Riv. Mat. Univ. Parma, Vol. **4** (2013), 197-213

Benjamin Cahen

Stratonovich-Weyl correspondence for the real diamond group

Abstract. We show that the usual Weyl quantization provides a Stratonovich-Weyl correspondence for the Schrödinger realization of a generic representation of the real diamond group . This Stratonovich-Weyl correspondence is related to that obtained by polarization of the Berezin map in [B. Cahen, *Berezin quantization and holomorphic representations,* Rend. Sem. Mat. Univ. Padova **129** (2013), 277-297].

Keywords. Stratonovich-Weyl correspondence; Berezin quantization; diamond group; coherent states; coadjoint orbit; unitary representation; holomorphic representation; Schrödinger representation; Bargmann-Fock representation; Segal-Bargmann transform.

Mathematics Subject Classification (2010): 22E27, 22E25, 81S10, 81R30.

1 - Introduction

The notion of Stratonovich-Weyl correspondence was introduced in [**33**] as a generalization of the classical Weyl correspondence [**1**]. The systematic study of the Stratonovich-Weyl correspondences began with the work of J. M. Gracia-Bondía, J. C. Várilly and their co-workers (see [**23**], [**20**], [**18**] and [**22**]).

Definition 1.1. [**22**] Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and let μ be a (suitably normalized) G-invariant measure on M. Then a *Stratonovich-Weyl correspondence* for the triple (G, π, M) is a linear isomorphism W from a space of operators on $\mathcal{H}$ to a

Received: September 27, 2012; accepted in revised form: June 3, 2013.

space of (generalized) functions on M satisfying the following properties:

1. W maps the identity operator of $\mathcal{H}$ to the constant function 1;
2. the function $W(A^*)$ is the complex-conjugate of $W(A)$;
3. W is G-covariant with respect to π: For each $g \in G$ and $x \in M$, we have

$$W(\pi(g) A\, \pi(g)^{-1})(x) = W(A)(g^{-1} \cdot x);$$

4. Traciality: We have

$$\int_M W(A)(x) W(B)(x)\, d\mu(x) = \mathrm{Tr}(AB).$$

Stratonovich-Weyl correspondences were constructed for various Lie group representations, in particular for the unitary irreducible representations of the Poincaré group [**18**]. In [**16**], Stratonovich-Weyl correspondences for holomorphic representations of quasi-Hermitian Lie groups were obtained by taking the isometric part in the polar decomposition of the Berezin quantization map (see also [**20**], [**14**], [**15**] and, for the case of Hermitian symmetric domains, [**2**], [**3**]).

Let us consider the case of the $(2n+1)$-dimensional Heisenberg group G_0. Each non-degenerate unitary irreducible representation of G_0 has two classical realizations: the Schrödinger realization on $L^2(\mathbb{R}^n)$ and the Bargmann-Fock realization on the Fock space [**21**]. An intertwining operator between these realizations is the Segal-Bargmann transform [**21**], [**19**]. In the setting of the orbit method, the Schrödinger realization can be obtained from a real polarization of the corresponding coadjoint orbit of G_0 and the Bargmann-Fock realization from a complex polarization [**5**], [**6**]. Moreover, the usual Weyl correspondence provides a Stratonovich-Weyl correspondence for the Schrödinger realization [**4**]. It is also known that this Stratonovich-Weyl correspondence is related, by the Segal-Bargmann transform, to the Stratonovich-Weyl correspondence for the Bargmann-Fock realization which was obtained by polarization of the Berezin quantization map [**29**], [**30**].

In the present paper, we show that the preceding results for the Heisenberg group can be extended to the $(2n+2)$-dimensional real diamond group G. The diamond group (also called oscillator group) is a solvable simply connected non-exponential Lie group which is a semi-direct product of the Heisenberg group with a real line. As the Heisenberg group, the diamond group plays an important role in harmonic analysis and in quantum mechanics [**28**], [**26**].

Each generic coadjoint orbit of G does not admit any real polarization and the corresponding representation is usually realized as a holomorphically induced representation on the Fock space. Schrödinger realization of such a representation can be nevertheless obtained from the Bargmann-Fock realization by conjugation with the Segal-Bargmann transform. We show that the usual Weyl correspondence is covariant with respect to this Schrödinger realization and then gives a Stratonovich-Weyl correspondence. We compare this Stratonovich-Weyl correspondence to that constructed by using the general method of [**16**].

This paper is organized as follows. In the Sections 2-4, we recall some known results about the unitary irreducible representations of the Heisenberg group and the Segal-Bargmann transform, and we interpret these results in the context of the method of orbits. In Section 5 and Section 6, we introduce the $(2n+2)$-dimensional real diamond group G, the generic coadjoint orbits of G, that is, the coadjoint orbits of maximal dimension, and the corresponding unitary irreducible representations. In Section 7, we describe a real parametrization and a complex parametrization of a generic coadjoint orbit of G, which are obtained by dequantizing the Schrödinger realization and the Bargmann-Fock realization of the corresponding representation. Finally, in Section 8, we establish our main results on Stratonovich-Weyl correspondences associated with these representations of G.

2 - Representations of the Heisenberg group

In this section, we introduce some notation relative to the Heisenberg group and the Schrödinger and Bargmann-Fock realizations of its unitary irreducible non-degenerated representations.

Let G_0 be the Heisenberg group of dimension $2n+1$ and $\mathfrak{g}_0$ be the Lie algebra of G_0. Let $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,\tilde{Z}\}$ be a basis of $\mathfrak{g}_0$ in which the only non trivial brackets are $[X_k\,,\,Y_k]=\tilde{Z}$, $1\leq k\leq n$ and let $\{X_1^*,\ldots,X_n^*,Y_1^*,\ldots,Y_n^*,\tilde{Z}^*\}$ be the corresponding dual basis of $\mathfrak{g}_0^*$.

