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Localization and stability of active scalar flows

Abstract. In this paper we discuss the time evolution of a two-dimensional active scalar flow. We extend to this dynamical system some rigorous results valid for a two-dimensional incompressible nonviscous fluid. In particular we study some characteristics of the dynamics when the initial field is concentrated in N small disjoint regions and we discuss the generalization of the localization. We investigate also the vanishing viscosity limit. Moreover we give some results of stability for the active scalar flow.

Keywords. Quasigeostrophic flow, localization, stability.

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1 - Introduction

In the present paper we discuss some properties of the following dynamical system: let $\theta(x, t)$, $x \in \mathbb{R}^2$ be the solution of the equation

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{1}$$

where

$$u = (u_1, u_2) = (\partial_2 \psi, -\partial_1 \psi), \tag{2}$$

$$\theta = \Lambda^{2-\alpha} \psi, \quad \alpha \in (0, 1), \tag{3}$$

and $\Lambda = \sqrt{-\Delta}$.

We denote this system as *active scalar flow.*

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We assume the boundary condition

$$u \to 0 \;\; as \;\; |x| \to \infty \; . \tag{4}$$

For $\alpha = 0$ we have the Euler equation in two dimensions in which θ has the meaning of vorticity, for $\alpha = 1$ the Surface Quasi-Geostrophic equation (SQG) in which θ has the meaning of temperature. This last system has relevance in geophysics (see for instance [**22**]). In this formulation the active scalar describes the dynamics of a rotating fluid over the *f-plane* with constant potential vorticity (PV) and stratification normalized to one (see the Appendix). This problem was firstly studied by Pierrhaumbert et. al ([**9**], [**23**]) in relation to the spectrum of turbulence in 2-d, beginning the research on $\alpha - turbulence$. On the other hand, from a mathematical point of view, this dynamical system was deeply studied (see [**6**], [**14**]) in relation to the problem of formation of singular fronts. The authors observed a formal analogy with the 3D Euler equation and studied the singular behaviour of the solutions in the more suitable framework of the 2D quasi-geostrophic equation. In this field there are numerous papers devoted to the global regularity and existence of solutions (see for example [**7**], [**12**], [**13**]). In [**8**] the *alpha-patch* model was studied, for $0 < \alpha \leq 1$ in the relation between the stream-function and the active scalar. Another recent generalization of the SQG equation is discussed in [**4**].

We study some mathematical properties of the problem for $\alpha \in (0, 1)$. The case with $\alpha = 1$ will be briefly discussed apart.

The aim of the present paper is to approach the SQG from a new different point of view with respect to the present literature, by studying the evolution of initially strong concentrated fields and extending some rigorous results of the classical point vortex model. The main part of these results are not trivial due to the different Green function for the SQG. We give a complete proof of the localization for strong concentrated active scalars, giving a connection between the SQG model and a system of ODEs with finite degrees of freedom. We also state a useful result of stability remarking the analogy with the two-dimensional Euler equation. Moreover we show that in the limit of point active scalars, we can introduce a dynamical system related to the SQG. To conclude we develop a complete study of the properties of the SQG in the limit of point fields, suggesting further work in this framework.

The plan of the paper is the following. In the next Section we study the localization: the proof of Theorem 2.1 is the main new result of the present paper. In Section 3 we discuss the point active scalar system, in Section 4 we present a stability result, which gives a property valid here and not in fluid mechanics (example at the end of the Section). In Section 5 we treat the vanishing viscosity limit for concentrated fields. Finally we give in the Appendix a short discussion of the physical root of the Surface Quasi-Geostrophic model with $\alpha = 1$.

2 - Localization

We denote by *localization* the following property: sharply concentrated initial data evolve in a concentrated state. For $\alpha = 0$ (two dimensional Euler equation) this property is well known and it gives a rigorous justification of the so-called *point vortex system* (see [**15**], [**18**], [**20**], [**21**]). In the present paper we discuss the case $0 < \alpha < 1$.

It is possible to introduce a weak form of eq.s (1)-(4):

$$\frac{d}{dt}\theta[f] = \theta[u \cdot \nabla f] + \theta[\partial_t f], \tag{5}$$

where $f(x,t)$ is a bounded smooth function and

$$\theta[f] = \int dx\ \theta(x,t) f(x,t)\ . \tag{6}$$

In (5) the velocity field u is given by

$$u(x,t) = \int K(x-y)\theta(y,t)dy, \tag{7}$$

where $K(x-y) = \nabla^{\perp} G(x-y)$, being $\nabla^{\perp} = (\partial_2, -\partial_1)$ and $G(x)$ the Green function of $(-\Delta)^{(1-\alpha/2)}$ with vanishing boundary condition at infinity.

Moreover the divergence-free of the velocity field and the fact that θ is transported by the flow show that the Lebesgue measure and the maximum of θ are conserved during the motion.

We consider an initial datum of the form:

$$\theta_\varepsilon(x,0) = \sum_{i=1}^{N} \theta_{\varepsilon;i}(x,0)\ , \tag{8}$$

where $\theta_{\varepsilon;i}(x,0)$ is a function with a definite sign supported in a region $\Lambda_{\varepsilon;i}$ such that

$$\Lambda_{\varepsilon;i} = supp\ \theta_{\varepsilon;i}(x,0) \subset \Sigma(z_i|\varepsilon)\ ,\ \Sigma(z_i|\varepsilon) \cap \Sigma(z_j|\varepsilon) = 0\ \ if\ \ i \neq j \tag{9}$$

for ε small enough. Here $\Sigma(z|r)$ denotes the circle of center z and radius r.

