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A study of the NS-$\overline{\boldsymbol{w}}$ model of turbulence

Abstract. In this paper, the NS-$\overline{\boldsymbol{w}}$ model with periodic boundary conditions is studied. This model is derived from the rotational Navier-Stokes equations by regularizing with an explicit spatial filter of width α the second term of the nonlinearity. It is first shown that the regular solution for NS-$\overline{\boldsymbol{w}}$ system verifies a sequence of energy inequalities called "ladder inequalities". These ladder inequalities give rise to series of time-averaged inverse square length-scales. These latter quantities are estimated in terms of the Reynolds number. Moreover, it is shown that the NS-$\overline{\boldsymbol{w}}$ model follows the usual $\kappa^{-\frac{5}{3}}$ Kolmogorov power law spectrum for wavenumbers smaller than $1/\alpha$ in the inertial range. However, this model has a steeper power law spectrum for wavenumbers greater than $1/\alpha$. Finally, the relation between the NS-$\overline{\boldsymbol{w}}$ model and the Navier-Stokes equations is discussed by proving a convergence theorem as the length scale α tends to zero.

Keywords. Turbulence models, Regularity, Navier-Stokes equations.

Mathematics Subject Classification (2010): 76B03, 76F05, 76D05, 35Q30.

1 - Introduction

This paper is devoted to the mathematical and physical study of the NS-$\overline{\boldsymbol{w}}$ model of turbulence

$$(1)\qquad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \overline{\boldsymbol{w}} - \nu \Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ -\alpha^2 \Delta \overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} = \boldsymbol{u} \qquad \text{in } \mathbb{T}_3, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \overline{\boldsymbol{u}} = 0, \\ \oint_{\mathbb{T}_3} \boldsymbol{u} = \oint_{\mathbb{T}_3} \overline{\boldsymbol{u}} = 0, \\ \boldsymbol{u}_{t=0} = \boldsymbol{u}^{in}. \end{cases}$$

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In equation (1), $\overline{\boldsymbol{w}} = \nabla \times \overline{\boldsymbol{u}}$ denotes the filtered averaged vorticity. The boundary conditions are periodic boundary conditions. Therefore, these equations are considered on the three dimensional torus $\mathbb{T}_3 = \left(\mathbb{R}^3/\mathcal{T}_3\right)$ where $\mathcal{T}_3 = \mathbb{Z}^3/L$, $\boldsymbol{x} \in \mathbb{T}_3$, and $t \in]0, +\infty[$. The unknowns are the velocity vector field $\boldsymbol{u}$ and the scalar Bernoulli or the dynamic pressure $P = p + \frac{1}{2}\boldsymbol{u}^2$. The viscosity ν, the initial velocity vector field $\boldsymbol{u}^{in}$, and the external force $\boldsymbol{f}$ with $\nabla \cdot \boldsymbol{f} = 0$ are given. In this paper, the force $\boldsymbol{f}$ does not depend on time.

As in [**3, 4, 8, 11, 12, 18, 24, 26**], the Helmholtz operator is used in the definition of the variable $\overline{\boldsymbol{w}}$. To obtain the NS-$\overline{\boldsymbol{w}}$ model, the nonlinear term $\overline{\boldsymbol{u}} \times \boldsymbol{w}$ in the Navier-Stokes-alpha (NS-α) model [**8, 7**] is replaced by $\boldsymbol{u} \times \overline{\boldsymbol{w}}$. The NS-$\alpha$ model has attracted much interest since it conserves the helicity [**26**], the Kelvin circulation theorem [**14**] and it is frame indifferent [**16**].

The model described by the system (1) has been introduced in [**23**] for numerical simulation purposes. The author in [**21**] proves the existence of a global attractor $\mathcal{A}$ to this model. In [**25**], the authors prove the global existence, uniqueness and the higher-order regularity of the solution similarly to the other alpha models [**3, 4, 8, 12, 18, 24, 26**].

By using the following identities [**16**],

$$\overline{\boldsymbol{w}} \times \boldsymbol{u} = \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{u}} - (\nabla \overline{\boldsymbol{u}})^T \boldsymbol{u} \tag{2}$$

$$\nabla(\boldsymbol{u} \cdot \overline{\boldsymbol{u}}) = (\nabla \overline{\boldsymbol{u}})^T \boldsymbol{u} + (\nabla \boldsymbol{u})^T \overline{\boldsymbol{u}}, \tag{3}$$

the model (1) can be written in the following equivalent form

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{u}} + (\nabla \boldsymbol{u})^T \overline{\boldsymbol{u}} - \nu \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ -\alpha^2 \Delta \overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} = \boldsymbol{u} \qquad \text{in } \mathbb{T}_3, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \overline{\boldsymbol{u}} = 0, \\ \displaystyle\oint_{\mathbb{T}_3} \boldsymbol{u} = \oint_{\mathbb{T}_3} \overline{\boldsymbol{u}} = 0, \\ \boldsymbol{u}_{t=0} = \boldsymbol{u}^{in}, \end{cases} \tag{4}$$

where $\pi = P - \boldsymbol{u} \cdot \overline{\boldsymbol{u}}$. Thus the system (4) is a perturbation of the Modified Leray-alpha model of turbulence which was introduced in [**18**]. This paper extends the work of [**1**] where the Modified Leray-alpha is studied and a sequence of energy inequalities for the unique regular solution of the Modified Leray-alpha is established. These energy inequalities are called “ladder inequalities”.

The ladder inequalities are studied first for the Navier-Stokes equations in [**6**] where the authors showed that for any C^∞ solution $(\boldsymbol{u}, p)$ to the (NSE), the velocity part $\boldsymbol{u}$ satisfies the following relation between its higher derivatives,

$$
(5)\quad \begin{aligned}
&\frac{1}{2}\frac{d}{dt}H_N \leq -\nu H_{N+1} + C_N H_N \|\nabla \boldsymbol{u}\|_\infty + H_N^{1/2} {\boldsymbol{\Phi}_N}^{1/2},\\
&\text{where } H_N = \int_{\mathbb{T}_3} \left|\nabla^N \boldsymbol{u}\right|^2 d\boldsymbol{x} \text{ and } \boldsymbol{\Phi}_N = \int_{\mathbb{T}_3} |\nabla^N \boldsymbol{f}|^2 d\boldsymbol{x}.
\end{aligned}
$$

These differential inequalities are used in [**6**] to show the existence of a lower bound on the smallest scale in the flow. Recently, the ladder inequalities have been used to study the intermittency of solutions to the Navier-Stokes equations [**13**]. The ladder inequalities are generalized to other alpha models in [**14, 15**]. The ladder inequalities for the Navier-Stokes equations are based on the assumption that a solution exists such that the higher order norms are finite. Note that, there is no such needed assumption in the case of the alpha regularization (1) where existence and uniqueness of a C^∞ solution are showed in [**25**]. The purpose of this paper is to study ladder inequalities for model (1) which is a continuation of Theorem 3.4 in [**25**].

1.1 - *Notations*

Let $L^p(\mathbb{T}_3)$ and $W^{s,p}(\mathbb{T}_3)$, $s \geq -1$, $1 \leq p \leq \infty$ be the usual Lebesgue and Sobolev spaces over $\mathbb{T}_3$. Bochner spaces $C(0,T;X), L^p(0,T;X)$ are defined in the standard way. To simplify notations, let $\boldsymbol{L}^p = L^p(\mathbb{T}_3)^3$, $\boldsymbol{H}^s = W^{s,2}(\mathbb{T}_3)^3$ and the subscript σ refers to divergence-free and zero-mean functions.

In the whole paper, $\alpha > 0$ is given. The expression $A \lesssim B$ means $A \leq cB$ where c is a constant. Such a constant c may depend on α, ν and L; and may blow-up when α tends to zero.

1.2 - *Main results*

One of the main results of this paper is:

Theorem 1.1. *Assume* $\boldsymbol{f} \in C^\infty(\mathbb{T}_3)^3$ *and* $\boldsymbol{u}^{in} \in C^\infty(\mathbb{T}_3)^3$ *such that* $\nabla \cdot \boldsymbol{u}^{in} = 0$. *Let* $(\boldsymbol{u}, p) := (\boldsymbol{u}^\alpha, p^\alpha)$ *be the unique solution to problem* (1). *Then the velocity part* $\boldsymbol{u}$ *satisfies the ladder inequalities,*

$$
(6)\quad \begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\overline{H_N} + 2\alpha^2 \frac{d}{dt}\overline{H_{N+1}} + \alpha^4 \frac{d}{dt}\overline{H_{N+2}}\right) \leq -\nu\Big(\overline{H_{N+1}} + 2\alpha^2\overline{H_{N+2}} + \alpha^4\overline{H_{N+3}}\Big)\\
&\quad + C_N\|\nabla\overline{\boldsymbol{u}}\|_\infty\big(\overline{H_N} + 2\alpha^2\overline{H_{N+1}} + \alpha^4\overline{H_{N+2}}\big) + {\overline{H_N}}^{1/2}{\boldsymbol{\Phi}_N}^{1/2} + \alpha^2{\overline{H_{N+1}}}^{1/2}{\boldsymbol{\Phi}_{N+1}}^{1/2},
\end{aligned}
$$

where

(7) $$\overline{H_N} = \int_{\mathbb{T}_3} \left|\nabla^N \overline{\boldsymbol{u}}\right|^2 d\boldsymbol{x}, C_0 = 0 \textit{ and } C_N \simeq 2^N \textit{ for all } N \geq 1.$$

The gradient symbol ∇^N here refers to all derivatives of every component of $\boldsymbol{u}$ of order N in $L^2(\mathbb{T}_3)$.

