_0, K) \times \{t = t_0 - 4\}, \\ \tilde{u}(x,t), & (x,t) \in \partial B(x_0, K) \times \{t : t_0 - 4 < t < t_0\} \cap \Omega_T. \end{cases}
$$

Note that h is continuous on $\partial_p C^-_{K,2}(x_0, t_0)$. We now assume, in order to reach a contradiction, that

$$
\sup_{\Omega_T \cap C^-_K(x_0,t_0) \cap \{(x,t): t = t_0 - 4\}} u \leq \frac{1}{A} \sup_{\Omega_T \cap C^-_1(x_0,t_0)} u, \tag{4.4}
$$

where $A = 10$. By construction (4.4) implies that

$$
\sup_{\Omega_T \cap C^-_K(x_0,t_0) \cap \{(x,t): t = t_0 - 4\}} \tilde{u} \leq \frac{1}{A} \sup_{\Omega_T \cap C^-_1(x_0,t_0)} \tilde{u}. \tag{4.5}
$$

We now note that

$$
\max\left\{ \tilde{u}(x,t) - \frac{1}{A} \sup_{\Omega_T \cap C^-_1(x_0,t_0)} \tilde{u}, 0 \right\} \tag{4.6}
$$

is a viscosity subsolution to the equation $\mathcal{H}^+_{\lambda,\Lambda,\eta} v = 0$ in $C^-_{K,2}(x_0, t_0)$ and that the boundary value of the function in (4.6), at $(x,t) \in \partial_p C^-_{K,2}(x_0, t_0)$, is bounded from above by $h(x,t)$. Hence, using the comparison principle it follows that

$$
\tilde{u}(x,t) - \frac{1}{A} \sup_{\Omega_T \cap C^-_1(x_0,t_0)} \tilde{u} \leq h(x,t), \quad \text{whenever } (x,t) \in C^-_{K,2}(x_0, t_0).
$$

Now using the Gaussian decay estimate of Lemma 3.2, and (4.3), we deduce that

$$
\begin{aligned}
\frac{(A-1)}{A} \sup_{\Omega_T \cap C_1^-(x_0,t_0)} \tilde{u} &\leq \sup_{C_1^-(x_0,t_0)} h \\
&\leq ce^{-K/c} \sup_{C_{K,2}^-(x_0,t_0)} h \\
&= ce^{-K/c} \sup_{\Omega_T \cap C_{K,2}^-(x_0,t_0)} \tilde{u} \\
&\leq ce^{-K/c}(2K)^{\gamma} \sup_{\Omega_T \cap C_1^-(x_0,t_0)} \tilde{u}.
\end{aligned} \tag{4.7}
$$

Since $A = 10$ we can now choose $K = K(\lambda, \Lambda, \eta, M, A)$ large enough to ensure that

$$ce^{-cK}(2K)^{\gamma} \leq \frac{1}{A}. \tag{4.8}$$

Using this choice for K we see that (4.7) can not hold and hence (4.5), and therefore (4.4), is contradicted. This completes the proof of the lemma. □

4.1 - *The final proof*

To begin the proof we let $0 < \delta \ll \sqrt{T}$ be a fixed constant, we let $(x_0, t_0) \in S_T$, $\delta^2 \leq t_0 \leq T - \delta^2$, and $d = \min\{r_0/2, \sqrt{(T - t_0 - \delta^2)/4}, \sqrt{(t_0 - \delta^2)/4}\}$. Obviously, to prove Theorem 1.2 it is enough to prove that

$$\sup_{\Omega_T \cap C_{2r}^-(x_0,t_0)} u(x,t) \leq cu(A_r^-(x_0,t_0)).$$

We now assume that $r < d$ and we let ρ be the largest number $r \leq \rho \leq d$ satisfying the inequality

$$\sup_{\Omega_T \cap C_{2r}^-(x_0,t_0)} u(x,t) \leq (r/\rho)^{\gamma} \sup_{\Omega_T \cap C_{2\rho}^-(x_0,t_0)} u(x,t), \tag{4.9}$$

where γ is the constant appearing in Lemma 2.7. Using Lemma 2.8 and the definition of the point $A_r^-(x_0,t_0)$ we see that

$$u(A_{2\rho}^-(x_0,t_0)) \leq c(\rho/r)^{\gamma} u(A_r^-(x_0,t_0)). \tag{4.10}$$

In the following we prove that

$$\sup_{\Omega_T \cap C_{2\rho}^-(x_0,t_0)} u(x,t) \leq cu(A_{2\rho}^-(x_0,t_0)) \tag{4.11}$$

for this particular choice of ρ. In fact, combining (4.9), (4.10) and (4.11) we see that

$$\sup_{\Omega_T \cap C_{2r}^-(x_0,t_0)} u(x,t) \leq cu(A_r^-(x_0,t_0)) \tag{4.12}$$

and hence the proof of Theorem 1.2 is complete once we have proved (4.11). To prove (4.11) we let $K \gg 1$ be given as in Lemma 4.4, and we divide the proof into two cases. First, we assume that $\delta/(2K) < \rho$. In this case ρ is large and combining Lemma 4.2 and Lemma 4.3 we see that

$$\sup_{\Omega_T \cap C^-_{2\rho}(x_0,t_0)} u(x,t) \leq cu(A^+_{2\rho}(x_0,t_0)) \leq c^2 u(A^-_{2\rho}(x_0,t_0)), \tag{4.13}$$

for some $c = c(\lambda, \Lambda, \eta, M, \operatorname{diam}(\Omega), T, \delta, K)$, $1 \leq c < \infty$. Hence, the proof is complete in this case. Second, we assume that $r \leq \rho \leq \delta/(2K)$ and we then first note, by the definition of ρ, that

$$\sup_{\Omega_T \cap C^-_{2\rho}(x_0,t_0)} u \geq (2K)^{-\gamma} \sup_{\Omega_T \cap C^-_{4K\rho}(x_0,t_0)} u. \tag{4.14}$$

Obviously (4.14) implies

$$\sup_{\Omega_T \cap C^-_{2\rho}(x_0,t_0)} u \geq (2K)^{-\gamma} \sup_{\Omega_T \cap C^-_{4K\rho,8\rho}(x_0,t_0)} u \geq (2K)^{-\gamma} \sup_{\Omega_T \cap C^-_{2K\rho,4\rho}(x_0,t_0)} u,$$

and hence we can use Lemma 4.4 to conclude that

$$\sup_{\Omega_T \cap C^-_{2K\rho}(x_0,t_0) \cap \{(x,t): t = t_0 - 16\rho^2\}} u \geq \frac{1}{10} \sup_{\Omega_T \cap C^-_{2\rho}(x_0,t_0)} u. \tag{4.15}$$

In particular, using if necessary Lemma 4.2, and the Harnack inequality in Lemma 2.6, we can now use (4.15) to conclude (4.11). This completes the proof of (4.11) and hence the proof of Theorem 1.2. □

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