For $a=(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n$, $b=(b_1,b_2,\ldots,b_n)\in\mathbb{R}^n$ and $c\in\mathbb{R}$, we denote by $[a,b,c]$ the element $\exp_{G_0}\Big(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k+c\tilde{Z}\Big)$ of G_0. The coadjoint action of G_0 is then given by

$$\mathrm{Ad}^*([a,b,c])\left(\sum_{k=1}^n\alpha_kX_k^*+\sum_{k=1}^n\beta_kY_k^*+\gamma\tilde{Z}^*\right)=\sum_{k=1}^n(\alpha_k+\gamma b_k)X_k^*+\sum_{k=1}^n(\beta_k-\gamma a_k)Y_k^*+\gamma\tilde{Z}^*.$$

Fix a real number $\lambda>0$ and denote by $\mathcal{O}_\lambda$ the orbit of the element $\xi_\lambda:=\lambda\tilde{Z}^*$ of $\mathfrak{g}_0^*$ under the coadjoint action of G_0 (the case $\lambda<0$ can be treated similarly). By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation of G_0 whose restriction to the center of G_0 is the character $[0,0,c] \to e^{i\lambda c}$ [**5**], [**21**]. This representation is associated with the coadjoint orbit $\mathcal{O}_\lambda$ by the Kirillov-Kostant method of orbits [**26**]. More precisely, if we choose the real polarization at ξ_λ to be the space spanned by the elements Y_k for $1 \le k \le n$ and $\tilde{Z}$ then we obtain the Schrödinger representation ρ_λ realized on $L^2(\mathbb{R}^n)$ as

$$\rho_\lambda([a,b,c])(f)(x) = e^{i\lambda(c - bx + \frac{1}{2}ab)} f(x-a),$$

see [**4**] for instance. Here we use the notation $xy := \sum_{k=1}^{n} x_k y_k$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$. The differential of ρ_λ is then given by

$$d\rho_\lambda(X_k)f(x) = -\partial_k f(x),\ d\rho_\lambda(Y_k)f(x) = -i\lambda x_k f(x),\ d\rho_\lambda(\tilde{Z})f(x) = i\lambda f(x)$$

where $k = 1, 2, \ldots, n$. Moreover, if we choose the complex polarization at ξ_λ to be the space spanned by the elements $X_k + iY_k$ for $1 \le k \le n$ and $\tilde{Z}$ then we obtain the Bargmann-Fock representation π_λ defined as follows [**10**], [**12**].

Let $\mathcal{H}_\lambda$ be the Hilbert space of holomorphic functions on $\mathbb{C}^n$ such that

$$\|F\|_\lambda^2 := \int_{\mathbb{C}^n} |F(z)|^2 \, e^{-|z|^2/2\lambda} \, d\mu_\lambda(z) < +\infty$$

where $d\mu_\lambda(z) := (2\pi\lambda)^{-n} \, dx \, dy$. Here $z = x + iy$ with x and y in $\mathbb{R}^n$.

Let us also introduce the action of G_0 on $\mathbb{C}^n$ defined by $g \cdot z := z + \lambda(b - ia)$ for $g = [a,b,c] \in G_0$ and $z \in \mathbb{C}^n$. Then π_λ is the representation of G_0 on $\mathcal{H}_\lambda$ given by

$$\pi_\lambda(g) \, F(z) = \alpha(g^{-1}, z) \, F(g^{-1} \cdot z)$$

where the map α is defined by

$$\alpha(g,z) = \exp(-ic\lambda + (1/4)(b + ai)(-2z + \lambda(-b + ai)))$$

for $g = [a,b,c] \in G_0$ and $z \in \mathbb{C}^n$.

The differential of π_λ is given by

$$\begin{cases} d\pi_\lambda(X_k)F(z) = \dfrac{1}{2} i z_k F(z) + \lambda i \dfrac{\partial F}{\partial z_k} \\ d\pi_\lambda(Y_k)F(z) = \dfrac{1}{2} z_k F(z) - \lambda \dfrac{\partial F}{\partial z_k} \\ d\pi_\lambda(\tilde{Z})F(z) = i\lambda F(z). \end{cases}$$

By following the same lines as in [**24**], Section 6 or [**19**], Section 1.3, we can verify that the Segal-Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{H}_\lambda$ defined by

$$B(f)(z) = (\lambda/\pi)^{n/4} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz - (\lambda/2)x^2} f(x) \, dx$$

is a (unitary) intertwining operator between ρ_λ and π_λ. The inverse Segal-Bargmann transform $B^{-1} = B^*$ is then given by

$$B^{-1}(F)(x) = (\lambda/\pi)^{n/4} \int_{\mathbb{R}^n} e^{(1/4\lambda)\bar{z}^2 - ix\bar{z} - (\lambda/2)x^2}\, F(z)\, e^{-|z|^2/2\lambda}\, d\mu_\lambda(z).$$

For $z \in \mathbb{C}^n$, introduce the coherent states $e_z(w) = \exp(\bar{z}w/2\lambda)$. We have the reproducing property $F(z) = \langle F, e_z\rangle_\lambda$ for each $F \in \mathcal{H}_\lambda$ where $\langle\cdot,\cdot\rangle_\lambda$ denotes the scalar product on $\mathcal{H}_\lambda$.

We can transfer the coherent states e_z to $L^2(\mathbb{R}^n)$ by putting $\varepsilon_z = B^{-1}e_z$.

Proposition 2.1.

(1) *For each $g \in G_0$ and each $z \in \mathbb{C}^n$, we have* $\pi_\lambda(g)e_z = \overline{\alpha(g,z)}e_{g\cdot z}$.

(2) *Similarly, for each $g \in G_0$ and each $z \in \mathbb{C}^n$, we have* $\rho_\lambda(g)\varepsilon_z = \overline{\alpha(g,z)}\varepsilon_{g\cdot z}$.

(3) *For each $z \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$, we have*

$$\varepsilon_z(x) = (\lambda/\pi)^{n/4}\, e^{(1/4\lambda)\bar{z}^2 - ix\bar{z} - (\lambda/2)x^2}.$$

Proof. (1) This is an immediate consequence of the reproducing property, see [**13**].

(2) Taking the equality $B^{-1}\pi_\lambda(g) = \rho_\lambda(g)B^{-1}$ into account, the result follows from (1).

(3) By the reproducing property again, we have $\langle f, \varepsilon_z\rangle_2 = B(f)(z)$ for each $f \in L^2(\mathbb{R}^n)$ and $z \in \mathbb{C}^n$ hence the result. □

3 - Berezin quantization

In this section, we introduce the Berezin quantization map and we review some of its properties, in connection with the representations of the Heisenberg group.