We take the intensity

$$\int dx\ \theta_{\varepsilon;i}(x,0) \equiv a_i \in \mathbb{R}\ , \tag{10}$$

independent of ε and we assume

$$|\theta_{\varepsilon;i}(x,0)| \leq M\varepsilon^{-\gamma}\ ,\ M > 0\ \ ,\ \gamma > 0\ . \tag{11}$$

We prove the following Theorem:

Theorem 2.1. *Denote by $\theta_\varepsilon(x,t)$ the time evolution of $\theta_\varepsilon(x,0)$ according to the active scalar flow with boundary condition $u \to 0$ as $|x| \to \infty$, then, for any fixed time T, for any $\alpha \in [0,1)$ and $0 \le t \le T$, $\forall d > 0$ $\exists$ $\varepsilon_0(d,T)$ such that, if $\varepsilon < \varepsilon_0$, then $\theta_\varepsilon(x,t) = \sum_{i=1}^{N} \theta_{\varepsilon,i}(x,t)$. Moreover, supp $\theta_{\varepsilon;i}(x,t) \subset \Sigma(z_i(t)|d)$, where $d \to 0$ as $\varepsilon \to 0$ and $z_i(t)$ are the solutions of the ordinary differential equations*

$$\begin{aligned} \dot{z}_i(t) &= \sum_{j=1;i\neq j}^{N} a_i \ \nabla^\perp G(|z_i - z_j|) \ , \ \nabla^\perp = (\partial_2, -\partial_1), \\ z_i(0) &= z_i, \end{aligned} \tag{12}$$

where $G(\,\cdot\,)$ is the Green function of $(-\Delta)^{(1-\alpha/2)}$ with vanishing boundary condition at infinity.

We remark that $T > 0$ must be such that $\forall t < T$ there are no collapses; a complete discussion of the existence of such T is given in Section 3.

We remember that the Green function of $(-\Delta)^{(1-\alpha/2)}$ in $\mathbb{R}^2$ with vanishing boundary conditions at infinity is (see for instance [**11**]):

$$G(r) = \psi(\alpha) r^{-\alpha} \quad , \quad \psi(\alpha) = -\frac{1}{\pi 2^{(2-\alpha)}} \frac{\Gamma(\alpha/2)}{\Gamma\left(\frac{2-\alpha}{2}\right)} \quad , \quad r = \sqrt{x_1^2 + x_2^2} \ , \tag{13}$$

where $\Gamma(\,\cdot\,)$ denotes the Euler Gamma function and $\alpha \in (0,1)$.

From the previous Theorem we obtain immediately the following corollary:

Corollary 2.1. *For any continuous bounded function $f(x)$*

$$\lim_{\varepsilon \to 0} \int dx \ \theta_\varepsilon(x,t) \, f(x) = \sum_{i=1}^{N} a_i f(z_i(t)) \ , \tag{14}$$

where $\theta_\varepsilon(x,t)$ is the time evolution of $\theta_\varepsilon(x,0)$.

In the sequel we call “vortices” these concentrated solutions.

The proof will be given in the sequel. We spend now few words on the interesting case $\alpha = 1$. The proof does not work directly because it is not uniform for $\alpha \to 1$. However the fact that the localization holds for any $\alpha < 1$ gives some suggestions also on the limiting case.

Proof of Theorem 2.1. The proof is quite similar to that discussed in the fluid mechanics case [**18, 20, 21**]. The main new and non trivial part is a sharp bound on the mass of the filaments of active scalar "far" from their center. We study the motion of a single blob of unitary intensity, immersed in an external, divergence-free, Lipschitz vector field $F(x,t)$,

$$|F(x,t) - F(y,t)| \leq L|x-y|. \tag{15}$$

This field $F(x,t)$ simulates the effect of the other $N-1$ concentrated vortices. Indeed we will show that the vortices remain disjoint and concentrated, generating a field with this property.

So the weak form of the active scalar equation is given by

$$\frac{d}{dt}\theta[f] = \theta[(u+F)\cdot\nabla f] + \theta[\partial_t f], \tag{16}$$

and defining the center of vorticity as

$$B_\varepsilon(t) = \int x\ \theta_\varepsilon(x,t)dx, \tag{17}$$

we prove the following auxiliary theorem to show that this single patch remains localized. Then it is simple to come back to the main theorem.

Theorem 2.2. *Suppose that*

$$supp\,|\theta_\varepsilon(x,0)| \subset \Sigma(x^*|\varepsilon) \tag{18}$$

and

$$|\theta_\varepsilon(x,0)| \leq M\varepsilon^{-\gamma}\ ,\ M>0\ \ ,\ \gamma>0, \tag{19}$$

$$\int dx\ \theta_\varepsilon(x,0) = 1, \tag{20}$$

then, there exists $C(\beta,T)>0$, *with* $\beta>0$, *such that for* $0\leq t\leq T$

$$supp\,|\theta_\varepsilon(x,t)| \subset \Sigma(B(t)|d) \tag{21}$$

where

$$d = C(\beta,T)\varepsilon^\beta, \tag{22}$$

and $B(t)$ *is the solution of the ordinary differential equation*

$$\frac{dB(t)}{dt} = F(B(t),t), \tag{23}$$

$$B(0) = x^*. \tag{24}$$

Proof of Theorem 2.2. For semplicity we take $B(0) = x^* = 0$.

First of all we have to show that the main part of the field remains concentrated around the center of vorticity. We introduce the moment of inertia with respect to the center of vorticity:

$$I_\varepsilon(t) = \int (x - B_\varepsilon(t))^2 \theta_\varepsilon(x,t)dx. \tag{25}$$

We study its growth in time. It is simple to prove that if F is null, i.e. having only one patch, $B_\varepsilon(t)$ and $I_\varepsilon(t)$ would be conserved along the motion. So it's clear that

$$\frac{dB_\varepsilon(t)}{dt} = \int F(x,t)\theta_\varepsilon(x,t)dx. \tag{26}$$

Now we give an estimate of the time variation of $I_\varepsilon(t)$ in the case $F \neq 0$,

$$\frac{dI_\varepsilon}{dt} = 2\int (x - B_\varepsilon(t))F(x,t)\theta_\varepsilon(x,t)dx. \tag{27}$$

