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Finite defective subsets of projective spaces

Abstract. We study finite sets $S \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_S(m)) > 0$ and either $\sharp(S) \le 4m + 2r - 15$ or $\sharp(S) \le mr + 1$ and a large subset of S is in linearly general position.

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1 - Introduction

In [**3**] A. Couvreur proved how to compute the minimal distance of the dual of the code obtained by evaluating all the homogeneous polynomials of degree m in $r+1$ variables at a finite subset S of $\mathbb{P}^r$. In [**1**] we considered the following classical question.

Question 1. *Fix positive integers r, m, z such that $r \ge 2$. Describe all subsets $S \subset \mathbb{P}^r$ such that $\sharp(S) \le z$ and $h^1(\mathcal{I}_S(m)) > 0$.*

For arbitrary r, m, z, Question 1 is hopeless. As in [**1**] we take z not too large with respect to m, r (in [**1**] with $z = 4m + r - 5$ and a few other assumptions). In this paper we prove the following results.

Theorem 1. *Fix integers $m \ge 2$, $r \ge 9$ and $8m \ge r + 22$. Let $S \subset \mathbb{P}^r$ be a finite subset such that $\sharp(S) \le 4m + 2r - 15$ and $\sharp(S \cap M) \le 4m - 5$ for each plane*

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M. We have $h^1(\mathcal{I}_S(m)) > 0$ *if and only if there is a hyperplane* $H \subset \mathbb{P}^r$ *such that* $h^1(\mathcal{I}_{S\cap H}(m)) > 0$.

We recall that a zero-dimensional scheme $Z \subset \mathbb{P}^r$ is said to be in linearly general position if for each $t \in \{1, \dots, r-1\}$ we have $\deg(Z \cap M) \le t+1$ for every t-dimensional linear subspace $M \subset \mathbb{P}^r$.

Theorem 2. *Fix integers* $m \ge 6$ *and* $r \ge 10$. *Let* $A, B \subset \mathbb{P}^r$, $r \ge 3$, *be finite subsets such that* $A \cap B = \emptyset$ *and* A *is in linearly general position. Set* $S := A \cup B$. *Assume* $\sharp(B) \le 8m-19$, $\sharp(B) < 1 + (m+1)(r+1-\lfloor (r+2)/2 \rfloor)/2$, $\sharp(S) \le mr+1$, $\sharp(S \cap H) \le 4m+r-6$ *for each hyperplane* $H \subset \mathbb{P}^r$ *and* $\sharp(S \cap M) \le 4m-5$ *for each plane* $M \subset \mathbb{P}^r$. *We have* $h^1(\mathcal{I}_S(m)) > 0$ *if and only if there is* $W \subseteq S$ *as in one of the following cases:*

(a) $\sharp(W) = m+2$ *and* W *is contained in a line;*
(b) $\sharp(W) = 2m+2$ *and* W *is contained in a plane conic;*
(c) $\sharp(W) = 3m$ *and* W *is the complete intersection of a degree* 3 *plane curve and a degree* m *surface;*
(d) $\sharp(W) \ge 3m+1$ *and* W *is contained in a degree* 3 *plane curve;*
(e) $\sharp(W) = 3m+2$ *and* W *is contained in a reduced and connected degree* 3 *curve spanning* $\mathbb{P}^3$.

Obviously $\sharp(A \cap W) \le 2$ in case (a), $\sharp(A \cap W) \le 3$ in cases (b), (c), (d) and $\sharp(A \cap W) \le 4$ in case (e). If $A \subset S$ and $h^1(\mathcal{I}_A(m)) > 0$, then $h^1(\mathcal{I}_S(m)) > 0$. Hence the "if"part of Theorems 1 and 2 is obvious. If $\sharp(S) \le 4m+r-5$, then Theorem 1 is true by [**1**], Theorem 1. We will use in an essential way the *statement* of [**1**], Theorem 1. Any improvement of [**1**], Theorem 1, would hopefully give a corresponding improvement of Theorem 1 and of Theorem 2.

