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The quasilinearity of a family of functionals in linear spaces with applications to inequalities

Abstract. The superadditivity and subadditivity properties of certain composite functionals are investigated. Applications in refining Jensen's, Hölder's, Minkowski's and Schwarz's inequalities are given.

Keywords. Additive, superadditive and subadditive functionals, Convex functions, Jensen's inequality, Hölder's inequality, Minkowski's inequality.

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1 - Introduction

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;
(ii) $x \in C$, $\alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \to \mathbb{R}$ is called *superadditive* (*subadditive*) on C if

(iii) $h(x+y) \geq (\leq) h(x) + h(y)$ for any $x, y \in C$

and *nonnegative* (*strictly positive*) on C if, obviously, it satisfies

(iv) $h(x) \geq (>) 0$ for each $x \in C$.

The functional h is $s-$*positive homogeneous* on C, for a given $s > 0$, if

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

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In [**3**], the following result has been obtained:

Theorem 1. *Let $x, y \in C$ and $h : C \to \mathbb{R}$ be a nonnegative, superadditive and s-positive homogeneous functional on C. If $M \geq m \geq 0$ are such that $x - my$ and $My - x \in C$, then*

$$M^s h(y) \geq h(x) \geq m^s h(y). \tag{1.1}$$

Now, consider $v : C \to \mathbb{R}$ an *additive* and *strictly positive* functional on C which is also *positive homogeneous* on C, i.e.,

(vi) $v(\alpha x) = \alpha v(x)$ for any $\alpha > 0$ and $x \in C$.

In [**4**] we obtained further results concerning the quasilinearity of some composite functionals:

Theorem 2. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive (subadditive) functional on C and $p, q \geq 1$ $(0<p, q<1)$ then the functional*

$$\Psi_{p,q} : C \to [0, \infty), \Psi_{p,q}(x) = h^q(x) v^{q\left(1-\frac{1}{p}\right)}(x) \tag{1.2}$$

is superadditive (subadditive) on C.

Theorem 3. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive functional on C and $0<p, q<1$ then the functional*

$$\Phi_{p,q} : C \to [0, \infty), \Phi_{p,q}(x) = \frac{v^{q\left(1-\frac{1}{p}\right)}(x)}{h^q(x)} \tag{1.3}$$

is subadditive on C.

Another result similar to Theorem 1 has been obtained in [**4**] as well, namely

Theorem 4. *Let $x, y \in C$, $h : C \to \mathbb{R}$ be a nonnegative, superadditive and s-positive homogeneous functional on C and v an additive, strictly positive and positive homogeneous functional on C. If $p, q \geq 1$ and $M \geq m \geq 0$ are such that $x - my$, $My - x \in C$, then*

$$M^{sq+q\left(1-\frac{1}{p}\right)} \Psi_{p,q}(y) \geq \Psi_{p,q}(x) \geq m^{sq+q\left(1-\frac{1}{p}\right)} \Psi_{p,q}(y) \tag{1.4}$$

where $\Psi_{p,q}$ is defined by (1.2).

As shown in [**3**] and [**4**], the above results can be applied to obtain refinements of the Jensen, Hölder, Minkowski and Schwarz inequalities for weights satisfying certain conditions.

The main aim of the present paper is to study quasilinearity properties of other composite functionals and to apply the obtained results in improving some classical inequalities as those mentioned above.

2 - General Quasilinearity Properties

The following result holds.

Theorem 5. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C.*

(i) *If $p \geq q \geq 0, p \geq 1$ and $h : C \to [0, \infty)$ is superadditive on C, then the new two parameter functional*

$$\Lambda_{p,q} : C \to [0, \infty), \Lambda_{p,q}(x) := v^{\frac{p-q}{p}}(x) h^q(x) \tag{2.1}$$

is superadditive on C;

(ii) *If $p \leq q, p \in (0, 1)$ and $h : C \to [0, \infty)$ is subadditive on C, then the functional $\Lambda_{p,q}$ is subadditive on C.*

Proof. First of all we observe that the following elementary inequality holds:

$$(\alpha + \beta)^p \geq (\leq) \alpha^p + \beta^p \tag{2.2}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function $f_p : [0, \infty) \to \mathbb{R}, f_p(t) = (t + 1)^p - t^p$ we have $f_p'(t) = p\left[(t + 1)^{p-1} - t^{p-1}\right]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0, \infty)$. Now for $t = \dfrac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\dfrac{\alpha}{\beta} + 1\right)^p - \left(\dfrac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (2.2).