Consider an operator (not necessarily bounded) A on $\mathcal{H}_\lambda$ whose domain contains e_z for each $z \in \mathbb{C}^n$. Then the Berezin symbol of A is the function $S_0(A)$ defined on $\mathbb{C}^n$ by

$$S_0(A)(z) := \frac{\langle A\, e_z\,,\, e_z\rangle_\lambda}{\langle e_z\,,\, e_z\rangle_\lambda}.$$

We can verify that each such operator is determined by its Berezin symbol and that if an operator A has adjoint A^* then we have $S_0(A^*) = \overline{S_0(A)}$, see [**7**], [**17**]. Moreover, we have the following result.

Proposition 3.1. *Let A be an operator on $\mathcal{H}_\lambda$ whose domain contains the coherent states e_z for each $z \in \mathbb{C}^n$. Then, for each $g \in G$, the domain of $\pi_\lambda(g^{-1})A\pi_\lambda(g)$ also contains e_z for each $z \in \mathbb{C}^n$ and we have*

$$S_0(\pi_\lambda(g)^{-1}A\pi_\lambda(g))(z) = S_0(A)(g \cdot z)$$

for each $g \in G$ and $z \in \mathbb{C}^n$, that is, S_0 is G_0-covariant with respect to π_λ.

Proof. This follows from (1) of Proposition 2.1. □

Let $\mathcal{L}_2(\mathcal{H}_\lambda)$ be the space of all Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. As a particular case of [**34**], Proposition 1.19, we have:

Proposition 3.2. *The map S_0 is a bounded operator from $\mathcal{L}_2(\mathcal{H}_\lambda)$ (endowed with the Hilbert-Schmidt norm) to $L^2(\mathbb{C}^n, \mu_\lambda)$ which is one-to-one and has dense range.*

Recall that the Berezin transform is the operator $\mathcal{B}$ on $L^2(\mathbb{C}^n)$ defined by $\mathcal{B} = S_0 S_0^*$ or, equivalently, by the integral formula

$$\mathcal{B}(F)(z) = \int_{\mathbb{C}^n} F(w)\, e^{|z-w|^2/2\lambda}\, d\mu_\lambda(w),$$

see [**7**], [**8**], [**34**], [**32**] for instance. Recall also that we have $\mathcal{B} = \exp(\lambda\Delta/2)$ where $\Delta = 4\sum_{k=1}^{n} \partial^2/\partial z_k \partial \bar{z}_k$, see [**34**], [**30**]. Another interesting property of S_0 is given by the following proposition. Here we denote by $\mathfrak{g}_0^c$ the complexification of $\mathfrak{g}_0$.

Proposition 3.3. *Let Φ_λ be the map defined by*

$$\Phi_\lambda(z) := \sum_{k=1}^{n} (\operatorname{Re} z_k X_k^* + \operatorname{Im} z_k Y_k^*) + \lambda \tilde{Z}^*.$$

Then

(1) *For each $X \in \mathfrak{g}_0^c$ and each $z \in \mathbb{C}^n$, we have*

$$S_0(d\pi_\lambda(X))(z) = i\langle \Phi_\lambda(z), X\rangle.$$

(2) *For each $g \in G_0$ and each $z \in \mathbb{C}^n$, we have $\Phi_\lambda(g \cdot z) = \operatorname{Ad}^*(g)\, \Phi_\lambda(z)$.*

(3) *The map Φ_λ is a diffeomorphism from $\mathbb{C}^n$ onto $\mathcal{O}_\lambda$.*

This proposition can be proved by a direct computation. It can be also considered as a particular case of [**16**], Proposition 5.1.

Now, we transfer S_0 to operators on $L^2(\mathbb{R}^n)$ by using B. We consider the map S defined by $S(A) := S_0(BAB^{-1})$ for A operator on $L^2(\mathbb{R}^n)$. Equivalently, we have

$$S(A)(z) := \frac{\langle A\,\varepsilon_z\,,\,\varepsilon_z\rangle_2}{\langle \varepsilon_z\,,\,\varepsilon_z\rangle_2}.$$

Clearly, the properties of S_0 give rise to similar properties of S. In particular, S is a bounded operator from $\mathcal{L}_2(L^2(\mathbb{R}^n))$ to $L^2(\mathbb{C}^n, \mu_\lambda)$ (Proposition 3.2) and S is G_0-covariant with respect to ρ_λ (Proposition 3.3). Also, by Proposition 3.3, we have $S(d\rho_\lambda(X))(z) = i\langle \Phi_\lambda(z), X\rangle$ for each $X \in \mathfrak{g}_0$ and $z \in \mathbb{C}^n$. Moreover, denoting by b the map from $\mathcal{L}_2(L^2(\mathbb{R}^n))$ onto $\mathcal{L}_2(\mathcal{H}_\lambda)$ defined by $b(A) = BAB^{-1}$, we have $S = S_0 b$ then

$$SS^* = (S_0 b)(S_0 b)^* = S_0 bb^* S_0^* = S_0 S_0^* = \mathcal{B},$$

that is, the Berezin transform corresponding to S is the same as the Berezin transform corresponding to S_0. Then we can write the polar decompositions of S_0 and S under the form $S_0 = \mathcal{B}^{1/2} U_0$ and $S = \mathcal{B}^{1/2} U$ where the maps $U_0 : \mathcal{L}_2(\mathcal{H}_\lambda) \to L^2(\mathbb{C}^n, \mu_\lambda)$ and $U : \mathcal{L}_2(L^2(\mathbb{R}^n)) \to L^2(\mathbb{C}^n, \mu_\lambda)$ are unitary. Thus, by following the same lines as in the proof of [**15**], Proposition 3.1, we see that U_0 is a Stratonovich-Weyl correspondence for $(G_0, \pi_\lambda, \mathbb{C}^n)$ and that U is a Stratonovich-Weyl correspondence for $(G_0, \rho_\lambda, \mathbb{C}^n)$. In particular, the G_0-covariance properties for U_0 and U follows from Proposition 3.1. Note also that we have $U = U_0 b$.

4 - Weyl quantization

In this section, we show that the Weyl correspondence gives a Stratonovich-Weyl correspondence for ρ_λ. Recall that the Weyl correspondence on $\mathbb{R}^{2n}$ is defined as follows.