Proposition 1. *Fix integers* $r \ge 10$ *and* $m \ge 6$. *Let* $S \subset \mathbb{P}^r$ *be a finite subset such that* $\sharp(S) \le 4m+3r-10$, $\sharp(S) < (m-2)(r+1)/2 + m/2 + 3$, $\sharp(S \cap M) \le 4m-5$ *for each plane* $M \subset \mathbb{P}^r$ *and either* $\sharp(S) \le 12m+2r-47$ *or* $\sharp(S) < r(m-1) - \lfloor (r+2)/2 \rfloor (m-4) + 2m - 4$. *Then there is a hyperplane* H *such that* $h^1(H, \mathcal{I}_{S\cap H,H}(m-1)) > 0$.

Proposition 1 is the first possibility: we could fix $t \in \{1, \dots, m-2\}$ and ask $h^1(H, \mathcal{I}_{S\cap H,H}(t)) = 0$ for every hyperplane H (the case $t=1$ is equivalent to the definition of *linearly general position* ([**4**])).

Remark 1. Results like Theorems 1 and 2 or Proposition 1 are interesting for arbitrary zero-dimensional schemes, not just for finite sets. In the applications on Goppa codes often zero-dimensional schemes must be used. When $r = 2$ everything is fine for arbitrary zero-dimensional schemes ([**5**]). The cases $r = 3$ and $r = 4$ of our proof of [**1**], Theorem 1, heavily use that S is reduced. Below, the proofs of Theorems 1 and 2 and of Proposition 1 do not use that S is reduced, except that we heavily use the statement of [**1**], Theorem 1.

2 - The proofs

Lemma 1. *Fix integers r, e, m such that $r \geq 6$ and $3 \leq e \leq m/2$. Then*

$$e + e(4(m - e) + r - 6) > 4m + 3r - 18. \tag{1}$$

Proof. Set $\psi(t) := t(4(m - t) + r - 5)$. The function ψ is increasing in the interval $0 \leq t \leq m/2$. Hence it is sufficient to prove the lemma when $e = 3$. We have $12m + 3r - 36 - 15 > 4m + 3r - 18$, because $m \geq 6$. $\square$

Proof of Theorem 1. Since $8m \geq r + 22$, we may assume $m \geq 4$.

If $r = 9$, then Theorem 1 is true by [**1**], Theorem 1. Hence we may assume $r \geq 10$. We cannot use induction on m and we do not use induction on r, but only use [**1**], Theorem 1, in $\mathbb{P}^k$, $2 \leq k \leq r - 1$. If $A \subset S$ and $h^1(\mathcal{I}_A(m)) > 0$, then $h^1(\mathcal{I}_S(m)) > 0$.

Now assume $h^1(\mathcal{I}_S(m)) > 0$ and that $h^1(\mathcal{I}_{S \cap H}(m)) = 0$ for every hyperplane $H \subset \mathbb{P}^r$ (i.e. $h^1(H, \mathcal{I}_{S \cap H, H}(m)) = 0$ for every hyperplane $H \subset \mathbb{P}^r$). Taking a subset of S if necessary we may assume $h^1(\mathcal{I}_A(m)) = 0$ for every $A \subsetneq S$. With this assumption we need to find a contradiction. We may also assume $\sharp(S) \geq 4m + r - 4$ ([**1**], Theorem 1).

Set $S_0 := S$. Let $H_1 \subset \mathbb{P}^r$ be a hyperplane such that $a_1 := \sharp(S_0 \cap H_1)$ is maximal. Set $S_1 := S_0 \setminus S_0 \cap H_1$. For each integer $i \geq 2$ define recursively the non-negative integer a_i, the hyperplane H_i and the set $S_i \subseteq S_{i-1}$ in the following way. Let H_i be any hyperplane such that $a_i := \sharp(H_i \cap S_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap H_i$. The sequence $\{a_i\}_{i \geq 1}$ is non-increasing. Since any r points of $\mathbb{P}^r$ are contained in a hyperplane, if $a_i \leq r - 1$, then $a_{i+1} = 0$. For each integer $i \geq 1$ we have an exact sequence

$$0 \to \mathcal{I}_{S_i}(m - i) \to \mathcal{I}_{S_{i-1}}(m + 1 - i) \to \mathcal{I}_{S_{i-1} \cap H_i, H_i}(m + 1 - i) \to 0 \tag{2}$$