For $p \in (0, 1)$ we have that f_p is strictly decreasing on $[0, \infty)$ which proves the second case in (2.2).

(i). If $h : C \to [0, \infty)$ is superadditive on C and $p \geq 1$, then by (2.2) we have

$$h^p(x + y) \geq [h(x) + h(y)]^p \geq h^p(x) + h^p(y)$$

for any $x, y \in C$, which provides that

$$\frac{h(x+y)}{v(x+y)} \geq \frac{[h^p(x)+h^p(y)]^{\frac{1}{p}}}{v(x)+v(y)}$$

$$= \left(\frac{v(x)\left[\frac{h(x)}{v^{\frac{1}{p}}(x)}\right]^p + v(y)\left[\frac{h(y)}{v^{\frac{1}{p}}(y)}\right]^p}{v(x)+v(y)} \right)^{\frac{1}{p}} [v(x)+v(y)]^{\frac{1}{p}-1} \tag{2.3}$$

for any $x, y \in C$ and $p \geq 1$.

Utilising the monotonicity property of *power means*, i.e.,

$$\left(\frac{\alpha x^p + \beta y^p}{\alpha+\beta}\right)^{\frac{1}{p}} \geq \left(\frac{\alpha x^q + \beta y^q}{\alpha+\beta}\right)^{\frac{1}{q}}$$

where $p \geq q \geq 0$, and $\alpha, \beta, x, y \geq 0$ with $\alpha + \beta > 0$, then we have

$$\left(\frac{v(x)\left[\frac{h(x)}{v^{\frac{1}{p}}(x)}\right]^p + v(y)\left[\frac{h(y)}{v^{\frac{1}{p}}(y)}\right]^p}{v(x)+v(y)} \right)^{\frac{1}{p}} \tag{2.4}$$

$$\geq \left(\frac{v(x)\left[\frac{h(x)}{v^{\frac{1}{p}}(x)}\right]^q + v(y)\left[\frac{h(y)}{v^{\frac{1}{p}}(y)}\right]^q}{v(x)+v(y)} \right)^{\frac{1}{q}}$$

$$= \left(\frac{v^{1-\frac{q}{p}}(x)h^q(x) + v^{1-\frac{q}{p}}(y)h^q(y)}{v(x)+v(y)} \right)^{\frac{1}{q}}.$$

By making use of (2.3) and (2.4), we get

$$\frac{h(x+y)}{v(x+y)} \geq \left(\frac{v^{1-\frac{q}{p}}(x)h^q(x) + v^{1-\frac{q}{p}}(y)h^q(y)}{v(x)+v(y)} \right)^{\frac{1}{q}} [v(x)+v(y)]^{\frac{1}{p}-1}$$

$$= \left(v^{1-\frac{q}{p}}(x)h^q(x) + v^{1-\frac{q}{p}}(y)h^q(y) \right)^{\frac{1}{q}} [v(x)+v(y)]^{\frac{1}{p}-\frac{1}{q}-1}$$

$$= \left(v^{1-\frac{q}{p}}(x)h^q(x) + v^{1-\frac{q}{p}}(y)h^q(y) \right)^{\frac{1}{q}} v^{\frac{1}{p}-\frac{1}{q}-1}(x+y),$$

which is equivalent, by taking the power q, with

$$\frac{h^q(x+y)}{v^q(x+y)} \geq \left(v^{1-\frac{q}{p}}(x)h^q(x) + v^{1-\frac{q}{p}}(y)h^q(y)\right)v^{\frac{q}{p}-q-1}(x+y). \tag{2.5}$$

Moreover, if we multiply (2.5) with $v^{1+q-\frac{q}{p}}(x+y)$ we get

$$h^q(x+y)v^{1-\frac{q}{p}}(x+y) \geq v^{1-\frac{q}{p}}(x)h^q(x) + v^{1-\frac{q}{p}}(y)h^q(y)$$

for any $x, y \in C$, which shows that $\Lambda_{p,q}$ is superadditive on C.