For each f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, we define the operator $W(f)$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$W(f)\varphi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{isq} f(p + (1/2)s, q)\, \varphi(p + s)\, ds\, dq.$$

It is well-known that the Weyl calculus can be extended to much larger classes of symbols (see for instance [**25**]). Here we only consider a class of C^∞-functions $f(p, q)$ which are polynomials in the variable q. If $f(p, q) = u(p)q^\alpha$ where $u \in C^\infty(\mathbb{R}^n)$ then we have

$$W(f)\varphi(p) = \left(i\frac{\partial}{\partial s}\right)^\alpha (u(p + (1/2)s)\, \varphi(p + s))\Big|_{s=0}, \tag{4.1}$$

see [**35**]. In particular, if $f(p,q)=u(p)$ then $W(f)\varphi(p)=u(p)\,\varphi(p)$ and if $f(p,q) = u(p)q_k$ then

$$W(f)\varphi(p)=i\big((1/2)\partial_k u(p)\,\varphi(p)+u(p)\partial_k\varphi(p)\big). \tag{4.2}$$

Let us introduce the action of G_0 on $\mathbb{R}^{2n}$ given by $g\cdot(p,q):=(p+a,q+\lambda b)$ where $g=[a,b,c]$. The following proposition is analogous to Proposition 3.1.

Proposition 4.1. *Let Ψ_λ be the map defined by*

$$\Psi_\lambda(p,q):=\sum_{k=1}^{n}(q_kX_k^*-\lambda p_kY_k^*)+\lambda\tilde{Z}^*.$$

Then

(1) *For each $X\in\mathfrak{g}_0^c$ and each $(p,q)\in\mathbb{R}^{2n}$, we have*

$$W^{-1}(d\rho_\lambda(X))(p,q)=i\langle\Psi_\lambda(p,q),X\rangle.$$

(2) *For each $g\in G_0$ and each $(p,q)\in\mathbb{R}^{2n}$, we have*

$$\Psi_\lambda(g\cdot(p,q))=\mathrm{Ad}^*(g)\Psi_\lambda(p,q).$$

(3) *The map Ψ_λ is a diffeomorphism from $\mathbb{R}^{2n}$ onto $\mathcal{O}_\lambda$.*

Proof. (1) By applying (4.2), we verify that $W(q_k)=-id\rho_\lambda(X_k)$, $W(\lambda p_k) = id\rho_\lambda(Y_k)$ for each $k=1,2,\dots,n$ and $W(\lambda)=-id\rho_\lambda(\tilde{Z})$. The result follows. Now, (2) is a consequence of the formula for the coadjoint action of G and it is easy to verify (3). □

Note that the maps Φ_λ and Ψ_λ are related by $\Phi_\lambda(q-\lambda pi)=\Psi_\lambda(p,q)$ for each $(p,q)\in\mathbb{R}^{2n}$.

We endow $\mathbb{R}^{2n}$ with the G_0-invariant measure $\tilde{\mu}:=(2\pi)^{-n}dp\,dq$. Then we have the following proposition.

Proposition 4.2. *The map W^{-1} is a Stratonovich-Weyl correspondence for $(G_0,\rho_\lambda,\mathbb{R}^{2n})$.*

Proof. We have just to prove that W^{-1} is G_0-covariant with respect to ρ_λ since the other properties of a Stratonovich-Weyl correspondence are easy to deduce from the classical properties of the usual Weyl correspondence. Covariance property can be deduced from [**21**], p. 83 (see also [**22**]) but the proof uses the Wigner transform. Here we sketch a more direct proof. Let $f\in L^2(\mathbb{R}^{2n},\tilde{\mu})$, $\varphi\in C_0^\infty(\mathbb{R}^n)$,

$g=[a,b,0]\in G_0$ and $p\in\mathbb{R}^n$. We can express $W(f)(\rho_\lambda(g)\varphi)(p)$ as an integral. By making the change of variables $(s,q)\to(s,q+\lambda b)$ in this integral, we get $W(f)\rho_\lambda(g)\varphi=\rho_\lambda(g)W(\tilde{f})\varphi$ where $\tilde{f}$ is defined $\tilde{f}(p,q)=f(p+a,q+\lambda b)$. Hence the result. $\square$

The following proposition asserts that if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by $(p,q)\to q-\lambda pi$ then the unitary part in the polar decomposition of S coincides with the inverse of the Weyl transform. This result was proved by S. Luo (see [**30**], Theorem 6). Here, we present a slightly different proof based on covariance.

Proposition 4.3. *For $F_0\in L^2(\mathbb{C}^n,\mu_\lambda)$, we have $SW(F)=\mathcal{B}^{1/2}F_0$ where $F(p,q)=F_0(q-\lambda pi)$. Consequently, denoting by j the map $F_0\to F$, we have $U=(Wj)^{-1}$.*

Proof. Let $F\in L^2(\mathbb{R}^{2n},\tilde{\mu})$ and $z\in\mathbb{C}^n$. We have to compute $SW(F)(z)$. As in the proof of Proposition 2.1, we fix $g=[a,b,0]\in G_0$ such that $g\cdot 0=z$, i.e. $\lambda(b-ia)=z$. By using the fact that W is G_0-covariant, we have

$$SW(F)(z)=e^{-(1/2\lambda)|z|^2}\langle W(F)\varepsilon_z,\varepsilon_z\rangle_2=\langle W(F)\rho(g)\varepsilon_0,\rho(g)\varepsilon_0\rangle_2=\langle W(\tilde{F})\varepsilon_0,\varepsilon_0\rangle_2$$

where $\tilde{F}(p,q):=F(p+a,q+\lambda b)$.

Now, from the definition of W it is clear that

$$\begin{aligned}&\langle W(\tilde{F})\varepsilon_0,\varepsilon_0\rangle_2\\ &\quad=(2\pi)^{-n}(\lambda\pi^{-1})^{n/2}\int_{\mathbb{R}^{2n}}e^{isq-(\lambda/2)((p+s)^2+p^2)}F(p+(1/2)s+a,q+\lambda b)\,dp\,ds\,dq.\end{aligned}$$

By performing the change of variables $(p,q)\to(p-(1/2)s-a,q-\lambda b)$ in this integral and by using the formula

$$\int_{\mathbb{R}^n}e^{isx-(\lambda/4)s^2}ds=\left(\frac{4\pi}{\lambda}\right)^{n/2}e^{-x^2/\lambda}$$

we get

$$SW(F)(z)=\langle W(\tilde{F})\varepsilon_0,\varepsilon_0\rangle_2=\pi^{-n}\int_{\mathbb{R}^{2n}}F(p,q)\,e^{-\lambda(p-a)^2-(1/\lambda)(q-\lambda b)^2}dp\,dq.$$