Remark 1. As $\alpha \to 0$, $\overline{H_N} \to H_N$, thus the inequality (5) is obtained.

Another task of this paper is to estimate quantities of physical relevance in terms of the Reynolds number Re (see (16) below). Since it is important to compare these estimates with other estimates for different alpha models from [**1, 13, 14**], these results are organized in the following Table 1, that is organized as follows.

Table 1. *Comparison of various upper bounds for the Navier-Stokes-α, Bardina, Leray-α and Modified Leray-α from* [**1, 14, 15**] *and* NS-$\overline{\boldsymbol{w}}$ *with constant omitted.*

	NS-α / Bardina	ML-α	Leray-α / NS-$\overline{\boldsymbol{w}}$	Eq.
$\ell\lambda_k^{-1}$	$Re^{5/8}$	$Re^{5/8}$	$Re^{7/12}$	(74)
$\langle \overline{H_1} \rangle$	$Re^{5/2}$	$Re^{5/2}$	$Re^{7/3}$	(72)
$\langle \overline{H_2} \rangle$	Re^3	Re^3	$Re^{8/3}$	(67)
$\langle \overline{H_3} \rangle$	- / -	Re^7	Re^3	(65)
$\ell^2 \langle \kappa_{N,r}^2 \rangle$	$Re^{11/4}$	$Re^{5/2}$	$Re^{17/12}$	(77)
$\ell^2 \langle \kappa_{1,0}^2 \rangle$	$Re \ln Re$	$Re \ln Re$	$Re \ln Re$	(63)
$\langle \Vert\overline{\boldsymbol{u}}\Vert_\infty^2 \rangle$	$Re^{11/4}$	$Re^{11/4}$	$Re^{5/2}$	(75)
$\langle \Vert\nabla\overline{\boldsymbol{u}}\Vert_\infty \rangle$	$Re^{35/16}$	$Re^{5/2}$	$Re^{17/12}$	(76)
$\ell^2 \langle \kappa_{N,0}^2 \rangle$	$Re^{\frac{11}{4}-\frac{7}{4N}}(\ln Re)^{\frac{1}{N}}$	$Re^{\frac{5}{2}-\frac{3}{2N}}(\ln Re)^{\frac{1}{N}}$	$Re^{\frac{17}{12}-\frac{5}{12N}}(\ln Re)^{\frac{1}{N}}$	(79)

The Estimates in terms of Reynolds number have been given for related models, including the Navier-Stokes-α, Bardina, Leray-α [**14, 15**] and Modified Leray-α [**1**]. To be consistent with these related studies the same exposition is used in this paper.

For simplicity the eqs. (1) will be considered with forcing $\boldsymbol{f}(\boldsymbol{x})$ to be taken in L^2 of narrow band type with a single length scale ℓ, (see [**13, 14**]) where $0<\ell<L$, such that

(8) $$\Vert\nabla^n \boldsymbol{f}\Vert_{L^2} \simeq \ell^{-n}\Vert\boldsymbol{f}\Vert_{L^2}.$$

The Grashof number Gr, defined below in terms of the forcing, has been used to express the most important estimates in Navier-Stokes theory. In order to compare these estimates with the results of Kolmogorov scaling theories [**10**], it is important to express these estimates in terms of Reynolds number Re_{NS} based on the Navier-

Stokes velocity $\boldsymbol{u}$. A good definition of this Reynolds number is

$$Re_{\mathcal{NS}} = \frac{U_{\mathcal{NS}}\ell}{\nu}, \quad U^2_{\mathcal{NS}} = L^{-3}\langle \|\boldsymbol{u}\|^2_{L^2}\rangle, \tag{9}$$

where $\langle\cdot\rangle$ is the long time average

$$\langle g(\,\cdot\,)\rangle = \mathrm{Lim}_{t\to\infty} \frac{1}{t}\int_0^t g(s)ds. \tag{10}$$

Where Lim indicates a generalized limit that extends the usual limits [**9**].

With $f_{rms} = L^{-3/2}\|\boldsymbol{f}\|_{L^2}$, the standard definition of the Grashof number in three dimensions is

$$Gr = \frac{\ell^3 f_{rms}}{\nu^2}. \tag{11}$$

Doering and Foias [**5**] have addressed the problem of how to relate $Re_{\mathcal{NS}}$ to Gr. They have shown that in the limit $Gr \to \infty$, solutions of the Navier-Stokes equations must satisfy

$$Gr \lesssim Re^2_{\mathcal{NS}} + Re_{\mathcal{NS}}. \tag{12}$$

Using the above relation (12), Doering and Gibbon [**13**] have reexpressed some Navier-Stokes estimates in terms of $Re_{\mathcal{NS}}$. In particular, they showed that the energy dissipation rate $\varepsilon_{\mathcal{NS}} = \nu\langle\|\nabla\boldsymbol{u}\|^2_{L^2}\rangle L^{-3}$ is bounded above by

$$\varepsilon_{\mathcal{NS}} \le c\nu^3\ell^{-4}\big(Re^3_{\mathcal{NS}} + Re_{\mathcal{NS}}\big), \tag{13}$$

and the inverse Kolmogorov length $\lambda^{-1}_{k,\mathcal{NS}} = (\varepsilon_{\mathcal{NS}}/\nu^3)^{1/4}$ is bounded above by

$$\ell\lambda^{-1}_{k,\mathcal{NS}} \le cRe^{3/4}_{\mathcal{NS}}. \tag{14}$$

The relation (12) is essentially a Navier-Stokes result. In [**14**, Appendix A.1], it has been shown that this property holds for the Navier-Stokes-alpha model [**7**].

In this paper, the following relation between Gr and the Reynolds number Re is established, where Re is based on the smoothed velocity $\boldsymbol{u}$.

Proposition 1.2. *Let $\boldsymbol{f} \in C^\infty(\mathbb{T}_3)^3$ be of narrow-band type and $\boldsymbol{u}^{in} \in C^\infty(\mathbb{T}_3)^3$ such that $\nabla\cdot\boldsymbol{u}^{in} = 0$. Let $\boldsymbol{u} := \boldsymbol{u}^\alpha$ be the velocity part of the solution to problem* (1). *Then*

$$Gr \lesssim Re^2 + Re, \tag{15}$$

where

$$Re = \frac{U\ell}{\nu}, \ \{U^2 = L^{-3}\Big\langle\|\boldsymbol{u}\|^2_{L^2}\Big\rangle = L^{-3}\Big\langle\|\overline{\boldsymbol{u}}\|^2_{L^2} + 2\alpha^2\|\nabla\overline{\boldsymbol{u}}\|^2_{L^2} + \alpha^4\|\Delta\overline{\boldsymbol{u}}\|^2_{L^2}\Big\rangle. \tag{16}$$

In order to estimate small length scales associated with higher order moments, the force $\boldsymbol{f}$ is combined with the higher derivative of the velocity [**6, 13, 14**], such that

$$L_N = \overline{F_N} + 2\alpha^2 \overline{F_{N+1}} + \alpha^4 \overline{F_{N+2}}, \tag{17}$$

where

$$\overline{F_N} = \overline{H_N} + \tau^2 \Phi_N, \tag{18}$$

the quantity $\overline{F_N}$ is different from zero and the quantity τ is defined by

$$\tau = \ell^2 \nu^{-1} (Gr \ln Gr)^{-1/2}, \tag{19}$$

where τ has the dimension of time and it is chosen in this way in order to ensure that the additional forcing term in (18) becomes negligible with respect to $\overline{H_N}$ when $Gr \to \infty$ (compare [**14,** Appendix A.2]). The L_N is used to define a set of time-dependent inverse length scales

$$\kappa_{N,r} = \left(\frac{L_N}{L_r}\right)^{\frac{1}{2(N-r)}}. \tag{20}$$

The following result is a consequence of Theorem 1.1 and (15).

Theorem 1.3. *As $Gr \to \infty$, for $N \geq 1$, $1 \leq p \leq N$ the unique solution to eqs.* (1) *satisfies*

$$\frac{1}{2}\frac{d}{dt} L_N \leq -\frac{\nu}{3} \frac{L_N^{1+\frac{1}{p}}}{L_{N-p}^{\frac{1}{p}}} + C_{N,\alpha} \|\nabla \overline{\boldsymbol{u}}\|_\infty L_N + C\nu\ell^{-2} Re(\ln Re) L_N \tag{21}$$

and for $N = 0$,

$$\frac{1}{2}\frac{d}{dt} L_0 \leq -\nu L_1 + C\nu\ell^{-2} Re(\ln Re) L_0 \,. \tag{22}$$

The second main result of the paper is the following Theorem and its proof follows closely to that of the Leray-alpha model in [**15**].