(ii). If $h : C \to [0,\infty)$ is subadditive on C and $0<p<1$, then by (2.2) we have

$$h^p(x+y) \leq [h(x)+h(y)]^p \leq h^p(x) + h^p(y)$$

for any $x, y \in C$, which provides

$$\frac{h(x+y)}{v(x+y)} \leq \frac{[h^p(x)+h^p(y)]^{\frac{1}{p}}}{v(x)+v(y)}$$

for any $x, y \in C$.

On utilizing a similar argument to the one outlined above, we deduce the desired result. □

The following result provides upper and lower bounds for $\Lambda_{p,q}(x)$ in terms of $\Lambda_{p,q}(y)$ when certain connections between the vectors x and y are provided:

Corollary 1. *Let $x, y \in C$, $h : C \to \mathbb{R}$ be a nonnegative, superadditive and s-positive homogeneous functional on C and v an additive, strictly positive and positive homogeneous functional on C. If $p \geq 1, p \geq q$ and $M \geq m \geq 0$ are such that $x - my$, $My - x \in C$, then*

$$M^{sq+1-\frac{q}{p}}\Lambda_{p,q}(y) \geq \Lambda_{p,q}(x) \geq m^{sq+1-\frac{q}{p}}\Lambda_{p,q}(y). \tag{2.6}$$

Proof. Observe that the functional $\Lambda_{p,q}(\cdot)$ is $\left(sq+1-\dfrac{q}{p}\right)$–positive homogeneous and superadditive on C. Applying Theorem 1 we deduce the desired double inequality (2.6). □

Remark 1. *Now, if we assume that $p \geq q \geq 0, p \geq 1$, by denoting $r := \dfrac{q}{p} \in [0,1]$, then the functional*

$$\Theta_{p,r}(x) := v^{1-r}(x)h^{pr}(x)$$

is superadditive, provided that v is additive and h is superadditive on C. In par-

ticular, the functional

$$\Upsilon_t(x) := v^{\frac{1}{2}}(x)h^t(x)$$

is superadditive for $t \geq \dfrac{1}{2}$.

If $p \leq q, p \in (0,1)$ *and if we denote* $s := \dfrac{q}{p} \in [1, \infty)$, *then the functional*

$$F_{p,s}(x) := \frac{h^{sp}(x)}{v^{s-1}(x)}$$

is subadditive provided that v *is additive and* h *is subadditive on* C. *In particular, the functional*

$$\Xi_z(x) := \frac{h^z(x)}{v(x)}$$

is subadditive for $z \in (0,2)$.

3 - Some Results Related to Jensen's Inequality

Let C be a convex subset of the real linear space X and let $f : C \to \mathbb{R}$ be a convex mapping. Here we consider the following well-known form of *Jensen's discrete inequality*:

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i f(x_i), \tag{3.1}$$

where I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C$, $p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i\in I} p_i > 0$.

Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i \in C$ $(i \in I)$. Now consider the functional $J : S_+(I) \to \mathbb{R}$ given by

$$J_I(\mathbf{p}) := \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \geq 0 \tag{3.2}$$

where $S_+(I) := \left\{\mathbf{p} = (p_i)_{i\in I} \middle| p_i \geq 0,\ i \in I \text{ and } P_I > 0\right\}$ and f is convex on C.

We observe that $S_+(I)$ is a convex cone and the functional J_I is nonnegative and positive homogeneous on $S_+(I)$.

Lemma 1 ([**6**]). *The functional* $J_I(\cdot)$ *is a superadditive functional on* $S_+(I)$.

Define the following functional

$$L_{p,q,I}(\mathbf{p}) := P_I^{\frac{p-q}{p}}\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^q \tag{3.3}$$

for $p \geq 1$ and , $p \geq q \geq 0$.

The following proposition can be stated:

Proposition 1. *The functional* $L_{p,q,I}(\cdot)$ *is superadditive on* $S_+(I)$ *for any* $p \geq 1$ *and* $p \geq q \geq 0$.