On the other hand, from the equality $\mathcal{B}=\exp(\lambda\Delta/2)$ (see Section 3), we deduce that

$$\mathcal{B}^{1/2}(F_0)(z) = (\lambda\pi)^{-n} \int_{\mathbb{C}^n} F_0(w)\, e^{|z-w|^2/\lambda}\, d\mu_{\lambda/2}(w)$$
$$= \pi^{-n} \int_{\mathbb{R}^{2n}} e^{-((\lambda b - q)^2 + (\lambda p - \lambda a)^2)/\lambda} F_0(q - \lambda p i)\, dp\, dq.$$

Then we obtain $SW(F) = \mathcal{B}^{1/2}(F_0)$. Thus the polar decomposition of S is $S = \mathcal{B}^{1/2}(Wj)^{-1}$ hence we have $U = (Wj)^{-1}$. □

Let ν_λ be the G_0-invariant measure on $\mathcal{O}_\lambda$ defined by $\nu_\lambda := (\Phi_\lambda^{-1})^*(\mu_\lambda) = (\Psi_\lambda^{-1})^*(\tilde{\mu})$. Then the maps $\tau_{\Phi_\lambda} : F \to F \circ \Phi_\lambda^{-1}$ from $L^2(\mathbb{C}^n, \mu_\lambda)$ onto $L^2(\mathcal{O}_\lambda, \nu_\lambda)$ and $\tau_{\Psi_\lambda} : F \to F \circ \Psi_\lambda^{-1}$ from $L^2(\mathbb{R}^{2n}, \tilde{\mu})$ onto $L^2(\mathcal{O}_\lambda, \nu_\lambda)$ are unitary and we have $\tau_{\Phi_\lambda} = \tau_{\Psi_\lambda} j$.

Proposition 4.4. *The map $\mathcal{W}_1 := \tau_{\Psi_\lambda} W^{-1}$ is a Stratonovich-Weyl correspondence for $(G_0, \rho_\lambda, \mathcal{O}_\lambda)$, the map $\mathcal{W}_2 := \tau_{\Phi_\lambda} U_0$ is a Stratonovich-Weyl correspondence for $(G_0, \pi_\lambda, \mathcal{O}_\lambda)$ and we have $\mathcal{W}_1 = \mathcal{W}_2 b$.*

Proof. By Section 3 and Proposition 4.2, it is clear that $\mathcal{W}_1$ and $\mathcal{W}_2$ are Stratonovich-Weyl correspondences. Moreover, we have $\mathcal{W}_1 = \tau_{\Phi_\lambda} j^{-1} W^{-1} = \tau_{\Phi_\lambda} U_0 b = \mathcal{W}_2 b$. □

5 - The diamond group

Let n be a positive integer. The $(2n+2)$-dimensional diamond (real) Lie algebra $\mathfrak{g}$ is the semi-direct product of $\mathbb{R}$ with the $(2n+1)$-dimensional Heisenberg Lie algebra. More precisely, $\mathfrak{g}$ has basis $\{H, \tilde{Z}, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$, the only non-trivial brackets being given by

$$[H, X_k] = -Y_k; \quad [H, Y_k] = X_k; \quad [X_k, Y_k] = \tilde{Z}.$$

Let G be the connected and simply connected (real) Lie group with Lie algebra $\mathfrak{g}$. Then G is a non-exponential solvable Lie group. Each $g \in G$ can be written uniquely as

$$g = \exp tH\, \exp\Big(\sum_{k=1}^{n} a_k X_k + \sum_{k=1}^{n} b_k Y_k + c\tilde{Z} \Big)$$

where t, c, a_k and b_k are real numbers. Then we denote $g = (t, (a_k), (b_k), c)$. The group law of G is given by

$$(t,(a_k),(b_k),c).(t',(a'_k),(b'_k),c')=(t'',(a''_k),(b''_k),c'')$$

where

$$\begin{aligned} t'' &= t+t' \\ a''_k &= a'_k + a_k\cos t' - b_k \sin t' \\ b''_k &= b'_k + a_k \sin t' + b_k\cos t' \\ c'' &= c+c'+\frac{1}{2}\sum_{k=1}^{n}\big(\cos t'(a_kb'_k - a'_kb_k) - \sin t'(a_ka'_k + b_kb'_k)\big). \end{aligned}$$

Note that the Heisenberg group G_0 is the subgroup of G consisting of all elements of G of the form $(0,a,b,c)=[a,b,c]$ in the notation of Section 2.

Let $\{H^*,\tilde{Z}^*,X_1^*,\ldots,X_n^*,Y_1^*,\ldots,Y_n^*\}$ be the dual basis of $\mathfrak{g}^*$. The coadjoint action of $g=(t,(a_k),(b_k),c)\in G$ on $\xi=[d,(\alpha_k),(\beta_k),\gamma]:=dH^*+\gamma\tilde{Z}^*+\sum_k\alpha_kX_k^*+\sum_k\beta_kY_k^*$ is given by $\mathrm{Ad}^*(g)\xi=[d',(\alpha'_k),(\beta'_k),\gamma']$ where $\gamma'=\gamma$ and

$$\begin{aligned} d' &= d+\frac{\gamma}{2}\sum_{k=1}^{n}(a_k^2+b_k^2)+\sum_{k=1}^{n}(\alpha_kb_k - a_k\beta_k) \\ \alpha'_k &= \alpha_k\cos t+\beta_k\sin t-\gamma(a_k\sin t - b_k\cos t) \\ \beta'_k &= -\alpha_k\sin t+\beta_k\cos t-\gamma(a_k\cos t+b_k\sin t). \end{aligned}$$

From this, we easily deduce that if a coadjoint orbit of G contains a point of the form $[d,(\alpha_k),(\beta_k),0]$ then it is trivial or diffeomorphic to $\mathbb{R}\times\mathbb{S}^1$. Moreover, if a coadjoint orbit contains a point $[d,(\alpha_k),(\beta_k),\gamma]$ with $\gamma\neq 0$ then it has dimension $2n$ and contains a unique point of the form $[d,0,0,\gamma]$ with $\gamma\neq 0$. Such an orbit is called generic.