Theorem 1.4. *Let $\boldsymbol{f} \in C^\infty(\mathbb{T}_3)^3$ be of narrow-band type and $\boldsymbol{u}^{in} \in C^\infty(\mathbb{T}_3)^3$ such that $\nabla \cdot \boldsymbol{u}^{in} = 0$. Let $\boldsymbol{u} := \boldsymbol{u}^\alpha$ be the velocity part of the solution to problem* (1). *Then estimates in term of Reynolds number Re for the length scales associated with higher order moments solution $\kappa_{N,0}$ ($N \geq 2$), the inverse Kolmogorov length λ_k and the energy dissipation rate ε are given by*

$$\ell^2 \langle \kappa_{N,0}^2 \rangle \leq C(\alpha, \nu, \ell, L)^{(N-1)/N} Re^{17/12 - 5/12N} (\ln Re)^{1/N} + C\, Re \ln Re, \tag{23}$$

$$\ell\lambda_k^{-1} \lesssim Re^{7/12}, \tag{24}$$

$$\varepsilon \lesssim Re^{7/3}. \tag{25}$$

Where $\langle\cdot\rangle$ *is the long time average as defined above* (10).

The paper is organized as follows: Section 2 is devoted to prove Theorem 1.1. Section 3 establishes Proposition 1.2 that relates Gr to Re. This relation is used in Section 4 in order to prove Theorem 1.3. Section 5 is devoted to prove Theorem 1.4. Section 6 is devoted to study the kinetic energy spectrum $E(\kappa)$ for the NS-$\overline{\boldsymbol{w}}$ model. Section 7 relates the solutions of the NS-$\overline{\boldsymbol{w}}$ equations to those of the Navier-Stokes as $\alpha \to 0^+$. Finally, a summary and a discussion of the results are given.

2 - Ladder Inequalities: Proof of Theorem 1.1

The first step in the proof of Theorem 1.1 that has been expressed in Section 1, is to take $\boldsymbol{u} = \overline{\boldsymbol{u}} - \alpha^2 \Delta\overline{\boldsymbol{u}}$ as test function in (1). This gives after integrations by parts, the following energy inequality that corresponds to the case when $N = 0$ in the inequality (6)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|\overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2\Big) + \nu\Big(\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4\|\nabla^3\overline{\boldsymbol{u}}\|_{L^2}^2\Big) \\ \le \|\overline{\boldsymbol{u}}\|_{L^2}\|\boldsymbol{f}\|_{L^2} + \alpha^2\|\nabla\overline{\boldsymbol{u}}\|_{L^2}\|\nabla\boldsymbol{f}\|_{L^2}, \end{aligned} \tag{26}$$

or equivalently by using another integrations by parts combined with (8),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|\overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2\Big) + \nu\Big(\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4\|\nabla^3\overline{\boldsymbol{u}}\|_{L^2}^2\Big) \\ \le (1 + \alpha^2\ell^{-2})\|\overline{\boldsymbol{u}}\|\|\boldsymbol{f}\|, \end{aligned} \tag{27}$$

where the following equality is used

$$\|\boldsymbol{u}\|_{L^2}^2 = \|\overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2. \tag{28}$$

Since the NS-$\overline{\boldsymbol{w}}$ model has a unique regular solution, one can take the N derivative of (1). This leads, in the sense of distributions, to the following system for all $N \ge 1$,

$$\begin{cases} \dfrac{\partial\nabla^N\boldsymbol{u}}{\partial t} - \nabla^N(\boldsymbol{u}\times\overline{\boldsymbol{w}}) - \nu\nabla^N\Delta\boldsymbol{u} + \nabla^N\nabla P = \nabla^N\boldsymbol{f}, \\ \nabla\cdot\nabla^N\boldsymbol{u} = 0, \\ \nabla^N\boldsymbol{u}_{t=0} = \nabla^N\boldsymbol{u}^{in}, \end{cases} \tag{29}$$

where boundary conditions remain periodic and still with zero mean and the initial condition with zero divergence and mean. Taking $\nabla^N \boldsymbol{u}$ as a test function in (29) and using the higher order relation

$$\|\nabla^N \boldsymbol{u}\|_{L^2}^2 = \|\nabla^N \overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2 \|\nabla^{N+1} \overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4 \|\nabla^{N+2} \overline{\boldsymbol{u}}\|_{L^2}^2, \tag{30}$$

give the following equality:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^N \overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2 \|\nabla^{N+1} \overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4 \|\nabla^{N+2} \overline{\boldsymbol{u}}\|_{L^2}^2\Big) \\
&\quad = \nu\Big(\|\nabla^{N+1} \overline{\boldsymbol{u}}\|_{L^2}^2 + 2\alpha^2 \|\nabla^{N+2} \overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^4 \|\nabla^{N+3} \overline{\boldsymbol{u}}\|_{L^2}^2\Big) \\
&\qquad + \int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N (\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x} - \alpha^2 \int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N (\Delta \overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x} \\
&\qquad - \alpha^2 \int_{\mathbb{T}_3} \nabla^N \Delta \overline{\boldsymbol{u}} \nabla^N (\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x} + \alpha^4 \int_{\mathbb{T}_3} \nabla^N \Delta \overline{\boldsymbol{u}} \nabla^N (\Delta \overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x} \\
&\qquad + \int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N \boldsymbol{f} d\boldsymbol{x} - \alpha^2 \int_{\mathbb{T}_3} \nabla^N \Delta \overline{\boldsymbol{u}} \nabla^N \boldsymbol{f} d\boldsymbol{x},
\end{aligned} \tag{31}$$

where the pressure term vanishes as $\nabla \cdot \nabla^N \boldsymbol{u} = 0$.

Thus

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\overline{H_N} + 2\alpha^2 \frac{d}{dt}\overline{H_{N+1}} + \alpha^4 \frac{d}{dt}\overline{H_{N+2}}\right) \le -\nu\big(\overline{H_{N+1}} + 2\alpha^2 \overline{H_{N+2}} + \alpha^4 \overline{H_{N+3}}\big) \\
&\qquad + \left|\int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N (\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x}\right| + \alpha^2 \left|\int_{\mathbb{T}_3} \nabla^{N+1} \overline{\boldsymbol{u}} \nabla^{N-1} (\Delta \overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x}\right| \\
&\qquad + \alpha^2 \left|\int_{\mathbb{T}_3} \nabla^{N+1} \overline{\boldsymbol{u}} \nabla^{N+1} (\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x}\right| + \alpha^4 \left|\int_{\mathbb{T}_3} \nabla^{N+2} \overline{\boldsymbol{u}} \nabla^{N} (\Delta \overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x}\right| \\
&\qquad + \overline{H_N}^{1/2} \boldsymbol{\Phi}_N{}^{1/2} + \alpha^2 \overline{H_{N+1}}^{1/2} \boldsymbol{\Phi}_{N+1}{}^{1/2},
\end{aligned} \tag{32}$$

where the definition of $\overline{H_N}$ in (7) is used combining with integrations by parts in the Laplacien terms and in the last term.

The central terms are

$$\mathrm{NL}_1 = \left|\int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N (\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}}) d\boldsymbol{x}\right|, \tag{33}$$

(34) $$\mathrm{NL}_2 = \alpha^2 \left| \int_{\mathbb{T}_3} \nabla^{N+1}\overline{\boldsymbol{u}}\nabla^{N-1}(\Delta\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}})d\boldsymbol{x} \right|,$$

(35) $$\mathrm{NL}_3 = \alpha^2 \left| \int_{\mathbb{T}_3} \nabla^{N+1}\overline{\boldsymbol{u}}\nabla^{N+1}(\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}})d\boldsymbol{x} \right|$$

and

(36) $$\mathrm{NL}_4 = \alpha^4 \left| \int_{\mathbb{T}_3} \nabla^{N+2}\overline{\boldsymbol{u}}\nabla^{N}(\Delta\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}})d\boldsymbol{x} \right|.$$

These four terms NL_1, NL_2, NL_3 and NL_4 can be bounded using the following Gagliardo-Nirenberg interpolation inequality [**6**]:

Lemma 2.1. *Let $1 \leq q, r \leq \infty$, j and m such that $0 \leq j < m$ then the Gagliardo-Nirenberg interpolation inequality is:*

(37) $$\|\nabla^j \boldsymbol{v}\|_p \leq C \|\nabla^m \boldsymbol{v}\|_r^a \|\boldsymbol{v}\|_q^{1-a}$$

where

$$\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + \frac{1-a}{q}$$

for $\frac{j}{m} \leq a < 1$ *and* $a = \frac{j}{m}$ *if* $m - j - \frac{d}{r} \in N^*$.

The first nonlinear term NL_1 is estimated with the Gagliardo-Nirenberg inequality [**6**] by

(38) $$\mathrm{NL}_1 \leq c_N \|\nabla\overline{\boldsymbol{u}}\|_\infty \overline{H_N}, \text{ where } c_0 = 0 \text{ and } c_N \leq c2^N.$$

Indeed, the nonlinear first term NL_1 is found to satisfy

$$\mathrm{NL}_1 = \left| \int_{\mathbb{T}_3} \nabla^N\overline{\boldsymbol{u}}\nabla^N(\overline{\boldsymbol{u}} \times \overline{\boldsymbol{w}})d\boldsymbol{x} \right| \leq 2^N \overline{H_N}^{1/2} \sum_{l=1}^{N} \|\nabla^l \overline{\boldsymbol{u}}\|_{L^p} \|\nabla^{N+1-l}\overline{\boldsymbol{u}}\|_{L^q},$$

where p and q satisfy $1/p + 1/q = 1/2$ according to the Hölder inequality.