(often called the Castelnuovo's sequence or the Horace's lemma). By [**1**], Remark 1, there is an integer $i \geq 1$ such that $h^1(H_i, \mathcal{I}_{S_{i-1} \cap H_i, H_i}(m + 1 - i)) > 0$. Let e be the minimal such an integer. Notice that if $i \geq m + 2$, then $h^1(H_i, \mathcal{I}_{S_{i-1} \cap H_i, H_i}(m + 1 - i)) > 0$ if and only if $S_{i-1} \cap H_i \neq \emptyset$. Hence $e \leq m + 2$.

First assume $e = m+2$, then $\sharp(S) \ge (m+1)r+1$. Since $r \ge 5$ and $m \ge 4$, we have $(m+1)r+1 > 4m+2r-15$, a contradiction. Now assume $e = m+1$. Since $h^1(H_{m+1}, \mathcal{I}_{S_m \cap H_{m+1}, H_{m+1}}) > 0$, we have $a_{m+1} \ge 2$. Hence $\sharp(S) \ge rm+2$. Since $m \ge 4$ and $r \ge 6$, by induction on r we check that $rm+2 > 4m+2r-15$, a contradiction. Hence we may assume $e \le m$.

Since $h^1(H_1, \mathcal{I}_{H_1 \cap S, H_1}(m)) = 0$ by assumption, we have $e \ge 2$.

(a) Since $e \le m$, it is easy to check that $a_e \ge m+3-e$ and that equality holds if and only if there is a line $D \subset H_e$ such that $S_{e-1} \cap H_e \subset D$ ([**2**], Lemma 34). Let $U \subseteq H_e$ denote the linear subspace of H_e spanned by the set $S_{e-1} \cap H_e$. Set $\alpha := \dim(U)$. Since $e \le m$ and $h^1(H_1, \mathcal{I}_{H_1 \cap S, H_1}(m+1-e)) = 0$, we have $a_e \ge \alpha + 2$. Assume for the moment $\alpha \le r-2$. Since $S_{e-1} \ne \emptyset$, the set S_{e-2} spans $\mathbb{P}^r$. Hence there is a hyperplane H of $\mathbb{P}^r$ containing U and at least $r-1-\alpha$ points of S_{e-2}. Hence $a_{e-1} \ge a_e + r - 1 - \alpha$. Hence $a_i \ge a_e + r - 1 - \alpha \ge r+1$ for all $i < e$. Now assume $\alpha = r-1$. In this case we get $a_i \ge r+1$ for all $i \le e$, because $a_e \ge r+1$.

(b) In this step we assume $e \ge m/2$. Recall that $a_i \ge r+1$ for all $i < e$ and that $a_e \ge m+3-e$ (step (a)).

First assume $e \ge m/2+1$. We get $\sharp(S) \ge m(r+1)/2+3$. Hence

$$8m + 4r - 30 \ge mr + m + 6. \tag{3}$$

Obviously (3) is false if $r = 9$. Since $m \ge 4$, we get that (3) is false for $r > 9$ by induction on r.

Now assume $e = (m+1)/2$. We get $\sharp(S) \ge (r+1)(m-1)/2 + (m-1)/2 + 3$ and hence

$$8m + 4r - 30 \ge (m-1)(r+2) + 6. \tag{4}$$

Since $m \ge 4$ and m is odd, we have $m \ge 5$. Hence (4) fails if $r = 9$. Induction on r gives that (4) is false for all $r \ge 10$.