In this paper, we fix $\xi_0=[d_0,0,0,\lambda]$ with $\lambda>0$ (the case $\lambda<0$ can be treated similarly). As shown in [**6**], the method of orbits associates with the coadjoint orbit $\mathcal{O}(\xi_0)$ the unitary irreducible representation of G on $\mathcal{H}_\lambda$ defined by

$$(\pi(g)F)(z)=e^{-\lambda|u|^2/4}e^{ie^{it}\bar{u}z/2}e^{i(d_0t+\lambda c)}F(e^{it}z+\lambda iu)$$

where $g=(t,a,b,c)\in G$ and $u:=a+ib$.

The representation π can also be obtained by the general method of construction of holomorphic representations of quasi-Hermitian Lie groups described in [**31**], Chapter XII (see also [**16**]). Note that here we use a realization which is slightly different to that of [**16**], Section 7, an intertwining operator between these realizations being $F(z)\to F(i\lambda^{-1}z)$. Then the restriction $\pi|_{G_0}$ is precisely the representation π_λ of G_0 introduced in Section 3.

6 - Schrödinger representation of the diamond group

In this section, we introduce the Schrödinger representations of G by using the Segal-Bargmann transform B. Note that these Schrödinger representations cannot be obtained by a direct application of the method of orbits (as this is the case for G_0, see Section 3) since the corresponding coadjoint orbits do not admit real polarizations, see [**6**], p. 191. However, we can also introduce the Schrödinger representations of G by studying the extensions of ρ_λ to G [**6**], p. 192.

More precisely, we define the representation ρ of G on $L^2(\mathbb{R}^n)$ by setting $\rho(g) := B^{-1}\pi(g)B$ for $g \in G$.

Proposition 6.1.

(1) *The restriction of* ρ *to* G_0 *is* ρ_λ.

(2) *The operator* $\rho(\exp(tH))$ *is given by* $\rho(\exp(tH)) = \exp(d\rho(tH))$ *where*

$$d\rho(H)f(x) = i\left(d_0 - \frac{n}{2} + \frac{\lambda}{2}x^2\right)f(x) - i\frac{1}{2\lambda}\sum_{k=1}^{n}\partial^2_{kk}f.$$

Proof. Statement (1) clearly follows from the fact that B intertwines ρ_λ and π_λ, see Section 2. To prove (2), we have to compute $d\rho(H) = B^{-1}d\pi(H)B$. First, we note that

$$d\pi(H)F(z) = id_0F(z) + i\sum_{k=1}^{n} z_k\frac{\partial F}{\partial z_k}.$$

To simplify the notation, we set $C := (\lambda/\pi)^{n/4}$ and

$$\alpha(z,x) = \frac{1}{4\lambda}z^2 + ixz - \frac{\lambda}{2}x^2.$$

Then, for $f \in C_0(\mathbb{R}^n)$, we have

$$(d\pi(H)B)f(z) = id_0B(f)(z) + iC\int_{\mathbb{R}^n}\left(\frac{z^2}{2\lambda} + izx\right)e^{\alpha(z,x)}f(x)\,dx.$$

Thus, by integrating by parts, we obtain

$$(d\pi(H)B)f(z) = iC\int_{\mathbb{R}^n} e^{\alpha(z,x)}\left(\left(d_0 - \frac{n}{2} + \frac{\lambda}{2}x^2\right)f(x) - i\frac{1}{2\lambda}\sum_{k=1}^{n}\partial^2_{kk}f\right)dx.$$

Hence the result follows. □

Then we see that $d\rho(H)$ is the sum of a scalar operator and a Schrödinger operator. The problem of exponentiating a Schrödinger operator has been intensively studied, in particular in the context of the metaplectic representation, see for instance [**9**] and [**21**]. It is known that the exponential of a Schrödinger operator can be expressed as an integral operator, see [**21**], p. 193. We can also describe the spectral decomposition of $\exp(td\rho(H))$ as follows. Let us introduce the Hermite polynomials

$$H_l(x) := (-1)^l e^{x^2} \frac{d^l}{dx^l}(e^{-x^2})$$

for $l \in \mathbb{N}$ and define

$$\psi_l(x) := (\lambda/\pi)^{1/4}(2^l l!)^{-1/2} e^{-\lambda x^2/2} H_l(\sqrt{\lambda}x).$$

Then the functions $\psi_k(x) := \psi_{k_1}(x_1)\psi_{k_2}(x_2)\ldots\psi_{k_n}(x_n)$, $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$, form an orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of eigenvectors of $d\rho(H)$ hence of $\exp(td\rho(H))$, see [**21**], p. 52 and [**9**]. More precisely, we have $\rho(\exp(tH))\psi_k = e^{it(d_0+|k|)}\psi_k$ for each $k \in \mathbb{N}^n$.

7 - Dequantization of ρ and π

In this section, we compute $W^{-1}(d\rho(X))$ for $X \in \mathfrak{g}^c$, having in mind to extend Proposition 4.1 to G. The following lemma is an immediate consequence of Equation (4.2).

Lemma 7.1. *We have* $W(q_k) = -id\rho(X_k)$, $W(\lambda p_k) = id\rho(Y_k)$ *for* $k = 1, 2, \ldots, n$, $W(\lambda) = -id\rho(\tilde{Z})$ *and* $W\left(d_0 - \frac{n}{2} + \frac{\lambda}{2}p^2 + \frac{1}{2\lambda}q^2\right) = -id\rho(H)$.

Now, we consider

$$\Psi(p, q) := \sum_{k=1}^{n} q_k X_k^* - \lambda \sum_{k=1}^{n} p_k Y_k^* + \lambda \tilde{Z}^* + \left(d_0 + \frac{\lambda}{2}p^2 + \frac{1}{2\lambda}q^2\right)H^*.$$

Then we have $\Psi(0, 0) = \xi_0$. Let us also consider the action of G on $\mathbb{R}^{2n}$ given by

$$g \cdot (p, q) := \big((p + a)\cos t + (\lambda^{-1}q + b)\sin t, (q + \lambda b)\cos t - \lambda(p + a)\sin t\big)$$

for $g = (t, a, b, c)$.

Proposition 7.2.

(1) *For each* $g \in G$ *and each* $(p, q) \in \mathbb{R}^{2n}$, *we have* $\Psi(g \cdot (p, q)) = \mathrm{Ad}^*(g)\Psi(p, q)$.
(2) *The map* Ψ *is a diffeomorphism from* $\mathbb{R}^{2n}$ *onto* $\mathcal{O}(\xi_0)$.