Proof. Define $v(\mathbf{p}) = P_I$ and $h(\mathbf{p}) = J_I(\mathbf{p})$. Then for $p \geq 1$ and $p \geq q \geq 0$ we have

$$\Lambda_{p,q}(\mathbf{p}) = v^{\frac{p-q}{p}}(\mathbf{p}) h^q(\mathbf{p}) = P_I^{\frac{p-q}{p}} J_I^q(\mathbf{p}) = L_{p,q,I}(\mathbf{p})$$

for any $\mathbf{p} \in S_+(I)$.

Since $v(\cdot)$ is additive and $J_I(\cdot)$ is superadditive on $S_+(I)$ on applying Corollary 1 we conclude that $L_{p,q,I}(\cdot)$ is also superadditive on $S_+(I)$. □

Remark 2. *We observe that, in particular, the following functionals*

$$L_{p,\alpha,I}(\mathbf{p}) := P_I^{1-\alpha}\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{\alpha p}$$

and

$$\tilde{L}_{p,I}(\mathbf{p}) := P_I^{\frac{1}{2}}\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{\frac{p}{2}}$$

are superadditive on $S_+(I)$ *for any* $p \geq 1$ *and* $\alpha \in (0,1)$.

We can state the following result that provides a refinement and a reverse for the Jensen's inequality when bounds for the weights are known:

Proposition 2. *If* $\mathbf{p}, \mathbf{q} \in S_+(I)$ *and* $M \geq m \geq 0$ *are such that* $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, *i.e.,* $Mp_i \geq q_i \geq mp_i$ *for each* $i \in I$, *then:*

$$\begin{aligned}(3.4)\qquad & M^{\frac{1+\alpha(p-1)}{\alpha p}}\left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}}\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right] \\ & \geq \sum_{i\in I} q_i f(x_i) - Q_I f\left(\frac{1}{Q_I}\sum_{i\in I} q_i x_i\right) \\ & \geq m^{\frac{1+\alpha(p-1)}{\alpha p}}\left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}}\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right],\end{aligned}$$

for any $p \geq 1$ *and* $\alpha \in (0,1)$.

Proof. Follows by Corollary 1 applied for the functional $L_{p,\alpha,I}(\cdot)$. The details are omitted. □

The above Proposition 2 can be utilized to obtain various inequalities generated by the appropriate choices of the convex function f.

1. If $f : X \to \mathbb{R}$, $f(x) = \|x\|^r$, $r \geq 1$, $p \geq 1$ and $\alpha \in (0,1)$ where $(X, \|\cdot\|)$ is a normed linear space, then we can state the inequality:

$$M^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \left[\sum_{i\in I} p_i \|x_i\|^r - P_I^{1-r} \left\|\sum_{i\in I} p_i x_i\right\|^r\right] \tag{3.5}$$
$$\geq \sum_{i\in I} q_i \|x_i\|^r - Q_I^{1-r} \left\|\sum_{i\in I} q_i x_i\right\|^r$$
$$\geq m^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \left[\sum_{i\in I} p_i \|x_i\|^r - P_I^{1-r} \left\|\sum_{i\in I} p_i x_i\right\|^r\right]$$

and, in particular,

$$M^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \left[\sum_{i\in I} p_i \|x_i\| - \left\|\sum_{i\in I} p_i x_i\right\|\right] \tag{3.6}$$
$$\geq \sum_{i\in I} q_i \|x_i\| - \left\|\sum_{i\in I} q_i x_i\right\|$$
$$\geq m^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \left[\sum_{i\in I} p_i \|x_i\| - \left\|\sum_{i\in I} p_i x_i\right\|\right]$$

for $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{p}, \mathbf{q} \in S_+(I)$ with $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$ and $M \geq m > 0$ and for any vectors $x_i \in X$, $i \in I$.

2. For $x_i > 0$ and $p_i \geq 0$, $(i \in \mathbb{N})$ so that $P_I > 0$, let us denote

$$A(I, \mathbf{p}, x) := \frac{1}{P_I} \sum_{i\in I} p_i x_i, \qquad G(I, \mathbf{p}, x) := \left(\prod_{i\in I} (x_i)^{p_i}\right)^{\frac{1}{P_I}},$$

the *weighted arithmetic* and *geometric means,* respectively.