Noting that

$$\int (x - B_\varepsilon(t)) \cdot F(B_\varepsilon(t),t)\theta_\varepsilon(x,t)dx = 0, \tag{28}$$

we have

$$(27) = 2\int (x - B_\varepsilon(t)) \cdot (F(x,t) - F(B_\varepsilon(t),t)\theta_\varepsilon(x,t))dx, \tag{29}$$

and using the Lipschitz condition we have

$$\frac{dI_\varepsilon(t)}{dt} \leq 2L\int dx\theta_\varepsilon(x,t)(x - B_\varepsilon(t))^2 = 2LI_\varepsilon(t), \tag{30}$$

and finally

$$I_\varepsilon(t) \leq I_\varepsilon(0)e^{2Lt}, \tag{31}$$

which means that

$$\lim_{\varepsilon\to 0} I_\varepsilon(t) = 0 \quad \text{at least as } \varepsilon^2. \tag{32}$$

So the main part of the scalar field is concentrated near the center of vorticity. To state the localization property, we have to prove that the mass of filaments is null. First of all we study the growth of the distance of a fluid particle from the center of vorticity $B_\varepsilon(t)$.

By using (26) and (7) we have that

$$
\begin{aligned}
&\left|\left(u(x,t)+F(x,t)-\frac{dB_\varepsilon(t)}{dt}\right)\cdot\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\right| \\
&\le\left|(F(x,t)-\int\theta_\varepsilon(y,t)F(y,t)dy)\cdot\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\right| \\
&+\left|\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\cdot\int dyK(x-y)\theta_\varepsilon(y,t)\right|,
\end{aligned}
\tag{33}
$$

and using the assumption of unitary intensity the right hand side is

$$
\begin{aligned}
&\left|\int\theta_\varepsilon(y,t)(F(x,t)-F(y,t))dy)\cdot\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\right| \\
&+\left|\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\cdot\int dyK(x-y)\theta_\varepsilon(y,t)\right|.
\end{aligned}
\tag{34}
$$

The first term is obviously estimated by using Lipschitz condition

$$
\begin{aligned}
&\left|\int\theta_\varepsilon(y,t)(F(x,t)-F(y,t))dy)\cdot\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\right|\le const\ R, \\
&R\equiv|x-B_\varepsilon(t)|.
\end{aligned}
\tag{35}
$$

Following [**18**] we estimate the second term dividing the circle $\Sigma(B_\varepsilon(t)|R)$ in different annulii to control the contribution of fluid particles going away from the center of vorticity:

$$
\Sigma(B_\varepsilon(t)|R)=\sum_{k=1}^{k^*}[(\Sigma(B_\varepsilon(t)|a_k)-\Sigma(B_\varepsilon(t)|a_{k-1})\cup(\Sigma(B_\varepsilon(t)|R)-\Sigma(B_\varepsilon(t)|a_{k^*}))],
\tag{36}
$$

with $a_0=0$, $a_1=\varepsilon$, $a_k=2a_{k-1}$. The cut-off k^* is such that $a_{k^*+1}\le R$ and $a_{k^*+2}\ge R$. We give an estimate to the contribution of fluid particles in each ring:

$$
\begin{aligned}
&\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\cdot\int_{D_i}dyK(x-y)\theta_\varepsilon(y,t) \\
&=\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\cdot\int_{D_i}dyK(x-B_\varepsilon(t))\theta_\varepsilon(y,t) \\
&+\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\cdot\int_{D_i}dy[K(x-y)-K(x-B_\varepsilon(t))]\theta_\varepsilon(y,t)
\end{aligned}
\tag{37}
$$

where $D_i\equiv\Sigma(B_\varepsilon(t)|a_k)-\Sigma(B_\varepsilon(t)|a_{k-1})$.

The first term is null because $(x - B_\varepsilon(t)) \cdot K(x - B_\varepsilon(t)) = 0$. Recalling (7) we have

$$|K(x-y) - K(x)| \leq const \, \frac{\rho}{|x|(|x| - \rho)^s}, \tag{38}$$

with $|y| < \rho < \dfrac{|x|}{2}$ and $s = \alpha + 1$. Substituting this result in the second term of (37):

$$\begin{aligned} &\left| \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int_{D_i} dy [K(x-y) - K(x - B_\varepsilon(t))] \theta_\varepsilon(y,t) \right| \\ &\leq const \, \frac{a_k}{R(R-a_k)^s} \int_{D_i} \theta_\varepsilon(y,t) dy, \end{aligned} \tag{39}$$

where

$$\int_{D_i} \theta_\varepsilon(y,t) dy \tag{40}$$

is the vorticity mass in the ring D_i. Defining the vorticity mass outside $\Sigma(B_\varepsilon(t)|R)$,

$$m_t(r) = 1 - \int_{\Sigma(B_\varepsilon(t)|R)} \theta_\varepsilon(y,t) dy, \tag{41}$$

we have the following bound

$$I_\varepsilon \geq r^2 m_t(r). \tag{42}$$

Recalling that the moment of inertia is bounded by a term of order ε^2, we find

$$m_t(r) \leq const \, \frac{\varepsilon^2}{r^2}, \tag{43}$$

so we obtain a bound on the mass of vorticity contained in each ring

$$\int_{D_i} \theta_\varepsilon(y,t) dy \leq const \, \frac{\varepsilon^2}{a_{k-1}^2}. \tag{44}$$

Coming back to (39) we finally have

$$const \, \frac{a_k}{R(R-a_k)^s} \int_{D_i} \theta_\varepsilon(y,t) dy \leq const \, \frac{\varepsilon^2}{a_{k-1}^2} \frac{a_k}{R(R-a_k)^s}. \tag{45}$$

Hence adding the contributions from each ring we obtain the bound on the radial velocity produced by the fluid particles far from the boundary:

$$const \, \frac{\varepsilon}{R^{1+s}}, \quad s \in [1, 2). \tag{46}$$

To complete the proof we have to estimate the contribution of vorticity mass near the boundary of the support.