Now assume $e = m/2$. We get $\sharp(S) \ge (m/2-1)(r+1) + (m/2+3)$. Hence

$$8m + 4r - 30 \ge (m-2)(r+1) + m + 6. \tag{5}$$

If $m = 8$, then (5) fails. Since $r \ge 9$, we see by induction on m that (5) fails for all $m \ge 8$. Hence we only need to check the cases with $m \le 7$. Since $e = m/2$, we only need to do the cases $(m, e) = (6, 3)$ and $(m, e) = (4, 2)$. Assume $m = 4$. Since $8m \ge r + 22$, we get $r \in \{9, 10\}$. We have $\sharp(S) \le 4m + 2r - 15 = 2r + 1$. Since $r \le 10$, we have $2r + 1 \le r + 11 = 4m + r - 5$. In this case we may apply [**1**], Theorem 1. Now assume $m = 6$ and $e = 3$. Since $4m + 2r - 15 = 2r + 9 \ge \sharp(S) \ge (r+1) + (r+1) + a_3$, we get $a_3 \le 7$. Recall that $h^1(H_3, \mathcal{I}_{H_3 \cap S_2, H_3}(4)) > 0$. Since $a_3 \le 2 \cdot 4 + 1$, there is a line $J \subset H_3$ such that $\sharp(J \cap S_2) \ge 6$ ([**2**], Lemma 34). Hence $a_3 \ge 6$. Since S_1 spans $\mathbb{P}^r$ and 6 of its points are contained in a line, the maximality property of a_2 implies $a_2 \geq r+4$. Since $a_1 \geq a_2$, we get $\sharp(S) \geq 2r+14 > 2r+9$, a contradiction.

From now on we assume $e < m/2$. In particular we assume $m \geq 5$ and $m-e \geq 3$. Hence $4(m-e+1)-4 \geq 2(m-e+1)+2$.

(c) Assume $a_e \geq 4(m-e+1)+\alpha-4$. Since $a_e > 0$, the set S_{e-2} spans $\mathbb{P}^r$. Hence there is a hyperplane $M \subset \mathbb{P}^r$ containing U and any $r-1-\alpha$ points of S_{e-2}. The maximality property of a_{e-1} gives $a_{e-1} \geq 4(m-e+1)+r-5$. Hence $\sharp(S) \geq e(4(m-e+1)+r-5)+\alpha-r+1 \geq e(4(m-e+1)+r-5)-r+2$. Lemma 1 gives a contradiction if $e \geq 3$. Now assume $e=2$. Let $M_1 \supset U$ be a hyperplane such that the integer $m_1 := M_1 \cap S_0$ is maximal among all the hyperplanes containing U. We just saw that $m_1 \geq 4m-9+r$. Set $S_1' := S \setminus S \cap M_1$. For all integers $i \geq 2$ define recursively the integer m_i, the hyperplane M_i and the set $S_i' \subseteq S_{i-1}'$ in the following way. Let M_i be any hyperplane such that $m_i := \sharp(M_i \cap S_{i-1}')$ is maximal and set $S_i' := S_{i-1}' \setminus S_{i-1}' \cap M_i$. If $S_{i+1}' \neq \emptyset$, then $m_i \geq r$. Since $\sharp(S) < m_1 + r$, we get $S_2' = \emptyset$, i.e. $S \subset M_1 \cup M_2$. Let V be the linear span of $S_1' \cap M_2$. Set $\beta := \dim(V)$. Since $m_2 < r$, we have $\beta \leq r-2$. Since $h^1(M_1, \mathcal{I}_{S\cap M_1, M_1}(m)) = h^1(\mathcal{I}_{S\cap M_1}(m)) = 0$, [**1**], Remark 1, gives $h^1(V, \mathcal{I}_{V\cap S_1', V}(m-1)) = h^1(M_2, \mathcal{I}_{M_2\cap S_1', M_2}(m-1)) > 0$. Since $a_1 \geq m_1 \geq 4m-9+r$, $a_2 \geq 4m+\alpha-8$ and $a_1+a_2 \leq 4m-15+2r$, we get $4m+\alpha \leq r+2$. Since $8m \geq r+22$, we get $2\alpha \leq r-18$ and hence $r \geq 20$. We have $m_1+m_2 \geq a_1+a_2$. Since $a_1 \geq m_1$, we get $m_2 \geq a_2$. Since S spans $\mathbb{P}^r$, we get $m_1 \geq m_2+(r-1-\beta)$. First assume $m_2 \geq 4m-8+\beta$. As above we get $2\beta \leq r-18$. Since $\alpha+\beta \leq r-2$, there is a hyperplane containing $U \cup V$. Hence $a_1 \geq 4m-8+\alpha+4m-8+\beta+(r-2-\alpha-\beta)$. Hence $12m-26+\alpha+r \leq 4m-15+2r$, contradicting the inequality $8m \geq r+22$. Now assume $m_2 \leq 4m-9+\beta$. By [**1**], Theorem 1, applied to the integer $m-1$ and the projective space V we get the existence of an integer $j \in \{1,2,3\}$, $j \leq \beta$, and a j-dimensional linear subspace N of V such that $\sharp(S_1' \cap N) \geq j(m-1)+2$. Since $\alpha+j+1 \leq r-1$, there is a hyperplane of $\mathbb{P}^r$ containing $U \cup V$. We take one such hyperplane, W_1, such that $n_1 := \sharp(W_1 \cap S)$ is maximal. Since S spans $\mathbb{P}^r$, we get $n_1 \geq 4m-8+\alpha+j(m-1)+2+r-2-j-\alpha$. Define the hyperplanes W_i, $i \geq 2$, in the following way. Fix $i \geq 2$ and assume defined the hyperplanes W_j, $1 \leq j \leq i-1$. Let W_i be a hyperplane containing the maximal number of points of $S \setminus (S \cap (\cup_{j=1}^{i-1} W_j))$. Notice that if $S \not\subseteq \cup_{i=1}^{i-1} W_j$, then $\sharp(S \cap W_{i-1}) - \sharp(S \cap (\cup_{j=1}^{i-2} W_j)) \geq r$. Hence $n_i = 0$ for all $i \geq 3$. Using (2) we get $h^1(W_2, \mathcal{I}_{S\setminus S\cap W_1, W_2}(m-1)) > 0$. First assume $\sharp(S \setminus S \cap W_1) \geq 4m-8+\beta$. As above we get $2\beta \leq r-18$. Hence there is a hyperplane containing $U \cup V$. Hence $4m-15+2r \geq 4m-8+\alpha+4m-8+\beta+(r-2-\alpha-\beta)$, a contradiction. Now assume $\sharp(S \setminus S \cap W_1) \leq 4m-9+\beta$. By [**1**], Theorem 1, there are $j' \in \{1,2,3\}$, $j' \leq \beta'$ and a j'-dimensional linear subspace N' such that $\sharp(N' \cap (S \setminus S \cap W_1)) \geq j'(m-1)+2$.