Applying the above Proposition 2 for the convex function $f(x) = -\ln x$, $x \in (0, \infty)$, we can state the following inequality:

$$\left[\frac{A(I,\mathbf{p},x)}{G(I,\mathbf{p},x)}\right]^{M^{\frac{1+\alpha(p-1)}{\alpha p}}\left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}}} \tag{3.7}$$
$$\geq \frac{A(I,\mathbf{q},x)}{G(I,\mathbf{q},x)} \geq \left[\frac{A(I,\mathbf{p},x)}{G(I,\mathbf{p},x)}\right]^{m^{\frac{1+\alpha(p-1)}{\alpha p}}\left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}}}$$

for $p \geq 1$ and $\alpha \in (0,1), I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{p},\mathbf{q} \in S_+(I)$ with $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$ and $M \geq m > 0$ and for any $x_i > 0$, $i \in I$.

4 - Some Results Related to Hölder's Inequality

Let $(X, \|\cdot\|)$ be a normed space and $I \in \mathcal{P}_f(\mathbb{N})$. We define

$$E(I) := \left\{x = (x_j)_{j\in I} \middle| x_j \in X,\ j \in I\right\}$$

and

$$\mathbb{K}(I) := \left\{\lambda = (\lambda_j)_{j\in I} \middle| \lambda_j \in \mathbb{K},\ j \in I\right\}.$$

We consider for $\gamma, \beta > 1, \frac{1}{\gamma} + \frac{1}{\beta} = 1$ the functional

$$H_I(\mathbf{p},\lambda,x;\gamma,\beta) := \left(\sum_{j\in I} p_j|\lambda_j|^\gamma\right)^{\frac{1}{\gamma}} \left(\sum_{j\in I} p_j\|x_j\|^\beta\right)^{\frac{1}{\beta}} - \left\|\sum_{j\in I} p_j\lambda_j x_j\right\|.$$

The following result has been proved in [**3**]:

Lemma 2. *For any* $\mathbf{p},\mathbf{q} \in S_+(I)$ *we have*

$$H_I(\mathbf{p}+\mathbf{q},\lambda,x;\gamma,\beta) \geq H_I(\mathbf{p},\lambda,x;\gamma,\beta) + H_I(\mathbf{q},\lambda,x;\gamma,\beta), \tag{4.1}$$

where $x \in E(I)$, $\lambda \in \mathbb{K}(I)$ *and* $\gamma, \beta > 1$ *with* $\frac{1}{\gamma} + \frac{1}{\beta} = 1$.

Remark 3. *The same result can be stated if* $(B, \|\cdot\|)$ *is a normed algebra and the functional* H *is defined by*

$$H_I(\mathbf{p},\lambda,x;\gamma,\beta) := \left(\sum_{i\in I} p_i\|x_i\|^\gamma\right)^{\frac{1}{\gamma}} \left(\sum_{i\in I} p_i\|y_i\|^\beta\right)^{\frac{1}{\beta}} - \left\|\sum_{i\in I} p_i x_i y_i\right\|,$$

where $x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \subset B$, $\mathbf{p} \in S_+(I)$ *and* $\gamma, \beta > 1$ *with* $\frac{1}{\gamma} + \frac{1}{\beta} = 1$.

Define the following functional on $S_+(I)$:

$$S_{p,\alpha,I}(\mathbf{p},\lambda,x;\gamma,\beta) := P_I^{1-\alpha} H_I^{\alpha p}(\mathbf{p},\lambda,x;\gamma,\beta), \tag{4.2}$$

where $\alpha \in [0,1]$ and $p \geq 1$.

We can state the following result.