We introduce a non negative mollifier, the function $W_R(r) \in C^\infty(\mathbb{R}^2)$, defined as

$$W_R(r) = \begin{cases} 1 & \text{if } |r|<R \\ 0 & \text{if } |r| > 2R, \end{cases} \tag{47}$$

such that

$$|\nabla W_R(r)| < \frac{C_1}{R}, \quad C_1 > 0, \tag{48}$$

$$|\nabla W_R(r) - \nabla W_R(r')| < \frac{C_1}{R^2}|r - r'|. \tag{49}$$

To control the vorticity mass near the boundary, we use the following regularized measure

$$\mu_t(R) = 1 - \int dx W_R(x - B_\varepsilon(t))\theta_\varepsilon(x,t). \tag{50}$$

As a matter of fact the regularization by the convolution with the mollifier is such that $\mu_t(R) = 0$ if $supp\,|\theta(x,t)| \subset \Sigma(B_\varepsilon(t)|R)$. So, giving an estimate on the time variation of $\mu_t(R)$, we have a measure of the localization of the active scalar near the center of vorticity.

By explicit calculation we have

$$\begin{aligned} \frac{d\mu_t(R)}{dt} &= -\int dx \nabla W_R(x - B_\varepsilon(t))(u(x,t) + F(x,t) - \frac{dB_\varepsilon(t)}{dt})\theta_\varepsilon(x,t) \\ &= -\int dx\theta_\varepsilon(x,t)\nabla W_R(x - B_\varepsilon(t))\int dy K(x-y)\theta_\varepsilon(y,t) \\ &\quad -\int dx\theta_\varepsilon(x,t)\nabla W_R(x - B_\varepsilon(t))\int (F(x,t) - F(y,t))\theta_\varepsilon(y,t), \end{aligned} \tag{51}$$

and the first integral in the r.h.s. of (51), using the antisymmetry of K, becomes

$$-\frac{1}{2}\int dx \int dy\theta_\varepsilon(x,t)\theta_\varepsilon(y,t)(\nabla W_R(x) - \nabla W_R(y))K(x-y). \tag{52}$$

To give an estimate of this integral, we split the integration domain in these sets

- if $h<n$ $T_h \equiv \{(x,y)|x \notin \Sigma(B_\varepsilon(t)|R), y \in \Sigma(B_\varepsilon(t)|a_h) - \Sigma(B_\varepsilon(t)|a_{h-1})\}$,
- if $h = n$ $T_n \equiv \{(x,y)|x \notin \Sigma(B_\varepsilon(t)|R), y \notin \Sigma(B_\varepsilon(t)|a_{n-1})\}$,
- if $h<n$ $S_h \equiv \{(x,y)|y \notin \Sigma(B_\varepsilon(t)|R), x \in \Sigma(B_\varepsilon(t)|a_h) - \Sigma(B_\varepsilon(t)|a_{h-1})\}$,
- if $h = n$ $S_n \equiv \{(x,y)|y \notin \Sigma(B_\varepsilon(t)|R), x \notin \Sigma(B_\varepsilon(t)|a_{n-1})\}$.

We notice that with this choice the integrand in (52) is null in the complementary of the domain $D \equiv \bigcup_{h=1}^{n} (T_h \cup S_h)$. Beginning from the set T_h, where $\nabla W_R(y) = 0$, we estimate the first term of the r.h.s. of (51) on D by

$$\begin{aligned} &\Bigg| \Bigg[\int_D dy \theta_\varepsilon(x,t)\theta_\varepsilon(y,t)\nabla W_R(x - B_\varepsilon(t))K(x - B_\varepsilon(t)) \Bigg] \\ &+ \Bigg[\int_D dy \theta_\varepsilon(x,t)\theta_\varepsilon(y,t)\nabla W_R(x - B_\varepsilon(t))[K(x-y) - K(x - B_\varepsilon(t))] \Bigg] \Bigg| \end{aligned} \tag{53}$$

where $D \equiv \Sigma(B_\varepsilon(t)|a_h) - \Sigma(B_\varepsilon(t)|a_{h-1})$. The first term is null because $\nabla W_R(x) \cdot K(x) = 0$. The estimate on the second term comes directly from (39), (45) and (46) giving

$$(52) \le const\, \frac{m_t(R)}{R} \frac{\varepsilon}{R^{1+s}} + \sum_{h=2}^{n-1} \frac{a_h}{R(R-a_h)^s} \frac{\varepsilon^2}{a_{h-1}^2} \le \frac{m_t(R)\varepsilon}{R^{2+s}}. \tag{54}$$

To give an estimate on T_n, recalling (49), we note that the integrand in (52) satisfies the following inequality:

$$|(\nabla W_R(x) - \nabla W_R(y))K(x-y)| \le \frac{const}{R^2|x-y|^\alpha}, \tag{55}$$

and substituting in (52) we obtain,

$$(52) \le \frac{const}{R^2} \int dx \int dy \theta_\varepsilon(x,t)\theta_\varepsilon(y,t)(x-y)^{-\alpha}. \tag{56}$$

We finally give an estimate on the last integral rearranging the vorticity mass as close as possible to the singularity, where $y \to x$. We take a circle of radius η around the origin and use the hypothesis that $|\theta_\varepsilon(x,0)| \le M\,\varepsilon^{-\gamma}$.

Being θ conserved along the motion we obtain

$$\left| \int_D \theta_\varepsilon(r,t)|r|^{-\alpha} r dr d\phi \right| \le M\pi\eta^{(2-\alpha)}\varepsilon^{-\gamma}, \tag{57}$$

where η is the radius of the circle $\Sigma(0|\eta)$, such that $m_t(a_{k^*}) = M\pi\eta^2\varepsilon^{-\gamma}$.