The following two Gagliardo-Nirenberg inequalities

$$\|\nabla^l \overline{\boldsymbol{u}}\|_{L^p} \leq c_1 \|\nabla^N \overline{\boldsymbol{u}}\|_{L^2}^a \|\nabla\overline{\boldsymbol{u}}\|_\infty^{1-a},$$

$$\|\nabla^{N+1-l} \overline{\boldsymbol{u}}\|_{L^q} \leq c_2 \|\nabla^N \overline{\boldsymbol{u}}\|_{L^2}^b \|\nabla\overline{\boldsymbol{u}}\|_\infty^{1-b},$$

where a anb b must satisfy

$$\frac{1}{p} = \frac{l-1}{3} + a\left(\frac{1}{2} - \frac{N-1}{3}\right),$$

$$\frac{1}{q} = \frac{N-l}{3} + b\left(\frac{1}{2} - \frac{N-1}{3}\right),$$

and $a + b = 1$ lead to (38).

In the same way, the other non linear terms NL_2, NL_3 and NL_4 can be estimated with Gagliardo-Nirenberg inequality in order to have

$$\mathrm{NL}_2 \leq c_N \alpha^2 \|\nabla \overline{\boldsymbol{u}}\|_\infty \overline{H_{N+1}}, \tag{39}$$

$$\mathrm{NL}_3 \leq c_N \alpha^2 \|\nabla \overline{\boldsymbol{u}}\|_\infty \overline{H_{N+1}}, \tag{40}$$

$$\mathrm{NL}_4 \leq c_N \alpha^4 \|\nabla \overline{\boldsymbol{u}}\|_\infty \overline{H_{N+2}}, \tag{41}$$

where $c_N \leq c2^N$.

The result (6) then follows.

3 - Proof of Proposition 1.2

Given $\boldsymbol{u} = \overline{\boldsymbol{u}} - \alpha^2 \Delta \overline{\boldsymbol{u}}$, the Poincaré inequality $\|\boldsymbol{u}\|_{L^2} \leq L/2\pi \|\nabla \boldsymbol{u}\|_{L^2}$ immediately leads to

$$\alpha^2 \|\overline{\boldsymbol{u}}\|_{H^2} \leq \|\boldsymbol{u}\|_{L^2} \leq \left(\frac{L^2}{4\pi^2} + \alpha^2\right) \|\overline{\boldsymbol{u}}\|_{H^2}, \tag{42}$$

thus

$$\|\overline{\boldsymbol{u}}\|_{H^2} \simeq \|\boldsymbol{u}\|_{L^2}. \tag{43}$$

In particular,

$$Re^2 \simeq \left\langle \|\overline{\boldsymbol{u}}\|_{H^2}^2 \right\rangle \simeq \left\langle \|\boldsymbol{u}\|_{L^2}^2 \right\rangle. \tag{44}$$

Let us take $\boldsymbol{f}$ as a test function in (1). This gives, by using $\boldsymbol{f}$ which is a divergence free function that verifies (8) and that is time independent, the following inequality

$$\|\boldsymbol{f}\|_{L^2}^2 \lesssim \|\boldsymbol{u}\|_{L^2} \|\overline{\boldsymbol{w}}\|_{H^1} \|\boldsymbol{f}\|_{H^1} + \|\nabla \boldsymbol{u}\|_{H^{-1}} \|\nabla \boldsymbol{f}\|_{H^1} + \frac{d}{dt}(\boldsymbol{u}, \boldsymbol{f}). \tag{45}$$

The following equality (see [**6**]),

$$\|\overline{\boldsymbol{w}}\|_{H^1} = \|\overline{\boldsymbol{u}}\|_{H^2}$$

combined with an integration with respect to time of (45) gives

$$t\|\boldsymbol{f}\|_{L^2}^2 \lesssim \int_0^t \|\boldsymbol{u}\|_{L^2}^2\|\boldsymbol{f}\|_{H^1}dt + \int_0^t \|\boldsymbol{u}\|_{L^2}\|\boldsymbol{f}\|_{H^2}dt + \|\boldsymbol{u}\|_{L^\infty((0,t),L^2)}\|\boldsymbol{f}\|_{L^2}. \tag{46}$$

Thus, given that $\|\boldsymbol{u}\|_{L^\infty((0,t),L^2)} \le k < \infty$ ([**21**]), equation (46) yields to

$$\|\boldsymbol{f}\|_{L^2} \lesssim \langle \|\boldsymbol{u}\|_{L^2}^2 \rangle + \langle \|\boldsymbol{u}\|_{L^2} \rangle. \tag{47}$$

This implies

$$Gr \lesssim Re^2 + Re. \tag{48}$$

Remark 2. In reference [**17**], it has been shown that this relation holds for a generalized alpha model.

4 - Proof of Theorem 1.3.

Let

$$\overline{F_N} = \overline{H_N} + \tau^2 \Phi_N$$

where the quantity τ is defined by

$$\tau = \ell^2 \nu^{-1}(Gr \ln Gr)^{-1/2}.$$

Theorem 1.1 gives

$$\begin{aligned} \frac{1}{2}\left(\frac{d}{dt}\overline{H_N} + 2\alpha^2\frac{d}{dt}\overline{H_{N+1}} + \alpha^4\frac{d}{dt}\overline{H_{N+2}}\right) &\le -\nu\left(\overline{H_{N+1}} + 2\alpha^2\overline{H_{N+2}} + \alpha^4\overline{H_{N+3}}\right) \\ &+ C_N\|\nabla\overline{\boldsymbol{u}}\|_\infty\left(\overline{H_N} + 2\alpha^2\overline{H_{N+1}} + \alpha^4\overline{H_{N+2}}\right) \\ &+ \overline{H_N}^{1/2}\boldsymbol{\Phi}_N{}^{1/2} + 2\alpha^2\overline{H_{N+1}}^{1/2}\boldsymbol{\Phi}_{N+1}{}^{1/2} + \alpha^4\overline{H_{N+2}}^{1/2}\boldsymbol{\Phi}_{N+2}{}^{1/2}. \end{aligned} \tag{49}$$

By adding and substracting the following time independant quantity

$$\nu\tau^2\left(\boldsymbol{\Phi}_{N+1} + 2\alpha^2\boldsymbol{\Phi}_{N+2} + \alpha^4\Phi_{N+3}\right), \tag{50}$$

inequality (49) turns into an inequality for the F_N which is given by

$$\begin{aligned} \frac{1}{2}\left(\frac{d}{dt}\overline{F_N} + 2\alpha^2\frac{d}{dt}\overline{F_{N+1}} + \alpha^4\frac{d}{dt}\overline{F_{N+2}}\right) &\le -\nu\left(\overline{F_{N+1}} + 2\alpha^2\overline{F_{N+2}} + \alpha^4\overline{F_{N+3}}\right) \\ &+ C_N\|\nabla\overline{\boldsymbol{u}}\|_\infty\left(\overline{F_N} + 2\alpha^2\overline{F_{N+1}} + \alpha^4\overline{F_{N+2}}\right) + X_N \end{aligned} \tag{51}$$

where

$$X_N = \overline{H_N}^{1/2} \boldsymbol{\Phi}_N{}^{1/2} + 2\alpha^2 \overline{H_{N+1}}^{1/2} \boldsymbol{\Phi}_{N+1}{}^{1/2} + \alpha^4 \overline{H_{N+2}}^{1/2} \boldsymbol{\Phi}_{N+2}{}^{1/2} + \nu\tau^2\ell^{-2}\boldsymbol{\Phi}_N + 2\alpha^2\ell^{-2}\nu\tau^2\boldsymbol{\Phi}_{N+1} + \alpha^4\ell^{-2}\nu\tau^2\boldsymbol{\Phi}_{N+2}\,. \tag{52}$$

Young inequality implies

$$\begin{aligned} X_N \le & \left[\frac{g}{2}\overline{H_N} + \left(\frac{1}{2\tau^2 g} + \nu\ell^{-2}\right)\tau^2\boldsymbol{\Phi}_N\right] \\ & + 2\alpha^2\left[\frac{g}{2}\overline{H_{N+1}} + \left(\frac{1}{2\tau^2 g} + \nu\ell^{-2}\right)\tau^2\boldsymbol{\Phi}_{N+1}\right] \\ & + \alpha^4\left[\frac{g}{2}\overline{H_{N+2}} + \left(\frac{1}{2\tau^2 g} + \nu\ell^{-2}\right)\tau^2\boldsymbol{\Phi}_{N+2}\right]. \end{aligned} \tag{53}$$

To make the coefficients of $\overline{H_N}$ and $\tau^2\boldsymbol{\Phi}_N$ equal, choose g to satisfy

$$g - \frac{1}{\tau^2 g} - 2\nu\ell^{-2} = 0, \tag{54}$$

that gives

$$g = 2\nu\ell^{-2} + 2\sqrt{\nu^2\ell^{-4} + \frac{1}{\tau^2}}. \tag{55}$$

Now let $\tau = \ell^2\nu^{-1}(Gr \ln Gr)^{-1/2}$ as $Gr \to \infty$. Hence $g \simeq \tau^{-1}$ as $Gr \to \infty$. Consequently

$$\tau^{-1} = \ell^{-2}\nu(Gr \ln Gr)^{1/2} \le C\ell^{-2}\nu(Re \ln Re) \quad \text{as } Gr \to \infty, \tag{56}$$

and

$$X_N \le \frac{1}{2\tau}\left(\overline{F_N} + 2\alpha^2\overline{F_{N+1}} + \alpha^4\overline{F_{N+2}}\right). \tag{57}$$

Thus

$$\frac{1}{2}\frac{d}{dt}L_N \le -\nu L_{N+1} + C_{N,\alpha}\|\nabla\overline{\boldsymbol{u}}\|_\infty L_N + C\nu\ell^{-2}Re(\ln Re)L_N, \tag{58}$$

where

$$L_N = \overline{F_N} + 2\alpha^2\overline{F_{N+1}} + \alpha^4\overline{F_{N+2}}.$$

The inequality (21) follows from the inequality

$$L_{N+1} \ge \frac{1}{3}\frac{L_N^{1+\frac{1}{p}}}{L_{N-p}^{\frac{1}{p}}} \tag{59}$$

that can be showed in the same way as Lemma 1, p. 75 in [**14**].