Proposition 3. *If $\mathbf{p},\mathbf{q} \in S_+(I)$ and $M \geq m \geq 0$ with $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, then we have:*

$$\begin{aligned}
& M^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \\
& \times \left[\left(\sum_{i\in I} p_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i\in I} p_i\|x_i\|^\beta\right)^{\frac{1}{\beta}} - \left\|\sum_{i\in I} p_i\lambda_i x_i\right\|\right] \\
& \geq \left(\sum_{i\in I} q_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i\in I} q_i\|x_i\|^\beta\right)^{\frac{1}{\beta}} - \left\|\sum_{i\in I} q_i\lambda_i x_i\right\| \\
& \geq m^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \\
& \times \left[\left(\sum_{i\in I} p_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i\in I} p_i\|x_i\|^\beta\right)^{\frac{1}{\beta}} - \left\|\sum_{i\in I} p_i\lambda_i x_i\right\|\right]
\end{aligned} \tag{4.3}$$

for $x \in E(I)$, $\lambda \in \mathbb{K}(I)$, $\alpha \in (0,1)$, $p \geq 1$ and $\alpha,\beta > 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$.

Proof. Follows from Corollary 1 applied to the superadditive functional $S_{p,\alpha,I}(\cdot,\lambda,x;\gamma,\beta)$. The details are omitted. □

5 - Some Results Related to Minkowski's Inequality

Let $(X,\|\cdot\|)$ be a normed space and $I \in \mathcal{P}_f(\mathbb{N})$. We define the functional:

$$\begin{aligned}
M_I(\mathbf{p},x,y;\delta) = & \left[\left(\sum_{i\in I} p_i\|x_i\|^\delta\right)^{\frac{1}{\delta}} + \left(\sum_{i\in I} p_i\|y_i\|^\delta\right)^{\frac{1}{\delta}}\right]^\delta \\
& - \sum_{i\in I} p_i\|x_i+y_i\|^\delta,
\end{aligned} \tag{5.1}$$

where $\mathbf{p} \in S_+(I)$, $\delta \geq 1$ and $x,y \in E(I)$.

The following result concerning the superadditivity of the functional $M_I(\cdot, x, y; \delta)$ holds [3]:

Lemma 3. *For any* $\mathbf{p}, \mathbf{q} \in S_+(I)$, *we have*

$$M_I(\mathbf{p} + \mathbf{q}, x, y; \delta) \geq M_I(\mathbf{p}, x, y; \delta) + M_I(\mathbf{q}, x, y; \delta),$$

where $x, y \in E(I)$ *and* $\delta \geq 1$.

Since the functional $M_I(\cdot, x, y; \delta)$ is positive homogeneous on $S_+(I)$, then on utilizing Theorem 1 for the superadditive functional

$$V_{p,\alpha,I}(\mathbf{p}, \lambda, x; \gamma, \beta) := P_I^{1-\alpha} M_I^{\alpha p}(\mathbf{p}, x, y; \alpha),$$

we can state the following proposition.

Proposition 4. *If* $\mathbf{p}, \mathbf{q} \in S_+(I)$ *and* $M \geq m \geq 0$ *with* $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, *then we have:*

$$\begin{aligned}
(5.2)\qquad & M^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \\
& \times \left\{ \left[\left(\sum_{i \in I} p_i \|x_i\|^\delta \right)^{\frac{1}{\delta}} + \left(\sum_{i \in I} p_i \|y_i\|^\delta \right)^{\frac{1}{\delta}} \right]^\delta - \sum_{i \in I} p_i \|x_i + y_i\|^\delta \right\} \\
& \geq \left[\left(\sum_{i \in I} q_i \|x_i\|^\delta \right)^{\frac{1}{\delta}} + \left(\sum_{i \in I} q_i \|y_i\|^\delta \right)^{\frac{1}{\delta}} \right]^\delta - \sum_{i \in I} q_i \|x_i + y_i\|^\delta \\
& \geq m^{\frac{1+\alpha(p-1)}{\alpha p}} \left(\frac{P_I}{Q_I}\right)^{\frac{1-\alpha}{\alpha p}} \\
& \times \left\{ \left[\left(\sum_{i \in I} p_i \|x_i\|^\delta \right)^{\frac{1}{\delta}} + \left(\sum_{i \in I} p_i \|y_i\|^\delta \right)^{\frac{1}{\delta}} \right]^\delta - \sum_{i \in I} p_i \|x_i + y_i\|^\delta \right\},
\end{aligned}$$

where $x, y \in E(I)$, $\alpha \in (0, 1)$, $p \geq 1$ *and* $\delta \geq 1$.