Recalling (43), i.e. $m_t(r) \le const \frac{\varepsilon^2}{r^2}$, η is given by:

$$\eta = const\, \frac{1}{\sqrt{M\pi}} \frac{\varepsilon^{1+\frac{\gamma}{2}}}{R}. \tag{58}$$

Finally substituting in (57) we have that

$$(56) \leq const\, m_t\left(\frac{R}{2}\right)\frac{\varepsilon^{(2-\alpha-\frac{\alpha\gamma}{2})}}{R^{4-\alpha}}. \tag{59}$$

The same method can be applied to the subsets S_h and we finally obtain

$$\begin{aligned}&\frac{1}{2}\int dx\int dy\theta_\varepsilon(x,t)\theta_\varepsilon(y,t)(\nabla W_R(x)-\nabla W_R(y))K(x-y)\\&\leq\left[const\,\frac{\varepsilon^{(2-\alpha-\frac{\alpha\gamma}{2})}}{R^{4-\alpha}}+\frac{\varepsilon}{R^{s+2}}\right]m_t\left(\frac{R}{2}\right).\end{aligned} \tag{60}$$

Now we study the second term, in (51). We consider two cases:

- if $|y-B_\varepsilon(t)|>R$

$$\begin{aligned}&\int dx\theta_\varepsilon(x,t)\nabla W_R(x-B_\varepsilon(t))\cdot\int dy\theta_\varepsilon(x,t)(F(x,t)-F(y,t))\\&\leq\frac{const}{R}\int dx\theta_\varepsilon(x,t)\int dy\theta_\varepsilon(x,t)F(x,t),\end{aligned} \tag{61}$$

recalling that $m_t(R)\leq\dfrac{\varepsilon^2}{R^2}$, we have

$$(61)\leq const\|F\|_\infty\frac{m_t(r)\varepsilon^2}{R^3}. \tag{62}$$

- If $|y-B_\varepsilon(t)|\leq R$, using the Lipschitz condition on $F(x,t)$ we find,

$$\int dx\theta_\varepsilon(x,t)\nabla W_R(x-B_\varepsilon(t))\cdot\int dy\theta_\varepsilon(x,t)(F(x,t)-F(y,t))\leq const\, m_t(R). \tag{63}$$

Finally, recollecting all these results, we find the following estimate for the time variation of $\mu_t(R)$,

$$\left|\frac{d\mu_t(R)}{dt}\right|\leq\left[const+const\,\frac{\varepsilon^2}{R^3}+\frac{\varepsilon^{(2-\alpha-\frac{\alpha\gamma}{2})}}{R^{4-\alpha}}+\frac{\varepsilon}{R^{s+2}}\right]m_t(R). \tag{64}$$

Being $\int_{\Sigma(0|\frac{R}{2})}dxW_R(x)\theta_\varepsilon(x,t)\leq\int_{\Sigma(0|R)}\theta_\varepsilon(x,t)$, we can observe that

$$m_t(R)\leq\mu_t(R/2). \tag{65}$$

Using (65) and integrating (64) we have

$$\mu_t(R) \leq \mu_0(R) + A(R) \int_0^t dt \mu_t\left(\frac{R}{2}\right), \tag{66}$$

where $A(R) = const + const\, \frac{\varepsilon^2}{R^3} + \frac{\varepsilon^{(2-\alpha-\frac{\alpha\gamma}{2})}}{R^{4-\alpha}} + \frac{\varepsilon}{R^{s+2}}$.

Now we can use an iterative method,

$$\begin{aligned} &\mu_t(R) \leq \mu_0(R) + A(R) \int_0^t dt \mu_t\left(\frac{R}{2}\right) \\ &\leq \mu_0(R) + \mu_0\left(\frac{R}{2}\right) A(R) \int_0^t dt + A(R) A\left(\frac{R}{2}\right) \int_0^t dt_1 \int_0^{t_1} dt \mu_t\left(\frac{R}{4}\right) + \dots . \end{aligned} \tag{67}$$

We iterate n times, where $n \to \infty$ as $\varepsilon \to 0$. We choose n such that $A(R2^{-k})$ is bounded for all $k \leq n$ and $\mu_0(R2^{-n}) = 0$. Under these constraints we choose

$$R \sim \varepsilon^\beta, \quad \beta \leq \frac{1}{3+\alpha}, \tag{68}$$

$$n = \text{Integer part of } \left[-\frac{1-\beta(s+2)}{s+2}\, log_2\, \varepsilon \right], \text{ with } \varepsilon \in (0,1). \tag{69}$$

Using (65) and iterating n times we find

$$m_t(R) \leq \frac{(const)^n}{n!} \to 0 \quad \text{as } \varepsilon \to 0 \text{ faster than any power in } \varepsilon. \tag{70}$$

This means that the vorticity mass becomes very small near the boundary. To conclude the proof we give an estimate on the velocity field generated by the vorticity mass near the boundary,

$$\left| \int_D dy K(x-y) \theta_\varepsilon(y,t) \right| \leq 1/2\pi \int_D dy \theta_\varepsilon(y,t) |y|^{-\alpha-1}, \tag{71}$$

where $D = \Sigma(B_\varepsilon(t)|R) - \Sigma(B_\varepsilon(t)|a_k^*)$. We use again the arrangement of the vorticity mass near the singularity,

$$\int_D dy \theta_\varepsilon(y,t) |y|^{-\alpha-1} \leq const\; \eta^{(1-\alpha)} \varepsilon^{-\gamma}, \tag{72}$$

being $M\pi\eta^2\varepsilon^{-\gamma} = m_t(a_{k^*})$, and using (70) for which $m_t(a_{k^*}) \to 0$ when $\varepsilon \to 0$, we obtain

$$\left| \int_D dy K(x-y)\theta_\varepsilon(y,t) \right| \to 0 \quad as \quad \varepsilon \to 0, \tag{73}$$

i.e. the velocity generated by the fluid particles near the boundary is small as $\varepsilon \to 0$. Finally we can prove (21), giving a global estimate of the radial velocity of the fluid particles going away from the center of vorticity. Recalling (46) and (70), we have

$$\begin{aligned} \left| \frac{dR}{dt} \right| &\leq const\, R + const \frac{\varepsilon}{R^{s+1}} \\ &+ \text{terms smaller than any power in } \varepsilon \text{ when } R > const\ \varepsilon^\beta. \end{aligned} \tag{74}$$

We have proved that the evolution of a concentrated initial field evolves in a concentrated field, because the radial velocity of the fluid particles is very small. As a matter of fact for $R > const\ \varepsilon^\beta$ the last two terms of the right hand side are negligible and, by using Gromwall Lemma, we obtain (22). □

Remark 2.1. *We note that in general the number of steps in the previous proof depends on the value of* α*, and it goes to infinity when* $\alpha \to 1$.