Since $j+j' \le 6$, $\alpha \le r/2-9$ and $r \ge 20$, there is a hyperplane E_1 of $\mathbb{P}^r$ containing $U \cup N \cup N'$. If $j+j' \ge 4$, then we get $4m-15+2r \ge 4m-8+\alpha+(j+j')(m-1)+4+(r-3-\alpha-j-j')+4m-8+\alpha$, a contradiction. Now assume $j+j' \le 3$. As above there is a hyperplane E_2 containing $S \setminus S \cap E_1$. Call β' the dimension of the linear span of $S \setminus S \cap E_1$. If $\sharp(S)-\sharp(S \cap E_1) \ge 4m-8+\beta'$, then as above we get a contradiction. Now assume $\sharp(S)-\sharp(S \cap E_1) \le 4m-9+\beta'$. By [**1**], Theorem 1, there are an integer $j'' \in \{1,2,3\}$, $j'' \le \beta'$, and a j''-dimensional linear subspace N'' of E_2 such that $\sharp(N'' \cap (S \setminus S \cap W_1)) \ge j'(m-1)+2$. Since $j+j'+j'' \le 3+j'' \le 6$, $\alpha \le r/2-9$ and $r \ge 20$, there is a hyperplane $\mathbb{P}^r$ containing $U \cup N \cup N' \cup N''$. Call F_1 any such hyperplane with $\sharp(S \cap F_1)$ maximal. If $j+j'+j'' \ge 4$, then we get $a_1 \ge 4m-8+\alpha+(j+j'+j'')(m-1)+6+(r-1-\alpha-j-j'-j''-3)$ and hence we get a contradiction. Now assume $j+j'+j'' \le 3$, i.e. $j=j'=j''=1$. As above we get the existence of a hyperplane F_2 containing $S \setminus S \cap F_1$. Call β'' the dimension of the linear span of $S \setminus S \cap F_1$. If $\sharp(S)-\sharp(S \cap F_1) \ge 4m-8+\beta''$, then as above we get a contradiction. Now assume $\sharp(S)-\sharp(S \cap F_1) \le 4m-9+\beta''$. By [**1**], Theorem 1, there are an integer $j_1 \in \{1,2,3\}$, $j_1 \le \beta''$, and a j_1-dimensional linear subspace N_+ of E_2 such that $\sharp(N_+ \cap (S \setminus S \cap F_1)) \ge j_1(m-1)+2$. Since $\alpha+j+j'+j''+3+3 \le r-1$, there is a hyperplane containing $U \cup N \cup N' \cup N'' \cup N_+$. We get $a_1 \ge 4m-8+\alpha+(j+j'+j''+j_1)(m-1)+6+(r-1-\alpha-j-j'-j''-j_1-4)$, a contradiction.