When $N = 0$ the only difference is that the nonlinear term vanishes and in the same way the following inequality holds

$$\frac{1}{2}\frac{d}{dt}L_0 \leq -\nu L_1 + C\nu\ell^{-2}Re(\ln Re)L_0 \tag{60}$$

which is (22).

5 - Estimates in terms of Reynolds number: Proof of Theorem 1.4

To obtain length scales estimates let us define the quantities

$$\kappa_{N,r} = \left(\frac{L_N}{L_r}\right)^{\frac{1}{2(N-r)}}.$$

In order to find an estimate for the length scales associated with higher order moments solution $\kappa_{N,0}$ ($N \geq 2$), upper bounds for $\langle \kappa^2_{N,r} \rangle$, $\langle \kappa^2_{1,0} \rangle$ and $\langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle$ are first established. Then the following identity is used

$$\kappa^2_{N,0} \; = \; \kappa^{2(N-1)/N}_{N,1} \kappa^{2/N}_{1,0} \tag{61}$$

in order to deduce the result.

The first two bounds are obtained by dividing by L_N in Theorem 1.3 and time averaging to obtain

$$\langle \kappa^2_{N,r} \rangle \leq C_{N,\alpha}\nu^{-1}\langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle + C\ell^{-2}Re(\ln Re) \tag{62}$$

and

$$\left\langle \kappa^2_{1,0} \right\rangle \leq C\ell^{-2}Re(\ln Re). \tag{63}$$

Then time averaging (27) and using (15) in order to obtain

$$\begin{aligned} \nu\langle \overline{H_1} + 2\alpha^2\overline{H_2} + \alpha^4\overline{H_3} \rangle \; &\leq \; (1 + \alpha^2\ell^{-2})\langle \overline{H_0}^{1/2} \boldsymbol{\Phi}_0{}^{1/2} \rangle \\ &\leq \; (1 + \alpha^2\ell^{-2})\langle \overline{H_0}^{1/2} \boldsymbol{\Phi}_0{}^{1/2} \rangle \\ &\leq \; C(1 + \alpha^2\ell^{-2})\nu^3\ell^{-4}L^3Re^3. \end{aligned} \tag{64}$$

Thus

$$\langle \overline{H_3} \rangle \;\; \lesssim \;\; Re^3. \tag{65}$$

The same estimate holds true for $\overline{H_1}$ and $\overline{H_2}$, hence

$$\langle \overline{H_1} \rangle \;\; \lesssim \;\; Re^3, \tag{66}$$

and

$$\langle \overline{H_2} \rangle \quad \lesssim \quad Re^3. \tag{67}$$

Thus (66) implies that the energy dissipation rate $\varepsilon = \nu \langle \|\nabla \overline{\boldsymbol{u}}\|_{L^2}^2 \rangle L^{-3}$ is bounded above by

$$\varepsilon \lesssim Re^3, \tag{68}$$

and the inverse Kolmogorov length $\lambda_k^{-1} = (\varepsilon/\nu^3)^{1/4}$ is bounded above by

$$\ell \lambda_k^{-1} \lesssim Re^{3/4}. \tag{69}$$

Inequality (67) combined with the following interpolation inequality

$$\overline{H_N} \leq \overline{H_{N-s}}^{\frac{r}{r+s}} \, \overline{H_{N+r}}^{\frac{s}{r+s}}, \tag{70}$$

that is

$$\begin{aligned} \overline{H_1} &\leq \overline{H_0}^{\frac{2}{3}} \, \overline{H_3}^{\frac{1}{3}}, \\ \overline{H_2} &\leq \overline{H_0}^{\frac{1}{3}} \, \overline{H_3}^{\frac{2}{3}}, \end{aligned} \tag{71}$$

are used to improve (66) and (67) in order to obtain

$$\langle \overline{H_1} \rangle \leq \langle \overline{H_0} \rangle^{\frac{2}{3}} \langle \overline{H_3} \rangle^{\frac{1}{3}} \lesssim Re^{7/3}, \tag{72}$$

and

$$\langle \overline{H_2} \rangle \leq \langle \overline{H_0} \rangle^{\frac{1}{3}} \langle \overline{H_3} \rangle^{\frac{2}{3}} \lesssim Re^{8/3}. \tag{73}$$

Inequality (72) improves the above result (69) for the inverse Kolmogorov length to

$$\ell \lambda_k^{-1} \lesssim Re^{7/12}. \tag{74}$$

This also implies that the energy dissipation rate $\varepsilon = \nu \langle \|\nabla \overline{\boldsymbol{u}}\|_{L^2}^2 \rangle L^{-3}$ is also bounded by $Re^{7/3}$ but all the improved estimates blow up when α tends to zero.

The following estimates for $\langle \|\overline{\boldsymbol{u}}\|_\infty \rangle$ and $\langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle$ are obtained from (65), (72) and (73). In fact, Agmon's inequality [**9**]

$$\|\boldsymbol{u}\|_\infty \leq \|\boldsymbol{u}\|_{H^1}^{1/2} \|\boldsymbol{u}\|_{H^2}^{1/2}$$

says that

$$\begin{aligned} \langle \|\overline{\boldsymbol{u}}\|_\infty^2 \rangle \quad &\leq \quad \langle \overline{H_1} \rangle^{1/2} \langle \overline{H_2} \rangle^{1/2} \\ &\lesssim \quad Re^{5/2}, \end{aligned} \tag{75}$$

and

$$\begin{aligned} \langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle \quad &\leq \quad \langle \overline{H_3} \rangle^{1/4} \langle \overline{H_2} \rangle^{1/4} \\ &\lesssim \quad Re^{17/12}. \end{aligned} \tag{76}$$

The estimate for $\langle \kappa_{N,0}^2 \rangle$ is obtained directly from the $\|\nabla \overline{\boldsymbol{u}}\|_\infty$ term in the ladder inequalities (6) as opposed to the $\nu^{-1}\|\overline{\boldsymbol{u}}\|_\infty^2$ term in [**14**]. Inequality (62) combined with (76) gives that

$$\ell^2 \langle \kappa_{N,r}^2 \rangle \leq C(\alpha, \nu, \ell, L) Re^{17/12} + CRe(\ln Re). \tag{77}$$

By writing

$$\begin{aligned} \langle \kappa_{N,0}^2 \rangle &= \left\langle \kappa_{N,1}^{2(N-1)/N} \kappa_{1,0}^{2/N} \right\rangle \\ &\leq \langle \kappa_{N,1}^2 \rangle^{(N-1)/N} \langle \kappa_{1,0}^2 \rangle^{1/N}, \end{aligned} \tag{78}$$

and then using the above estimates for $\langle \kappa_{N,1}^2 \rangle$ and $\langle \kappa_{1,0}^2 \rangle$, one gets for $N \geq 2$,

$$\ell^2 \langle \kappa_{N,0}^2 \rangle \leq C(\alpha, \nu, \ell, L)^{(N-1)/N} Re^{17/12 - 5/12N} (\ln Re)^{1/N} + C\, Re \ln Re. \tag{79}$$

Note that when $N = 1$ the following inequality holds $\ell^2 \langle \kappa_{1,0}^2 \rangle \leq C\, Re \ln Re$.

6 - Energy spectrum

The goal of this section is to study the energy spectrum for the NS-$\overline{\omega}$ model. It is shown in this section that the NS-$\overline{\boldsymbol{w}}$ model follows the usual $\kappa^{-\frac{5}{3}}$ Kolmogorov power law spectrum for wavenumbers smaller than $1/\alpha$ in the inertial range. This provides physical fidelity to the model on the large scales.