Moreover we note that if the active scalar has always a definite positive sign, as in the physical model where it represents a temperature field, we can obtain a concentration in average by stopping the iterative procedure at the first step as in [**15**].

We return to the proof of Theorem 2.1. We take the minimal distance between active scalars $R_m >> \varepsilon$. At the beginning we have N separated fields and simulate the influence of other active scalars as an external field. From Theorem 2.2 we find that these fields remain separated. Then it is easy to prove the localization and the convergence (14) in the limit of strong concentration. □

In the next section we prove that it exists a minimum distance between active scalars, in relation to the so called problem of ε-collapses.

3 - Point active scalar system

We can study the point active scalar system (12), interesting by itself behind its physical origin. First we note that it is an Hamiltonian system, with Hamiltonian function H,

$$H = -\frac{1}{2\pi} \sum_{i,j;i\neq j}^{N} a_i a_j |x_i - x_j|^{-\alpha}, \tag{75}$$

and conjugate variables x_{i1} and $a_i x_{i2}$, such that

$$a_i \frac{dx_{i_1}}{dt} = \frac{\partial H}{\partial x_{i_2}}, \tag{76}$$

$$a_i \frac{dx_{i_2}}{dt} = -\frac{\partial H}{\partial x_{i_1}}. \tag{77}$$

The independence of H on time implies the constancy of H. The translation invariance implies the constancy of M, a quantity proportional to the center of vorticity B,

$$M = \sum_{i=1}^{N} a_i x_i = \left(\sum_{i=1}^{N} a_i \right) B\ , \tag{78}$$

the rotation invariance implies the constancy of the moment of Inertia I

$$I = \sum_{i=1}^{N} a_i x_i^2. \tag{79}$$

Furthermore, given the Hamiltonian structure, we could do a qualitative study of the system (76), as in the case of the point vortex system, studying the explicit solution for $N = 2$, the integrability for $N = 3$, regions of integrability for $N = 4$ and regions of chaotic motions.

Here we discuss the existence of the solution for any time for the system of N ODEs (12). This is not trivial because the singularity of the interaction produces a divergence when two point particles are in the same position (collapse). When all the intensities a_i have the same sign and $\alpha > 0$, the conservation of energy forbids any collapse. For $\alpha = 0$ we add the conservation of moment of inertia to forbid any collapse. Otherwise they can happen.

We define ε-collapse as an event in which two vortices arrive at a distance less than ε. We want to give an estimate of the probability of occurence of ε-collapse. We will prove that the Lebesgue measure of initial conditions leading to a collapse in finite time vanishes when $\varepsilon \to 0$.

It is possible to extend to the case $\alpha > 0$ the machinery discussed in [**21**] and to prove that the point particles remain in a bounded region of the plane and the collapses are exceptional. The proof is similar to that discussed in [**21**], with only a different choice of the "Lyapunov" function. We sketch it, to remark the difference. First of all we introduce a regularization of the dynamics defined by (12). The regularized Green function $G_\varepsilon(x) \in C^\infty(\mathbb{R}^2)$ is given by

$$G(x) = G_\varepsilon(x) = |x|^{-\alpha} \quad \text{if } |x| > \varepsilon, \tag{80}$$

$$G_\varepsilon(x) \leq G(x) \quad \forall x \in \mathbb{R}^2, \tag{81}$$

and there exists a constant B_1, independent of ε such that

$$\left|\frac{dG_\varepsilon(x)}{dx}\right| < \frac{B_1}{|x|} \quad \forall x \in \mathbb{R}. \tag{82}$$

We obtain a regularized dynamics, defined by the following Cauchy problem for a system of N ODEs

$$\frac{dx_i^\varepsilon(t)}{dt} = \nabla_i^\perp \sum_{j=1;j\neq i}^{N} a_j G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t)), \tag{83}$$

$$x_i^\varepsilon(t=0) = x_i. \tag{84}$$

It is clear that the dynamics defined by (83) is again Hamiltonian and it coincides with (12) up to the first ε-collapse.

We want to prove that the measure of the initial condition leading to an ε-collapse in (83), is infinitesimal in ε.

Following [**21**] an intermediate result on the boundedness property of the dynamics of N concentrated fields, is given by the following theorem

Theorem 3.1. *Given a system of N concentrated fields of different signs such that* $\sum_{i\in P(N)} a_i \neq 0$, *where P(N) is the family of all subsets of the first N integers, there exists a constant C depending on N, T and* a_i, *independent of* ε *and the initial condition, such that:*

$$\max_{i=1,\dots,N} \sup_{0\le t\le T} |x_i^\varepsilon(t) - x_i| \le C, \tag{85}$$

where $x_i^\varepsilon(t)$ *is the solution of the Cauchy problem given by* (83).