(d) From now we assume $a_e \le 4(m-e+1)+\alpha-5$. By [**1**], Theorem 1, applied to U we get either the existence of an integer $j \in \{1,2,3\}$ and a j-dimensional linear subspace $N_1 \subseteq U \subseteq H_e$ such that $\sharp(N_1 \cap S_{e-1}) \ge j(m-e+1)+2$ or the existence of a plane containing at least $4(m-e+1)-4$ points of S_{e-1} (we may take $N_1 := U$ if $\alpha \le 2$ by [**2**], Lemma 34). In the latter case we may take $j=2$ and take this plane as N_1. Set $r_1 := \dim(N_1)$. Let $H_{1,1}$ be a hyperplane containing N_1 and such that $a_{1,1} := \sharp(S_{1,0} \cap H_{1,1})$ is maximal. Set $S_{1,1} = S_{1,0} \setminus S_{1,0} \cap H_{1,1}$. For each integer $i \ge 2$ define recursively the non-negative integer $a_{1,i}$, the hyperplane $H_{1,i}$ and the set $S_{1,i} \subseteq S_{1,i-1}$ in the following way. Let $H_{1,i}$ be any hyperplane such that $a_{1,i} := \sharp(H_{1,i} \cap S_{1,i-1})$ is maximal. Set $S_{1,i} := S_{1,i-1} \setminus S_{1,i-1} \cap H_{1,i}$. The sequence $\{a_{1,i}\}_{i \ge 2}$ is non-increasing. As for the integer a_i we see that $a_{1,i+1} = 0$ if $a_{1,i} \le r-1$. We have an exact sequence similar to (2) with $H_{1,i}, S_{1,i}$ and $S_{1,i-1}$ instead of H_i, S_i and S_{i-1}. From this exact sequence we get the existence of an integer $e(1) \ge 1$ such that $h^1(H_{1,e(1)}, \mathcal{I}_{S_{1,e(1)-1} \cap H_{1,e(1)}, H_{1,e(1)}}(m+1-e(1))) > 0$. Since $h^1(H_{1,1}, \mathcal{I}_{S \cap H_{1,1}, H_{1,1}}(m)) = 0$, we have $e(1) \ge 2$. As for e we first see that $e(1) \le m$ and then use steps (a) and (b) to exclude the case $e(1) \ge m/2$. Now assume $e(1) < m/2$. Let α_1 denote the dimension of the linear span U_1 of $S_{1,e(1)-1} \cap H_{1,e(1)}$. As in step (c) we exclude the case $a_{e(1)} \ge 4(m-e(1)+1)-4+\alpha_1$. Hence we may assume $a_{e(1)} \le 4(m-e(1)+1)-5+\alpha_1$. Hence there are an integer $j \in \{1,2,3\}$ and a j-dimensional linear subspace