Following [**4**], let

$$\boldsymbol{u}_\kappa = \sum_{\kappa \leq |\boldsymbol{j}| \leq 2\kappa} \hat{\boldsymbol{u}}_{\boldsymbol{j}} e^{i\boldsymbol{j}\cdot\boldsymbol{x}} \tag{80}$$

$$\overline{\boldsymbol{\omega}}_\kappa = \sum_{\kappa \leq |\boldsymbol{j}| \leq 2\kappa} \hat{\overline{\boldsymbol{\omega}}}_{\boldsymbol{j}} e^{i\boldsymbol{j}\cdot\boldsymbol{x}}, \tag{81}$$

$$\boldsymbol{u}_\kappa^< = \sum_{j<\kappa} \boldsymbol{u}_j, \quad \boldsymbol{u}_\kappa^> = \sum_{j \geq 2\kappa} \boldsymbol{u}_j, \tag{82}$$

$$\overline{\boldsymbol{\omega}}_\kappa^< = \sum_{j<\kappa} \overline{\boldsymbol{\omega}}_j, \quad \boldsymbol{\omega}_\kappa^> = \sum_{j \geq 2\kappa} \boldsymbol{\omega}_j, \tag{83}$$

where

$$\hat{\phi}_{\boldsymbol{j}} = \frac{1}{(2\pi L)^3} \int_{\mathbb{T}_3} \phi(\boldsymbol{x}) e^{-i\boldsymbol{j}\cdot\boldsymbol{x}} d\boldsymbol{x}$$

denote the Fourier coefficients of the function ϕ. The energy balance for the NS-$\overline{\omega}$ is given by

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{u}_\kappa, \boldsymbol{u}_\kappa) + \nu(-\Delta \boldsymbol{u}_\kappa, \boldsymbol{u}_\kappa) = T_\kappa - T_{2\kappa}, \tag{84}$$

where

$$T_\kappa = \left(\boldsymbol{u}_\kappa^< \times \overline{\boldsymbol{w}}_\kappa^<, \boldsymbol{u}_\kappa\right) - \left(\left(\boldsymbol{u}_\kappa + \boldsymbol{u}_\kappa^>\right) \times \left(\overline{\boldsymbol{w}}_\kappa + \overline{\boldsymbol{w}}_\kappa^>\right), \boldsymbol{u}_\kappa^<\right).$$

Time-averaging equation (84) gives

$$\langle \nu(-\Delta \boldsymbol{u}_\kappa, \boldsymbol{u}_\kappa)\rangle = \langle T_\kappa\rangle - \langle T_{2\kappa}\rangle. \tag{85}$$

Let

$$E_\alpha^{\boldsymbol{u}}(\kappa) = \sum_{|\boldsymbol{j}|=\kappa} |\hat{\boldsymbol{u}}_{\boldsymbol{j}}|^2. \tag{86}$$

Consequently, (85) implies

$$\nu\kappa^3 E_\alpha^{\boldsymbol{u}}(\kappa) \approx \nu \int_\kappa^{2\kappa} \eta^2 E_\alpha^{\boldsymbol{u}}(\eta) d\eta \approx \langle T_\kappa\rangle - \langle T_{2\kappa}\rangle. \tag{87}$$

As long as

$$\nu\kappa^3 E_\alpha^{\boldsymbol{u}}(\kappa) \ll \langle T_\kappa\rangle \tag{88}$$

and

$$\langle T_\kappa\rangle \approx \langle T_{2\kappa}\rangle, \tag{89}$$

the wavenumber κ belongs to the inertial range, see [**4**, **8**] for more details. Since there are two different velocities in the NS-$\overline{\boldsymbol{w}}$ model, the average velocity on an eddy of length size $\frac{1}{\kappa}$ can be evaluated in three different ways:

$$U_\kappa^0 = \left\langle \frac{1}{L^3} \int_{\mathbb{T}_3} \boldsymbol{u}_\kappa \cdot \boldsymbol{u}_\kappa \right\rangle \sim \kappa^{\frac{1}{2}} E_\alpha^{\boldsymbol{u}}(\kappa)^{\frac{1}{2}}, \tag{90}$$

$$U_\kappa^1 = \left\langle \frac{1}{L^3} \int_{\mathbb{T}_3} \boldsymbol{u}_\kappa \cdot \overline{\boldsymbol{u}}_\kappa \right\rangle \sim \frac{\kappa^{\frac{1}{2}} E_\alpha^{\boldsymbol{u}}(\kappa)^{\frac{1}{2}}}{(1+\alpha^2\kappa^2)^{\frac{1}{2}}}, \tag{91}$$

$$U_\kappa^2 = \left\langle \frac{1}{L^3} \int_{\mathbb{T}_3} \overline{\boldsymbol{u}}_\kappa \cdot \overline{\boldsymbol{u}}_\kappa \right\rangle \sim \frac{\kappa^{\frac{1}{2}} E_\alpha^{\boldsymbol{u}}(\kappa)^{\frac{1}{2}}}{(1+\alpha^2\kappa^2)}. \tag{92}$$

Then, for different definitions of U_κ^n, the corresponding turnover time τ_κ^n for an eddy of the size $\dfrac{1}{\kappa}$ will be (cf. [**4, 20**])

$$\tau_\kappa^n \approx \frac{(1+\alpha^2\kappa^2)^{\frac{n}{2}}}{\kappa^{\frac{3}{2}} E_\alpha^{\boldsymbol{u}}(\kappa)^{\frac{1}{2}}} \quad n = 0, 1, 2. \tag{93}$$

Following [**4**], as $\boldsymbol{u}$ is the velocity of the model, by analogy with the definition of the Navier-Stokes energy dissipation rate ε_{NS}, the corresponding mean energy dissipation rate of the variable $\boldsymbol{u}$ of the model (1) is given by,

$$\varepsilon^{\boldsymbol{u}} = \nu \langle \|\nabla \boldsymbol{u}\|_{L^2}^2 \rangle L^{-3}.$$

Therefore, the energy dissipation rate $\varepsilon^{\boldsymbol{u}}$ is

$$\varepsilon^{\boldsymbol{u}} = \frac{1}{\tau_\kappa^n} \int_\kappa^{2\kappa} E_\alpha^{\boldsymbol{u}}(\eta) d\eta \sim \frac{\kappa^{\frac{5}{2}} E_\alpha^{\boldsymbol{u}}(\kappa)^{\frac{3}{2}}}{(1+\alpha^2\kappa^2)^{\frac{n}{2}}}, \tag{94}$$

which implies the following spectral scaling law for the NS-$\overline{\boldsymbol{w}}$ inertial range,

$$E_\alpha^{\boldsymbol{u}}(\kappa) \sim \frac{{\varepsilon^{\boldsymbol{u}}}^{\frac{2}{3}} (1+\alpha^2\kappa^2)^{\frac{n}{3}}}{\kappa^{\frac{5}{3}}}. \tag{95}$$

Thus, the kinetic energy spectrum $E_\alpha^{\overline{\boldsymbol{u}}}(\kappa) = \dfrac{E_\alpha^{\boldsymbol{u}}(\kappa)}{(1+\alpha^2\kappa^2)^2}$ of the variable $\overline{\boldsymbol{u}}$ is given by

$$E_\alpha^{\overline{\boldsymbol{u}}}(\kappa) \sim \frac{{\varepsilon^{\boldsymbol{u}}}^{\frac{2}{3}}}{\kappa^{\frac{5}{3}}} \quad \text{for } \alpha \leq \frac{1}{\kappa}, \tag{96}$$

$$E_\alpha^{\overline{\boldsymbol{u}}}(\kappa) \sim \frac{{\varepsilon^{\boldsymbol{u}}}^{\frac{2}{3}}}{\kappa^{\frac{17-2n}{3}}} \quad \text{for } \alpha \geq \frac{1}{\kappa}. \tag{97}$$

Therefore, this energy spectra provides physical fidelity to the model on the large scales and has a much faster decaying power law $\kappa^{\frac{2n-17}{3}}$, $(n = 0, 1, 2)$ than the usual Kolmogorov $\kappa^{-5/3}$ power law, in the subrange $\kappa\alpha \geq 1$. This signifies that the NS-$\overline{\boldsymbol{w}}$ model, like the other alpha models [**3, 4, 8, 18**], is a good candidate subgrid scale model of turbulence.

Table 2. *Energy spectra for the Navier-Stokes-α, Bardina, Leray-α and Modified Leray-α from* [**3, 4, 8, 18**] *and NS-*$\overline{\boldsymbol{w}}$ based on the eddy turnover time τ_κ^2.

	NS-α / Bardina / ML-α	Leray-α / NS-$\overline{\boldsymbol{w}}$	Eq.
$E_\alpha^{\overline{\boldsymbol{u}}}(\kappa)$ for $\alpha \geq \frac{1}{\kappa}$	κ^{-3}	$\kappa^{-\frac{13}{3}}$	(97)

In Table 2, we compare the energy spectra $E_\alpha^{\overline{u}}(\kappa)$, for $\alpha \geq \frac{1}{\kappa}$, for the NS-$\overline{\boldsymbol{w}}$ with the energy spectra of Navier-Stokes-α, Bardina, Leray-α and Modified Leray-α models [**3, 4, 8, 18**]. The energy spectra here is based on the eddy turnover time τ_κ^2 which is obtained by relation (93) for $n = 2$. The decay of the energy spectra for the NS-$\overline{\boldsymbol{w}}$ is κ^{-3} for $\alpha \geq \frac{1}{\kappa}$, which is equal to that of Leray-α model [4]. Nevertheless, it has been shown in [**3, 8, 18**] that for Navier-Stokes-α, Bardina, and Modified Leray-α, the power laws for the energy spectra are κ^{-3} for $\alpha \geq \frac{1}{\kappa}$.

7 - Accuracy and limit consistency of the NS-$\overline{\boldsymbol{w}}$ model

In the first part of this section, the accuracy of the NS-$\overline{\boldsymbol{w}}$ model as $\alpha \to 0^+$ is studied by computing the model's consistency error. The following definition of consistency error can be found in [**26**].

Definition 7.1. The *consistency error* of a Navier-Stokes regularization is the residual of a Navier-Stokes solution in the model.