We omit the complete proof, it is fundamentally the same proved in [**21**], using as Lyapunov function the quantity

$$\sum_{i,j=1,i\neq j}^{N} G_\varepsilon|x_i - x_j|. \tag{86}$$

We have to notice that the main constraint to state this theorem is that

$$\sum_{i\in P(N)} a_i \neq 0,$$

generally it is not true for all choices of signs of a_i. Moreover, a corollary of this theorem is

Corollary 3.1. *If* $\sum_{i \in P(N)} a_i \neq 0$, *for all* R *and* T, *there exists* $R^*(T)$ *for which* N *concentrated active scalars, initially contained in* $\Sigma(0|R^*)$ *cannot leave* $\Sigma(0|R^*(T))$ *for all initial data and* $\varepsilon \in (0,1)$, *where* $\Sigma(0|r)$ *denotes the circle of radius* r *around the origin.*

Through this result of boundedness and the invariance of the Lebesgue measure under the flow generated by (83), we can state the following theorem

Theorem 3.2. *Being* $\lambda(dX) = \dfrac{dx_1, \dots, dx_N}{(\pi R^2)^N}$ *the normalized Lebesgue measure on* Σ_R^N, *where* $X = \{x_1, \dots, x_N\}$ *is the set of initial conditions. Then:*

$$\lim_{\varepsilon \to 0} \lambda(\{X | d_T^\varepsilon(X) < \varepsilon\}) = 0, \tag{87}$$

where

$$d_T^\varepsilon(X) = \min_{i \neq j} \inf_{0 \leq t \leq T} |x_i^\varepsilon(t) - x_j^\varepsilon(t)|. \tag{88}$$

Hence the collapses are exceptional.

4 - A stability result

We want to extend to the active scalar case the stability results stated in [**19**] for the incompressible inviscid fluid in the plane, where the authors found sufficient conditions for nonlinear stability in norm L_1 on the whole plane for a vast class of initial conditions, including patches and non-smooth functions.

It is well known that the classical Arnol'd stability results [**1**] need the presence of a boundary and so it can not be applied in the whole plane (for a review see [**10**], [**21**]). Vice versa results of [**19**] remain valid in our case. They are related to the symmetries of the problem, i.e. to the constancy of the moment of inertia I

$$I = \int x^2 \theta(x,t) dx \ . \tag{89}$$

In general in an infinite dimensional space the definition of stability depends on the norms we choose. In our case we adopt the L_1 norm and we state the following

Theorem 4.1. *Let be* $\bar{\theta} \in L_\infty$ *the stationary solution of the active scalar equation such that it is a monotone non increasing function, with* $|\bar{\theta}(r)| < const\ r^{-\gamma}$, *where* $\gamma > 4,\ r = |x|$. *Consider a perturbation in the familiy of isorotational perturbations*

$$P = \Big\{\ \theta \in L_\infty,\ |\theta| < const\ r^{-\gamma},\ \gamma > 4|\forall\lambda, \\ \mu\{(x,y)|\theta(x,y) > \lambda)\} = \mu\{(x,y)|\bar\theta(x,y) > \lambda)\}\ \Big\} \tag{90}$$

where $\mu(\,\cdot\,)$ *is the Lebesgue measure.*

Then $\bar\theta$ *is stable in* L_1 *with respect to perturbations* $\theta \in P$, *i.e.* $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ *such that if*

$$\theta_0 \in P \quad and \quad \|\bar\theta - \theta_0\|_1 < \delta_\varepsilon, \tag{91}$$

then

$$\sup_{t\geq 0}\ \|\bar\theta - \theta_t\|_1 < \varepsilon, \tag{92}$$

where θ_t *is the time evolution of* θ_0 *according to the active scalar flow.*

The proof is similar to that discussed in [**19**] and we do not write it explicitly. The main idea of the proof is that any stationary monotone solution $\bar\theta$ of the active scalar flow is an absolute minimum for the functional I in the class of all isorotational fields $\theta \in P$. We remark that the result formally holds in the geophysical case $\alpha = 1$.

From a physical point of view this result is interesting. For $\alpha = 0$ this is a classical, already applied, result of stability of oceanic vortices (see for example [**3**]). For $\alpha = 1$ it states that a hot patch of active scalar field dipped in a cold background remains stable in average.

4.1 - *Example: stability of sharp fronts*

Recent works by Rodrigo et al. (see for example [**24**]) are devoted to the analysis of the evolution of sharp fronts for the Surface Quasi-Geostrophic equation. They were interested in the evolution of a periodic front, given by a smooth periodic function $\phi(x)$. This means that the initial condition on the scalar function $\theta(x,y)$ is given by

$$\begin{cases} \theta(x,y) = 1 & y \geq \phi(x) \\ \theta(x,y) = 0 & y < \phi(x). \end{cases} \tag{93}$$

The physical meaning of this condition is the following: the temperature takes two different values in complementary domains, modelling the evolution of a sharp front. We can treat the stability problem of such model. The Green function in the periodic domain conserves the behavior in the origin and at infinity of the Green function in the plane (as we can verify by a symmetric summation on the periodic strips of the domain). We can prove again the nonlinear stability, in the L_1 sense, of such condition by using the translational symmetry of the problem in the y direction. We observe that the variation of the height of the center of mass is a constant of motion, which has a minimum in the stationary state. Moreover the time evolution is well defined for $0 < \alpha \leq 1$, so the state is stable in average. Here we notice that this result is valid for all $\alpha > 0$, including the case $\alpha = 1$ of the Surface Quasi-Geostrophic Equation. However this is not still true for the case $\alpha = 0$ of the Euler equation, because in this case the Green function decays too slowly at infinity.

5 - Limit of vanishing viscosity for concentrated scalar active fields

In recent works (see for example [**5**]), in relation to the Surface Quasi-Geostrophic equation, it was studied the *Dissipative Quasi-Geostrophic Equation*, a family of equation for an active scalar, where the dissipative term is given by a fractional power of the Laplacian. These models are defined by

$$\partial_t\theta + (u \cdot \nabla)\theta + \nu(-\Delta)^{\lambda}\theta = 0 \quad \lambda \in [0,1],\ \nu > 0. \tag{94}$$

In ref. [**25**] was studied the inviscid limit for such models, with smooth initial condition. In this paper we treat the inviscid limit for strong concentrated field in the special case $\lambda = 1$ of Dissipative Quasi-Geostrophic Equation.