$N_2 \subseteq H_{1,e(1)}$ such that $\sharp(N_2 \cap S_{1,e(1)-1}) \geq j(m-e(1)+1)+2$ ([**1**], Theorem 1). Set $r_2 := \dim(N_2)$. Notice that $S_{1,e(1)-1} \cap N_2 \cap N_1 = \emptyset$, because $N_1 \subset H_{1,1}$ and $e(1) \geq 2$. Hence $N_2 \cap S_{1,e(1)-1}$ and $N_1 \cap S_{e-1}$ are disjoint subsets of S. Since $r \geq 10$, we have $\dim(N_1) + \dim(N_2) \leq r-2$. Hence there is a hyperplane of $\mathbb{P}^r$ containing $N_1 \cup N_2$. Set $S_{2,0} := S$. Let $H_{2,1}$ be a hyperplane containing $N_1 \cup N_2$ and such that $a_{2,1} := \sharp(S_{1,1} \cap H_{2,1})$ is maximal among all hyperplanes containing $N_1 \cup N_2$. Set $S_{2,1} = S_{2,0} \setminus S_{2,0} \cap H_{2,1}$. For each integer $i \geq 2$ define recursively the non-negative integer $a_{2,i}$, the hyperplane $H_{2,i}$ and the set $S_{2,i} \subseteq S_{2,i-1}$ in the following way. Let $H_{2,i}$ be any hyperplane such that $a_{2,i} := \sharp(H_{2,i} \cap S_{2,i-1})$ is maximal. Set $S_{2,i} := S_{2,i-1} \setminus S_{2,i-1} \cap H_{2,i}$. The sequence $\{a_{2,i}\}_{i\geq 2}$ is non-increasing. We have an exact sequence similar to (2) with $H_{2,i}$, $S_{2,i}$ and $S_{2,i-1}$ instead of H_i, S_i and S_{i-1}. From this exact sequence we get the existence of an integer $e(2) \geq 2$ such that $h^1(H_{2,e(2)}, \mathcal{I}_{S_{2,e(2)-1} \cap H_{2,e(2)}, H_{2,e(2)}}(m+1-e(2))) > 0$. As for e we first see that $e(2) \leq m$ and that $a_{2,e(2)-1} \geq r+1$, and then (step (b)) exclude the case $e(2) \geq m/2$. Now assume $e(2) < m/2$. Let α_2 denote the dimension of the linear span U_2 of $S_{2,e(2)-1} \cap H_{2,e(2)}$. As in step (c) we see that $a_{2,e(2)} \leq 4(m-e(2)+1)-5+\alpha_2$. Hence there are an integer $j \in \{1,2,3\}$ and a j-dimensional linear subspace $N_3 \subseteq H_{1,e(2)}$ such that $\sharp(N_3 \cap S_{1,e(2)-1}) \geq j(m-e(2)+1)+2$. Set $r_3 := \dim(N_3)$. If $r_1+r_2+r_3+2 \geq r$, then we set $s := 3$. Assume for the moment $r_1+r_2+r_3+2 \leq r-1$. Take a hyperplane containing $N_1 \cup N_2 \cup N_3$. And so on. We continue in the same way until we get a linear subspace N_s of dimension $r_s \in \{1,2,3\}$ with $\sharp(N_i \cap (S \setminus S \cap (N_1 \cup \cdots \cup N_{i-1}))) \geq (m-e(i)+1)r_i + 2$ for all $i \in \{2,\dots,s\}$ and $s-1+r_1+\cdots+r_s \geq r$. Fix an integer $i \in \{1,\dots,s\}$ such that $e(i) \geq 4$. Since $a_{i,e(i)} > 0$, the set $S_{i,e(i)-2}$ spans $\mathbb{P}^r$. Hence there is a hyperplane containing N_i and at least $r-1-r_i$ further points of $S_{i,e(i)-2}$. Hence $a_{i,e(i)-1} \geq 2 + r_i(m-e(i)+1)+r-1-r_i = r+3+r_i(m-e(i))$. Since $a_{i,h} \geq a_{i,x}$ if $2 \leq h < x$, we get