(3) *For each* $X \in \mathfrak{g}^c$ *and each* $(p,q) \in \mathbb{R}^{2n}$, *we have*

$$W^{-1}(d\rho(X))(p,q) = i\langle \Psi(p,q) - (n/2)H^*, X\rangle.$$

Proof. Statement (1) follows from the formula for the coadjoint action of G, see Section 5. Statement (2) is a consequence of Statement (1). Statement (3) follows from Lemma 7.1. □

On the other hand, let us introduce the action of G on $\mathbb{C}^n$ defined by $g \cdot z = e^{-it}(z + \lambda(b - ia))$ for $g = (t,a,b,c)$. In [**16**], we showed the following proposition which is analogous to Proposition 3.3.

Proposition 7.3. *Let* $\Phi(z) := \sum_{k=1}^{n} (\mathrm{Re}\, z_k X_k^* + \mathrm{Im}\, z_k Y_k^*) + \lambda \tilde{Z}^* + (d_0 + (1/2\lambda)|z|^2)H^*$. *Then*

(1) *For each* $X \in \mathfrak{g}^c$ *and each* $z \in \mathbb{C}^n$, *we have*

$$S_0(d\pi(X))(z) = i\langle \Phi(z), X\rangle.$$

(2) *For each* $g \in G$ *and each* $z \in \mathbb{C}^n$, *we have* $\Phi(g \cdot z) = \mathrm{Ad}^*(g)\,\Phi(z)$.

(3) *The map* Φ *is a diffeomorphism from* $\mathbb{C}^n$ *onto* $\mathcal{O}(\xi_0)$.

Clearly, the maps Φ and Ψ are connected by $\Phi(q - i\lambda p) = \Psi(p,q)$. Then we recover the following result of [**16**].

Proposition 7.4. *For each* $X \in \mathfrak{g}^c$ *and each* $z \in \mathbb{C}^n$ *we have*

$$U_0(d\pi(X))(z) = i\langle \Phi(z) - (n/2)H^*, X\rangle.$$

Proof. In Section 4, we proved that $U_0 b = U = j^{-1}W^{-1}$. Then, for each $X \in \mathfrak{g}^c$, we have $U_0(d\pi(X)) = j^{-1}W^{-1}b^{-1}(d\pi(X)) = j^{-1}W^{-1}(d\rho(X))$, hence the result by (3) of Proposition 7.2. □

8 - Stratonovich-Weyl correspondence for G

As seen in [**16**], it follows from the classical properties of S_0 that U_0 is a Stratonovich-Weyl correspondence for $(G, \pi, \mathbb{C}^n)$. In particular, U_0 is G-covariant with respect to π since S_0 is.

Proposition 8.1. *The map* W^{-1} *is a Stratonovich-Weyl correspondence for* $(G, \rho, \mathbb{R}^{2n})$.

Proof. As in the proof of Proposition 4.2, we have just to show that W^{-1} is G-covariant with respect to ρ. This can be deduced from the G-covariance of U_0 as follows. Denote by L the left-regular action of G on $C^\infty(\mathbb{C}^n)$ defined by $(L_gF)(z) = F(g^{-1}\cdot z)$ for $g \in G$, $F \in C^\infty(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. Similarly, denote by l the left-regular action of G on $C^\infty(\mathbb{R}^{2n})$. Then the G-covariance property of U_0 can be written as $U_0(\pi(g)A\pi(g)^{-1}) = L_gU_0(A)$ for each $A \in \mathcal{L}_2(\mathcal{H}_\lambda)$ and each $g \in G$. Since we have $U_0 = j^{-1}W^{-1}b^{-1}$, this is equivalent to the fact that $W^{-1}(\rho(g)A'\rho(g)^{-1}) = jL_gj^{-1}W^{-1}(A')$ for each $A' \in \mathcal{L}_2(L^2(\mathbb{R}^n))$ and $g \in G$. Since we have $l_g = jL_gj^{-1}$, the result follows. □

In Section 3 and Section 4, we showed that U_0 and W^{-1} are not only Stratonovich-Weyl correspondences for the representations ρ_λ and π_λ of G_0, but also adapted Weyl correspondences in the sense of [**11**], Section 6. By Proposition 7.2 and Proposition 7.3, we see that U_0 and W^{-1} are not adapted Weyl correspondences for π and ρ, respectively. It seems to be difficult to construct a Stratonovich-Weyl correspondence for π (or ρ) which is also an adapted Weyl correspondence. If, however, we associated ρ with the coadjoint orbit of $\xi_0 + \frac{n}{2}H^*$ (instead of ξ_0) then U_0 and W^{-1} would become also adapted Weyl correspondences. But such a way to associate representations with coadjoint orbits is not conform to the classical orbit method, see [**6**], [**26**] and might be justified by other arguments, see for instance the discussion in the case of the unitary irreducible representations of a compact semi-simple Lie group in [**26**], Chapter III.

Let ν be the G-invariant measure on $\mathcal{O}(\xi_0)$ defined by $\nu := (\Phi^{-1})^*(\mu_\lambda) = (\Psi^{-1})^*(\tilde{\mu})$. Clearly the maps $\tau_\Phi : F \to F\circ\Phi^{-1}$ from $L^2(\mathbb{C}^n, \mu_\lambda)$ onto $L^2(\mathcal{O}(\xi_0), \nu)$ and $\tau_\Psi : F \to F\circ\Psi^{-1}$ from $L^2(\mathbb{R}^{2n}, \tilde{\mu})$ onto $L^2(\mathcal{O}(\xi_0), \nu)$ are unitary and we have $\tau_\Phi = \tau_\Psi j$. By following the same lines as in the proof of Proposition 4.4, we obtain:

Proposition 8.2. *The map $\tilde{\mathcal{W}}_1 := \tau_\Psi W^{-1}$ is a Stratonovich-Weyl correspondence for $(G, \rho, \mathcal{O}(\xi_0))$, the map $\tilde{\mathcal{W}}_2 := \tau_\Phi U_0$ is a Stratonovich-Weyl correspondence for $(G, \pi, \mathcal{O}(\xi_0))$ and we have $\tilde{\mathcal{W}}_1 = \tilde{\mathcal{W}}_2 b$.*

References

[**1**] S. T. Ali and M. Engliš, *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys. **17** (2005), no. 4, 391-490.