In order to obtain the consistency error of the NS-$\overline{\boldsymbol{w}}$ model, the Navier-Stokes in its rotational form is rewritten to make the NS-$\overline{\boldsymbol{w}}$ model appear on the LHS as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \overline{\boldsymbol{w}} - \nu \Delta \boldsymbol{u} + \nabla P - \boldsymbol{f} = \boldsymbol{u} \times \boldsymbol{w} - \boldsymbol{u} \times \overline{\boldsymbol{w}}. \tag{98}$$

Thus, the model's consistency error is $\tau_{\overline{\boldsymbol{w}}} = \boldsymbol{u} \times \boldsymbol{w} - \boldsymbol{u} \times \overline{\boldsymbol{w}}$. A simple calculation implies that

$$\tau_{\overline{\boldsymbol{w}}} = \boldsymbol{u} \times \alpha^2 \Delta \overline{\boldsymbol{w}}. \tag{99}$$

Thus, $\|\tau_{\overline{\boldsymbol{w}}}\|_{L^2} = O(\alpha^2)$. Hence as the other alpha models [**22, 24, 26**], the NS-$\overline{\boldsymbol{w}}$ model has $O(\alpha^2)$ accuracy.

The next theorem shows that using energy estimates, one can extract subsequences of the weak solutions of (1) which converge, as $\alpha \to 0^+$ in the appropriate sense to a Leray-Hopf weak solution of the three-dimensional Navier-Stokes equations. For the definition and existence of weak solutions of the Navier-Stokes equations see, for instance, Refs. [**22, 24, 27**].

Theorem 7.2. *Let $T > 0$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma$, and denote by $(\boldsymbol{u}_\alpha, P_\alpha)$ the weak solution of* (1). *Then, there is a subsequence $(\boldsymbol{u}_{\alpha_j}, P_{\alpha_j})$ and a couple $(\boldsymbol{u}, P)$ such that, as $\alpha_j \to 0^+$*

$$(i) \quad \boldsymbol{u}_{\alpha_j} \rightharpoonup \boldsymbol{u} \textit{ weakly in } L^2(0,T;\boldsymbol{H}^1_\sigma) \cap L^{\frac{10}{3}}(0,T;\boldsymbol{L}^{\frac{10}{3}}),$$

$$(ii) \quad \boldsymbol{u}_{\alpha_j} \to \boldsymbol{u} \textit{ strongly in } L^q(0,T;\boldsymbol{L}^q) \textit{ for all } q < \frac{10}{3},$$

$$(iii) \quad P_{\alpha_j} \rightharpoonup P \textit{ weakly in } L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\mathbb{T}_3)).$$

Furthermore, the couple $(\boldsymbol{u}, P)$ *is a Leray-Hopf weak solution of the rotational Navier-Stokes equations*

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{w} - \nu \Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} \qquad in\ \mathbb{R}^+ \times \mathbb{T}_3, \tag{100}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{101}$$

with initial data $\boldsymbol{u}_{t=0} = \boldsymbol{u}_0$*, which satisfies the energy inequality*

$$\|\boldsymbol{u}(t)\|^2_{L^2} + \nu \int_0^t \|\boldsymbol{u}\|^2_{H^1} ds \le \frac{C}{\nu} \int_0^t \|\boldsymbol{f}\|^2_{H^{-1}} ds + \|\boldsymbol{u}_0\|^2_{L^2} \tag{102}$$

for almost every $t \in [0, T]$*.*

Proof. The proof of Theorem 7.2 follows the lines of the proof of the Theorem 4 in [**24**]. First, one needs to find estimates independent from α. The weak formulation of (1) can be written as

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial \boldsymbol{u}_\alpha}{\partial t}, \varphi \right\rangle - (\boldsymbol{u}_\alpha \times \overline{\boldsymbol{w}}_\alpha, \varphi) + \nu(\nabla \boldsymbol{u}_\alpha, \nabla \varphi) - (P_\alpha, \nabla \cdot \varphi)\, dt \\ &\quad = \int_0^T \langle \boldsymbol{f}, \varphi \rangle\, dt \qquad \text{for all } \varphi \in C^\infty_c([0,T] \times (\mathbb{T}_3))^3. \end{aligned} \tag{103}$$

By taking $\boldsymbol{u}$ as a test function in (103) and using the fact that the curl of two vectors is perpendicular to each of them, one finds that the solution of (1) satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_\alpha\|^2_{L^2} + \nu \|\boldsymbol{u}_\alpha\|^2_{H^1} \quad = \langle \boldsymbol{f}, \boldsymbol{u}_\alpha \rangle \tag{104}$$

which implies after using Cauchy Schwarz and Young inequality the following inequality

$$\|\boldsymbol{u}_\alpha(t)\|^2_{L^2} + \nu \int_0^t \|\boldsymbol{u}_\alpha\|^2_{H^1} ds \quad \le \frac{C}{\nu} \int_0^t \|\boldsymbol{f}\|^2_{H^{-1}} ds + \|\boldsymbol{u}_0\|^2_{L^2} \tag{105}$$

for almost every $t \in [0, T]$.

Notice that the above estimates are formal and can be made rigorous by using the Galerkin approximating scheme and then passing to the limit by using an Aubin compactness theorem (see, for example, [**2, 7, 22**]). The right hand side in (105) is bounded by a constant C which is independent from α. Therefore $\boldsymbol{u}_\alpha$ belongs to the energy space $L^\infty([0,T];\boldsymbol{L}^2)\cap L^2([0,T];\boldsymbol{H}^1_\sigma)$ and $\boldsymbol{w}_\alpha \in L^2([0,T];\boldsymbol{L}^2)$ uniformly with respect to α. Recall that $\|\overline{\boldsymbol{w}}_\alpha\|^2_{L^2} \le \|\boldsymbol{w}_\alpha\|^2_{L^2}$ hence

$$\overline{\boldsymbol{w}}_\alpha \in L^2([0,T];\boldsymbol{L}^2) \quad \text{uniformly w.r.t } \alpha. \tag{106}$$

Then a standard interpolation argument yields to

$$\boldsymbol{u}_\alpha \in L^{\frac{10}{3}}(0,T;\boldsymbol{L}^{\frac{10}{3}}) \quad \text{uniformly w.r.t } \alpha. \tag{107}$$

The above inequality (107) combined with (106) and Hölder inequality implies that

$$\boldsymbol{u}_\alpha \times \overline{\boldsymbol{w}}_\alpha \in L^{\frac{5}{4}}(0,T;\boldsymbol{L}^{\frac{5}{4}}) \quad \text{uniformly w.r.t } \alpha. \tag{108}$$

This gives by using the Sobolev embedding $\boldsymbol{L}^{\frac{5}{4}} \hookrightarrow \boldsymbol{H}^{-\frac{9}{10}}$ that

$$\boldsymbol{u}_\alpha \times \overline{\boldsymbol{w}}_\alpha \in L^{\frac{5}{4}}(0,T;\boldsymbol{H}^{-\frac{9}{10}}) \quad \text{uniformly w.r.t } \alpha. \tag{109}$$

Equations (1) implies that the pressure term P_α verifies the following equation

$$\Delta P_\alpha = \nabla\cdot(\boldsymbol{u}_\alpha \times \overline{\boldsymbol{w}_\alpha}), \tag{110}$$

consequently the classical elliptic theory combined with (108) implies that

$$P_\alpha \in L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\mathbb{T}_3)) \quad \text{uniformly w.r.t } \alpha. \tag{111}$$

Equations (1) combined with (109) implies that

$$\frac{\partial \boldsymbol{u}_{\alpha_j}}{\partial t} \in L^{\frac{5}{4}}(0,T;\boldsymbol{H}^{-\frac{9}{10}}) \quad \text{uniformly w.r.t } \alpha. \tag{112}$$

Thanks to the Aubin-Lions compactness Lemma [**28**], a subsequence $(\boldsymbol{u}_{\alpha_j},\boldsymbol{w}_{\alpha_j},P_{\alpha_j})$ and a triplet $(\boldsymbol{u},\boldsymbol{w},P)$ can be found such that the following properties hold when $\alpha_j \to 0^+$

$$\boldsymbol{u}_{\alpha_j} \rightharpoonup^* \boldsymbol{u} \quad \text{weakly}^* \text{ in } L^\infty(0,T;\boldsymbol{L}^2), \tag{113}$$

$$\boldsymbol{u}_{\alpha_j} \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(0,T;\boldsymbol{H}^1_\sigma)\cap L^{\frac{10}{3}}(0,T;\boldsymbol{L}^{\frac{10}{3}}), \tag{114}$$

$$\boldsymbol{w}_{\alpha_j} \rightharpoonup \boldsymbol{w} \quad \text{weakly in } L^2(0,T;\boldsymbol{L}^2), \tag{115}$$

$$\frac{\partial \boldsymbol{u}_{\alpha_j}}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{u}}{\partial t} \quad \text{weakly in } L^{\frac{5}{4}}(0,T;\boldsymbol{H}^{-\frac{9}{10}}), \tag{116}$$

$$P_{\alpha_j} \rightharpoonup P \quad \text{weakly in } L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\mathbb{T}_3)), \tag{117}$$

$$\boldsymbol{u}_{\alpha_j} \to \boldsymbol{u} \quad \text{strongly in } L^2(0,T;\boldsymbol{L}^2). \tag{118}$$

Thus (107) combined with (118) yields to

$$\boldsymbol{u}_{\alpha_j} \to \boldsymbol{u} \qquad \text{strongly in } L^q(0,T;\boldsymbol{L}^q) \text{ for all } q < \frac{10}{3}. \tag{119}$$

Moreover, it is not difficult to deduce that

$$\overline{\boldsymbol{w}}_{\alpha_j} \rightharpoonup \boldsymbol{w} \qquad \text{weakly in } L^2(0,T;\boldsymbol{L}^2), \tag{120}$$

and consequently

$$\boldsymbol{u}_{\alpha_j} \times \overline{\boldsymbol{w}}_{\alpha_j} \rightharpoonup \boldsymbol{u} \times \boldsymbol{w} \qquad \text{weakly in } L^q(0,T;\boldsymbol{L}^q), \text{ for all } q < \frac{5}{4}. \tag{121}$$