6 - Some Results Related to Schwarz's Inequality

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, nonnegative forms on X, i.e., the mapping $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions

(i) $\langle x,x\rangle\geq 0$ for all $x\in X$;

(ii) $\langle \alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$ for all $x,y\in X$ and $\alpha,\beta\in\mathbb{K}$;

(iii) $\langle y,x\rangle=\overline{\langle x,y\rangle}$ for all $x,y\in X$.

If $\langle\cdot,\cdot\rangle\in\mathcal{H}(X)$, then the functional $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$ is a semi-norm on X and the following version of *Schwarz's inequality* holds:

$$\|x\|\|y\|\geq|\langle x,y\rangle| \tag{6.1}$$

for each $x,y\in H$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$. Also, we can introduce on $\mathcal{H}(X)$ the following *binary relation* [**5**]

$$\langle\cdot,\cdot\rangle_2\geq\langle\cdot,\cdot\rangle_1 \quad\text{if and only if}\quad \|x\|_2\geq\|x\|_1 \quad\text{for any}\quad x\in H. \tag{6.2}$$

This is an *order relation* on $\mathcal{H}(X)$.

Consider the following functional [**5**]:

$$\sigma:\mathcal{H}(X)\times X^2\to\mathbb{R}_+,\qquad \sigma(\langle\cdot,\cdot\rangle;x,y):=\|x\|\|y\|-|\langle x,y\rangle|,$$

which is closely related to the Schwarz inequality in (6.1).

Lemma 4 ([**5**]). *The functional $\sigma(\cdot;x,y)$ is nonnegative, superadditive and positive homogeneous on $\mathcal{H}(X)$.*

The following proposition can be stated.

Proposition 5. *Let $M\geq m>0$ and $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ be two inner products on X such that $M\|x\|_1\geq\|x\|_2\geq m\|x\|_1$ for each $x\in X$. If $e\in X, e\neq 0$, $\alpha\in(0,1)$, and $p\geq 1$ then*

$$\begin{aligned} &M^{\frac{2[1+\alpha(p-1)]}{\alpha p}}\left(\frac{\|e\|_1}{\|e\|_2}\right)^{\frac{2(1-\alpha)}{\alpha p}}\left(\|x\|_1\|y\|_1-\left|\langle x,y\rangle_1\right|\right)\\ &\geq\|x\|_2\|y\|_2-\left|\langle x,y\rangle_2\right|\\ &\geq m^{\frac{2[1+\alpha(p-1)]}{\alpha p}}\left(\frac{\|e\|_1}{\|e\|_2}\right)^{\frac{2(1-\alpha)}{\alpha p}}\left(\|x\|_1\|y\|_1-\left|\langle x,y\rangle_1\right|\right),\end{aligned} \tag{6.3}$$

for any $x,y\in H$.

Proof. Consider the functional

$$\begin{aligned}\Omega_{p,\alpha,e}(\langle\cdot,\cdot\rangle) &= \langle e,e\rangle^{1-\alpha}\sigma^{\alpha p}(\langle\cdot,\cdot\rangle;x,y)\\ &= \|e\|^{2(1-\alpha)}[\|x\|\|y\| - |\langle x,y\rangle|]^{\alpha p}.\end{aligned}$$

Since this functional is superadditive, then by Theorem 1 we have

$$\begin{aligned}M^{2[1+\alpha(p-1)]}\|e\|_1^{2(1-\alpha)}&\left[\|x\|_1\|y\|_1 - \left|\langle x,y\rangle_1\right|\right]^{\alpha p}\\ &\geq \|e\|_2^{2(1-\alpha)}\left[\|x\|_2\|y\|_2 - \left|\langle x,y\rangle_2\right|\right]^{\alpha p}\\ &\geq m^{2[1+\alpha(p-1)]}\|e\|_1^{2(1-\alpha)}\left[\|x\|_1\|y\|_1 - \left|\langle x,y\rangle_1\right|\right]^{\alpha p}\end{aligned} \tag{6.4}$$

for x, y fixed in X.