[**2**] J. Arazy and H. Upmeier, *Weyl calculus for complex and real symmetric domains*, Harmonic analysis on complex homogeneous domains and Lie

groups (Rome, 2001), Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **13** (2002), no. 3-4, 165-181.

[3] J. ARAZY and H. UPMEIER, *Invariant symbolic calculi and eigenvalues of invariant operators on symmeric domains*, Function spaces, interpolation theory and related topics (Lund, 2000), 151-211, de Gruyter, Berlin 2002.

[4] D. ARNAL, M. CAHEN, S. GUTT and H. ZAHIR, *A Moyal type * product on Hermitian symmetric spaces*, Acad. Roy. Belg. Bull. Cl. Sci. (6) **2** (1991), no. 1-3, 91-103.

[5] L. AUSLANDER and B. KOSTANT, *Polarization and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255-354.

[6] P. BERNAT, N. CONZE, M. DUFLO, M. LÉVY-NAHAS, M. RAÏS, P. RENOUARD and M. VERGNE, *Représentations des groupes de Lie résolubles*, Dunod, Paris 1972.

[7] F. A. BEREZIN, *Quantization*, Math. USSR Izv. **8** (1974), no. 5, 1109-1165.

[8] F. A. BEREZIN, *Quantization in complex symmetric spaces*, Math. USSR Izv. **9** (1975), no. 2, 341-379 (1976).

[9] F. A. BEREZIN and M. A. SHUBIN, *The Schrödinger equation*, Kluwer Academic Publishers Group, Dordrecht 1991.

[10] B. CAHEN, *Contractions of $SU(1,n)$ and $SU(n+1)$ via Berezin quantization*, J. Anal. Math. **97** (2005), 83-102.

[11] B. CAHEN, *Weyl quantization for semidirect products*, Differential Geom. Appl. **25** (2007), 177-190.

[12] B. CAHEN, *Contraction of compact semisimple Lie groups via Berezin quantization*, Illinois J. Math. **53** (2009), no. 1, 265-288.

[13] B. CAHEN, *Berezin quantization on generalized flag manifolds*, Math. Scand. **105** (2009), 66-84.

[14] B. CAHEN, *Stratonovich-Weyl correspondence for compact semisimple Lie groups*, Rend. Circ. Mat. Palermo **59** (2010), 331-354.

[15] B. CAHEN, *Stratonovich-Weyl correspondence for discrete series representations*, Arch. Math. (Brno) **47** (2011), 41-58.

[16] B. CAHEN, *Berezin quantization and holomorphic representations*, Rend. Sem. Mat. Univ. Padova **129** (2013), 277-297.

[17] M. CAHEN, S. GUTT and J. RAWNSLEY, *Quantization on Kähler manifolds, I, Geometric interpretation of Berezin's quantization*, J. Geom. Phys. **7** (1990), 45-62.

[18] J. F. CARIÑENA, J. M. GRACIA-BONDÍA and J. C. VÁRILLY, *Relativistic quantum kinematics in the Moyal representation*, J. Phys. A **23** (1990), 901-933.

[19] M. COMBESCURE and D. ROBERT, *Coherent states and applications in mathematical physics*, Springer, Dordrecht 2012.

[20] H. FIGUEROA, J. M. GRACIA-BONDÍA and J. C. VÁRILLY, *Moyal quantization with compact symmetry groups and noncommutative harmonic analysis*, J. Math. Phys. **31** (1990), 2664-2671.

[21] G. B. FOLLAND, *Harmonic analysis in phase space*, Princeton Univ. Press, Princeton, NJ 1989.

[22] J. M. GRACIA-BONDÍA, *Generalized Moyal quantization on homogeneous symplectic spaces*, Deformation theory and quantum groups with applications to

mathematical physics (Amherst, MA, 1990), 93-114, Contemp. Math., **134**, Amer. Math. Soc., Providence, RI 1992.

[23] J. M. GRACIA-BONDÍA and J. C. VÁRILLY, *The Moyal representation for spin*, Ann. Physics **190** (1989), 107-148.

[24] B. C. HALL, *Holomorphic methods in analysis and mathematical physics*, First Summer School in Analysis and Mathematical Physics (Cuernavaca Morelos, 1998), 1-59, S. Pérez-Esteva and C. Villegas-Blas, eds., Contemp. Math., **260**, Amer. Math. Soc., Providence, RI 2000.

[25] L. HÖRMANDER, *The analysis of linear partial differential operators, III*, Grundlehren Math. Wiss., **274**, Section 18.5, Springer-Verlag, Berlin 1985.

[26] A. A. KIRILLOV, *Lectures on the orbit method*, Graduate Studies in Mathematics, **64**, American Mathematical Society, Providence, RI 2004.

[27] B. KOSTANT, *Quantization and unitary representations, I, Prequantization*, Lectures in modern analysis and applications, III, 87-207, Lecture Notes in Math., **170**, Springer-Verlag, Berlin 1970.

[28] J. LUDWIG, *Dual topology of diamond groups*, J. Reine Angew. Math. **467** (1995), 67-87.

[29] S. LUO, *On the Bargmann transform and the Wigner transform*, Bull. London Math. Soc. **30** (1998), 413-418.

[30] S. LUO, *Polar decomposition and isometric integral transforms*, Integral Transform. Spec. Funct. **9** (2000), no. 4, 313-324.

[31] K.-H. NEEB, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Mathematics, **28**, Walter de Gruyter & Co., Berlin 2000.

[32] B. ØRSTED and G. K. ZHANG, *Weyl quantization and tensor products of Fock and Bergman spaces*, Indiana Univ. Math. J. **43** (1994), no. 2, 551-583.

[33] R. L. STRATONOVICH, *On distributions in representation space*, Soviet Physics. JETP **4** (1957), 891-898.

[34] A. UNTERBERGER and H. UPMEIER, *The Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (1994), no. 3, 563-597.

[35] A. VOROS, *An algebra of pseudodifferential operators and the asymptotics of quantum mechanics*, J. Funct. Anal. **29** (1978), 104-132.

BENJAMIN CAHEN
Université de Metz, UFR-MIM
Département de Mathématiques
LMMAS, ISGMP-Bât. A
Ile du Saulcy 57045, Metz cedex 01, France
e-mail: cahen@univ-metz.fr