So we study the problem when the initial data are localized and the viscosity vanishes. We find the following

Theorem 5.1. *Consider the initial condition* $\theta_\varepsilon(x,0) = \sum_{i=1}^{N} \theta_{\varepsilon;i}(x,0)$ *where* $\theta_{\varepsilon;i}(x,0)$ *is a function with a definite sign supported in a region* $\Lambda_{\varepsilon;i}$ *such that*

$$\Lambda_{\varepsilon;i} = supp\ \theta_{\varepsilon;i}(x,0) \subset \Sigma(z_i|\varepsilon)\ ,\ \Sigma(z_i|\varepsilon) \cap \Sigma(z_j|\varepsilon) = 0\ if\ i \neq j, \tag{95}$$

for ε *small enough. Being* $\theta_\varepsilon(x,t)$ *the time evolution of* $\theta_\varepsilon(x,0)$ *according to*

$$\partial_t\theta + (u \cdot \nabla)\theta = \nu\Delta\theta, \quad \nu > 0, \tag{96}$$

given δ *and* ν_0 *any real positive numbers such that* $\nu \leq \nu_0\varepsilon^\delta$ *then for all fixed* T *smaller than the first collapse time and* $0 \leq t \leq T$,

$$\theta(x,t) \to \sum_{i=1}^{N} a_i\delta_{z_i(t)} \quad for\ \varepsilon \to 0, \tag{97}$$

where the point centers $z_i(t)$ move as:

$$\frac{dz_i}{dt} = \sum_{j=1;i\neq j}^{N} \nabla^{\perp} G(|z_i - z_j|), \nabla^{\perp} = (\partial_2, -\partial_1), \tag{98}$$

$$z_j(0) = z_i, \tag{99}$$

and $G(z)$ is the Green function of $(-\Delta)^{(1-\alpha/2)}$ with vanishing boundary condition at infinity. If we assume that all the active scalars have the same sign, then we find the same result making the two limits (strong concentration and vanishing viscosity) going to zero indipendently.

The last statement can be achieved in analogy to the result in [**16**]. We obtain the same limiting dynamical system as in Section 2. However the result is not trivial because the viscosity perturbation is singular, and a priori it could perturb the limit.

We do not give the explicit proof: it is a combination of the proof of Section 2 and the technique explained in [**17**].

Appendix. Surface Quasi-geostrophic equation: a derivation

In this short appendix we recall the physical root of our dynamical system, recalling the hypothesis of derivation of the Surface Quasi-Geostrophic Equation (SQG) firstly studied by Held et al. ([**23**]) for the 2D spectrum of turbulence and by Majda et al. ([**6**]) for the formation of singular fronts. The large scale dynamics of air mass in mid latitudes of terrestrial atmosphere, is often treated by using the quasi-geostrophic approximation (see [**22**]). The constitutive equations of this model, in the *f-plane* approximation, are the following:

$$\left(\partial_t + u_1\partial_x + u_2\partial_y\right)q = 0, \tag{100}$$

$$\left(\partial_t + u_1\partial_x + u_2\partial_y\right)\theta = s(z)u_3, \tag{101}$$

where

$$q = \nabla_H^2\psi + \partial_z\left(\frac{1}{s(z)}\partial_z\psi\right), \quad \nabla_H^2 = \partial_{xx} + \partial_{yy},$$

is the potential vorticity (PV), ψ is the stream function, θ the potential temperature, $u = (u_1, u_2, u_3)$ the velocity field,

$$s(z) \propto N^2(z),$$

being $N^2(z)$ the Brunt-Vaisala frequency (see [22]). In the quasi-geostrophic approximation, the stream function can be identified with the pressure field. Moreover, the vertical derivative of the pressure field is proportional to the potential temperature, i.e.

$$\partial_z \psi \propto \theta.$$

The first equation is the conservation of potential vorticity (PV) and the second is the conservation of energy. These are the constitutive equations of the Eady model, the complete derivation from the equations of an inviscid rotating fluid is in [22]. Assuming that the potential vorticity is initially constant (for semplicity null), we have from the conservation of PV

$$q = \nabla_H^2 \psi + \partial_z \left(\frac{1}{s(z)} \partial_z \psi \right) = 0. \tag{102}$$

If s is a constant normalized to one, we obtain the classical Laplace equation

$$\Delta \psi = 0. \tag{103}$$

Recalling for example [2], it is well known that the fractional Laplacian is the operator that maps Dirichlet boundary condition to the Neumann condition. Consider the following Dirichlet problem for the Laplace equation

$$\Delta \psi(x, y) = 0, \quad x \in \mathbb{R}^n, y > 0, \tag{104}$$

$$\psi(x, 0) = f(x), \quad x \in \mathbb{R}^n, \tag{105}$$

where f is a bounded smooth function. Then, $T = (-\Delta)^{1/2}$ maps Dirichlet condition into Neumann condition:

$$(-\Delta)^{1/2} f(x) = -\partial_y \psi(x, 0). \tag{106}$$

The proof is simple ([2]): applying T to the Dirichlet condition we obtain:

$$T(f)(x) = -\partial_y \psi(x, 0), \tag{107}$$

finding $-\partial_y \psi(x, y)$ as the solution of (104)-(105). Applying twice T to the boundary conditions we have:

$$T(T(f))(x) = T(-\partial_y \psi(x, 0))(x) = \partial_{yy} \psi(x, 0) = -\Delta_x f(x), \tag{108}$$

so, being $T^2 = -\Delta$, then $T = \sqrt{-\Delta}$.

Coming back to our original problem, we recall that $\theta = \partial_z \psi$. Finally neglecting vertical motions, i.e. taking $u_3 = 0$ in (101), we obtain the Surface Quasi-Geostrophic Equation (SQG) by a simple formal manipulation

$$\partial_t \theta + (u \cdot \nabla) \theta = 0, \quad u = (u_1, u_2) = (\partial_2 \psi, -\partial_1 \psi), \tag{109}$$

$$\theta = (-\nabla_H^2)^{1/2} \psi. \tag{110}$$

Finally we notice that our model is a generalization of this equation, initially studied by Held et al ([**23**]) in the framework of the 2D spectrum of turbulence (actually named α – *turbulence*).

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