Taking the power $\dfrac{1}{\alpha p} > 0$ in the inequality (6.4) we deduce the desired result (6.3). □

The above result can be used to obtain some inequalities for positive definite operators as follows:

Corollary 2. *Assume that $A : H \to H$ is a self-adjoint linear operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$ satisfying the property that there exist $\Gamma \geq \gamma > 0$ such that $\Gamma I \geq A \geq \gamma I$ in the operation order (i.e., $\Gamma\|x\|^2 \geq \langle Ax,x\rangle \geq \gamma\|x\|^2$ for any $x \in H$), then for $p > 1$ we have the inequality:*

$$\begin{aligned}\frac{\Gamma^{\frac{1+\alpha(p-1)}{\alpha p}}}{\langle Ae,e\rangle^{\frac{1-\alpha}{\alpha p}}}&(\|x\|\|y\| - |\langle x,y\rangle|)\\ &\geq \langle Ax,x\rangle^{1/2}\langle Ay,y\rangle^{1/2} - |\langle Ax,y\rangle|\\ &\geq \frac{\gamma^{\frac{1+\alpha(p-1)}{\alpha p}}}{\langle Ae,e\rangle^{\frac{1-\alpha}{\alpha p}}}(\|x\|\|y\| - |\langle x,y\rangle|),\end{aligned} \tag{6.5}$$

for any $x, y \in H$ and $e \in H$ with $\|e\| = 1$.

Remark 4. *Similar results can be stated if one uses the following nonnegative, superadditive and positive homogeneous functionals on $\mathcal{H}(X)$ (see* [**2**, pp. 8-15]):

$$\sigma_r(\langle\cdot,\cdot\rangle;x,y):=\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle;$$

$$\delta(\langle\cdot,\cdot\rangle;x,y):=\|x\|^2\|y\|^2-|\langle x,y\rangle|^2;$$

$$\delta_r(\langle\cdot,\cdot\rangle;x,y):=\|x\|^2\|y\|^2-(\operatorname{Re}\langle x,y\rangle)^2;$$

$$\gamma(\langle\cdot,\cdot\rangle;x,y):=\frac{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2}{\|y\|^2};$$

where in the definition of γ, $\langle\cdot,\cdot\rangle$ *is an inner product and* y *is not zero, and*

$$\beta(\langle\cdot,\cdot\rangle;x,y):=\left(\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right)^{\frac{1}{2}},$$

for each $x,y\in X$.

The details are left to the interested reader.

Remark 5. *For other examples of superadditive* (*subadditive*) *functionals that can provide interesting inequalities similar to the ones outlined above, we refer to* [**1**], [**7**], [**8**] [**9**] *and* [**10**].

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References

[**1**] H. Alzer, *Sub- and superadditive properties of Euler's gamma function*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3641-3648.

[**2**] S. S. Dragomir, *Advances in inequalities of the Schwarz, triangle and Heisenberg type in inner product spaces*, Nova Science Publishers, New York 2007.

[**3**] S. S. Dragomir, *Inequalities for superadditive functionals with applications*, Bull. Aust. Math. Soc. **77** (2008), 401-411.

[**4**] S. S. Dragomir, *Quasilinearity of some composite functionals with applications*, Bull. Aust. Math. Soc. **83** (2011), no. 1, 108-121.

[**5**] S. S. Dragomir and B. Mond, *On the superadditivity and monotonicity of Schwarz's inequality in inner product spaces*, Makedon. Akad. Nauk. Umet. Oddel. Mat.-Tehn. Nauk. Prilozi **15** (1994), no. 2, 5-22.

[**6**] S. S. Dragomir, J. Pečarić and L. E. Persson, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungar. **70** (1996), 129-143.

[**7**] L. Losonczi, *Sub- and superadditive integral means*, J. Math. Anal. Appl. **307** (2005), no. 2, 444-454.

[8] S. Y. Trimble, J. Wells and F. T. Wright, *Superadditive functions and a statistical application*, SIAM J. Math. Anal. **20** (1989), no. 5, 1255-1259.

[**9**] A. Vitolo and U. Zannier, *Inequalities for superadditive functions related to a property R [of] convex sets*, Boll. Un. Mat. Ital. A (7) **4** (1990), no. 3, 309-314.

[**10**] P. Volkmann, *Sur les fonctions simultanément suradditives et surmultiplicatives*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) **28**(**76**) (1984), no. 2, 181-184.

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