The above established convergences are clearly sufficient to take the limit in the weak formulation (103) and to conclude that $(\boldsymbol{u}, P)$ satisfies the weak formulation of the rotational Navier-Stokes equations

$$\begin{aligned} &\int_0^T \langle \frac{\partial \boldsymbol{u}}{\partial t}, \varphi\rangle - (\boldsymbol{u} \times \boldsymbol{w}, \varphi) + \nu(\nabla \boldsymbol{u}, \nabla\varphi) - (P, \nabla\cdot\varphi)\, dt \\ &\quad = \int_0^T \langle \boldsymbol{f}, \varphi\rangle\, dt \qquad \text{for all } \varphi \in C^\infty([0,T]\times(\mathbb{T}_3))^3. \end{aligned} \tag{122}$$

Notice also that every weak solution satisfies the energy equality (104) and hence the energy inequality (102) follows by passing to the lim inf as $\alpha \to 0^+$, see in [**24**] for more details. □

8 - Discussion

The NS-$\overline{\boldsymbol{w}}$ model considered in this paper is derived from the rotational Navier-Stokes equations by regularizing with the Helmholtz operator the second term of the nonlinearity. This model is a modification of the Navier-Stokes-alpha model and can also be seen as a perturbation of the modified Leray alpha. The ladder inequalities for this model are established and length-scale estimates are given in terms of the Reynolds number. All these estimates are listed in Table 1. These estimates coincide with the estimates to the Leray-alpha model given in [**15**]. The estimate for $\left\langle \kappa_{N,0}^2 \right\rangle$ is obtained directly from the $\|\nabla\overline{\boldsymbol{u}}\|_\infty$ term in the ladder inequalities (6) different from the $\nu^{-1}\|\overline{\boldsymbol{u}}\|_\infty^2$ term in [**14**]. Moreover, it has been shown here that the Energy spectra of the NS-$\overline{\boldsymbol{w}}$ model decays like the Leray-α and it decay faster than the Navier-Stokes-alpha model for $\alpha\kappa \geq 1$. In order to overcome the fact that the inertial range of the NS-$\overline{\boldsymbol{w}}$ is shorter than the one of the Navier Stokes equations, one may consider the following regularization of the Navier Stokes equations based on the numerical studies of Kim et al. [**19**] for the NS-$\alpha\beta$ model ($0<\beta<\alpha$)

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \overline{\boldsymbol{w}} - \nu\Big(1-\beta^2\Delta\Big)\Delta\overline{\boldsymbol{u}} + \nabla P = \boldsymbol{f} \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ -\alpha^2\Delta\overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} = \boldsymbol{u} \qquad \text{in } \mathbb{T}_3, \\ \nabla\cdot\boldsymbol{u} = \nabla\cdot\overline{\boldsymbol{u}} = 0, \\ \oint_{\mathbb{T}_3} \boldsymbol{u} = \oint_{\mathbb{T}_3} \overline{\boldsymbol{u}} = 0, \\ \boldsymbol{u}_{t=0} = \boldsymbol{u}^{in}. \end{cases} \tag{123}$$

The above equations (123) contain a dispersive term, of energetic origin, with coefficient α, and a dissipative term with coefficient β ([**19**]). Therefore, it is normal to call the above equations the NS-$\overline{\boldsymbol{w}}\beta$. It is clear that the NS-$\overline{\boldsymbol{w}}\beta$ model verifies the same mathematical proprieties as the NS-$\overline{\boldsymbol{w}}$, in particular, the NS-$\overline{\boldsymbol{w}}\beta$ model admits a unique solution verifying the length scale estimates proved above for the NS-$\overline{\boldsymbol{w}}$.

There are many different alpha models that have the same properties of existence, uniqueness, regularity and consistency. In order to continue the comparison between these models and decide which model offers a better computational large eddy simulation model of turbulence, it remains to establish sharp estimates for the dimension of the attractor of NS-$\overline{\boldsymbol{w}}$ model. Then determine the alpha model that reduces more the number of freedom degrees in the turbulent flow.

References

[**1**] H. Ali, *Ladder theorem and length-scale estimates for a Leray alpha model of turbulence*, Commun. Math. Sci. **10** (2012), no. 2, 477-491.

[**2**] H. Ali, *Large eddy simulation for turbulent flows with critical regularization*, J. Math. Anal. Appl. **394** (2012), 291-304.

[**3**] Y. Cao, E. M. Lunasin and E. S. Titi, *Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models*, Commun. Math. Sci. **4** (2006), 823-848.

[**4**] A. Cheskidov, D. D. Holm, E. Olson and E. S. Titi, *On a Leray-α model of turbulence*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), 629-649.

[**5**] C. R. Doering and C. Foias, *Energy dissipation in body-forced turbulence*, J. Fluid Mech. **467** (2002), 289-306.

[6] C. R. DOERING and J. D. GIBBON, *Applied analysis of the Navier-Stokes equations*, Cambridge University Press, Cambridge 1995.

[7] C. FOIAS, D. D. HOLM and E. S. TITI, *The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory*, J. Dynam. Differential Equations **14** (2002), 1-35.

[8] C. FOIAS, D. D. HOLM and E. S. TITI, *The Navier-Stokes-alpha model of fluid turbulence*, Phys. D **152/153** (2001), 505-519.

[9] C. FOIAS, O. MANLEY, R. ROSA and R. TEMAM, *Navier-Stokes equations and turbulence*, Cambridge University Press, Cambridge 2001.

[10] U. FRISCH, *Turbulence. The legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge 1995.

[11] M. GERMANO, *Differential filters for the large eddy numerical simulation of turbulent flows*, Phys. Fluids **29** (1986), 1755-1757.

[12] B. J. GEURTS and D. D. HOLM, *Leray and LANS-α modelling of turbulent mixing*, J. Turbul. **7** (2006), Paper 10, 33 pp. (electronic).

[13] J. D. GIBBON and C. R. DOERING, *Intermittency and regularity issues in 3D Navier-Stokes turbulence*, Arch. Ration. Mech. Anal. **177** (2005), 115-150.

[14] J. D. GIBBON and D. D. HOLM, *Length-scale estimates for the LANS-α equations in terms of the Reynolds number*, Phys. D **220** (2006), no. 1, 69-78.

[15] J. D. GIBBON and D. D. HOLM, *Estimates for the LANS-α, Leray-α and Bardina models in terms of a Navier-Stokes Reynolds number*, Indiana Univ. Math. J. **57** (2008), 2761-2773.

[16] J. L. GUERMOND, J. T. ODEN and S. PRUDHOMME, *An interpretation of the Navier-Stokes-alpha model as a frame-indifferent Leray regularization*, Phys. D **177** (2003), no. 1-4, 23-30.

[17] M. HOLST, E. LUNASIN and G.TSOGTGEREL, *Analysis of a general family of regularized Navier-Stokes and MHD models*, J. Nonlinear Sci. **20** (2010), no. 5, 523-567.

[18] A. A. ILYIN, E. M. LUNASIN and E. S. TITI, *A modified-Leray-α subgrid scale model of turbulence*, Nonlinearity **19** (2006), 879-897.

[19] T.-Y. KIM, M. CASSIANI, J. D. ALBERTSON, J. E. DOLBOW, E. FRIED and M. E. GURTIN, *Impact of the inherent separation of scales in the Navier-Stokes-αβ equations*, Phys. Rev. E (3) **79** (2009), no. 4, 045307, 4 pp.

[20] R. H. KRAICHNAN, *Inertial ranges in two-dimensional turbulence*, Phys. Fluids **10** (1967), no. 7, 1417-1423.

[21] W. LAYTON, *Existence of smooth attractors for the Navier-Stokes-omega model of turbulence*, J. Math. Anal. Appl. **366** (2010), no. 1, 81-89.

[22] W. LAYTON and R. LEWANDOWSKI, *A high accuracy Leray-deconvolution model of turbulence and its limiting behavior*, Anal. Appl. (Singap.) **6** (2008), no. 1, 23-49.

[23] W. LAYTON, C. C. MANICA, M. NEDA and L. G. REBHOLZ, *Numerical analysis and computational comparisons of the NS-alpha and NS-omega regularizations*, Comput. Methods Appl. Mech. Engrg. **199** (2010), 916-931.

[24] W. LAYTON and R. LEWANDOWSKI, *On a well-posed turbulence model*, Discrete Contin. Dyn. Syst. Ser. B **6** (2006), no. 1, 111-128 (electronic).

[25] W. LAYTON, I. STANCULESCU and C. TRENCHEA, *Theory of the NS-$\overline{w}$ model: a*

complement to the NS-α model, Commun. Pure Appl. Anal. **10** (2011), no. 6, 1763-1777.

[26] L. G. REBHOLZ, *A family of new, high order NS-α models arising from helicity correction in Leray turbulence models*, J. Math. Anal. Appl. **342** (2008), no. 1, 246-254.

[27] R. TEMAM, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland, Amsterdam 1984; reedited by AMS Chelsea Publishing, Providence, RI 2001.

[28] W. WALTER, *Differential and integral inequalities*, translated from the German by L. Rosenblatt and L. Shampine, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 55, Springer-Verlag, New York-Berlin 1970.

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