$$\sharp(S) \geq (e(i)-2)(r+3+r_i(m-e(i))) + r_i(m-e(i)+1)+2+a_{i,1}. \tag{6}$$

Obviously $a_{i,1} \geq r$. Assume for the moment $e(i) \geq 4$ (and hence $m > 2e(i) \geq 8$). Since $r_i \leq 3 < r$, the right hand side of (6) is an increasing function of $e(i)$ in the interval $[2, m/2)$. Hence $\sharp(S) \geq 2(r+1+r_i(m-4)+2)+2(m-4)+r \geq 3r+4m-9$, a contradiction. Now assume the existence of $i \in \{1,\dots,s\}$ such that $r_i \geq 2$ and $e(i) = 3$ (and hence $m \geq 7$). If $r_i = 2$ from (6) we get $\sharp(S) \geq (r+2m-3)+2m-4+r$, a contradiction. If $r_i = 3$ from (6) we get $\sharp(S) \geq (r+3m-6)+3m-7+r$, a contradiction. Now assume the existence of an integer $i \in \{2,\dots,s-1\}$ such that $r_i = 1$ and $e(i) = 3$. Hence $m \geq 7$. We have $a_{i,3} \geq m$ and $a_{i,2} \geq r-3+m$. Since $H_{i,1}$ contains $N_1 \cup N_2$, we have $a_{i,1} \geq r-5+2m$. Hence $\sharp(S) \geq 2r-8+4m$, a contradiction. Hence from now on we may assume $e(i) \leq 3$ for all i, $e(i) = 2$, if $2 \leq i \leq s-1$ and

$e(1) = 2$ if $r_1 \neq 1$. Set $\rho := \sum_{i=1}^{s-1} r_i$. We have

$$r - 1 - r_s \leq \rho + s - 2 \leq r - 1. \tag{7}$$

Since there is a hyperplane containing $N_1 \cup \cdots \cup N_{s-1}$, we have $a_{s,1} \geq \sum_{i=1}^{s-1} (r_i(m - e(i) + 1) + 2) \geq 2(s-1) + \rho(m-1) - 1$ (we use that $e(1) = 2$ if $r_1 \neq 1$). Since $s \leq \lfloor (r+2)/2 \rfloor$ and $r \geq 10$, we have $r \geq s + 4$.

(d1) First assume $e(s) = 2$ and $r_s = 1$. From (7) we get $\rho \geq r - s$. We have $a_{s,2} \geq m + 1$. We have $\sharp(S) \geq a_{s,1} + m + 1 \geq \rho(m-1) - 1 + 2(s-1) + m + 1 \geq r(m-1) - s(m-3) + m - 2 = 4m + 2r - 15 + [(r-s)(m-3) - 3m + 13] > 4m + 2r - 15$ (since $r \geq s + 3$), a contradiction.

(d2) Now assume $e(s) = 2$ and $r_s > 1$. We have $a_{s,2} \geq 2m$ (case $r_s = 2$) and $a_{s,2} \geq 3m - 1$ (case $r_s = 3$). Since $H_{s,1}$ contains $N_1 \cup N_2$ and S spans $\mathbb{P}^r$, we have $a_{s,1} \geq r - 1 - r_1 - r_2 - 1 + r_1(m - e(1) + 1) + r_2(m - e(2) + 1) + 4$, contradicting the assumption $\sharp(S) \leq 4m + 2r - 15$.

(d3) Now assume $e(s) = 3$. We have $a_{s,3} \geq r_s(m-2) + 2$ and hence $a_{s,2} \geq r - 1 - r_s + r_s(m-2) + 2 = r + 1 + r_s(m-3)$. If $r_s > 1$, we only use that $a_{s,1} \geq r$. Now assume $r_s = 1$. In this case we use that $a_{s,1} \geq r - 1 - r_1 - r_2 - 2 + r_1(m - e(1) + 1) + r_2(m - e(2) + 1) + 4$ (step (d2)). □

Proof of Proposition 1. The proof is absolutely similar to the one of Theorem 1. It only requires few obvious numerical adjustments, due to the new assumptions. □

Proof of Theorem 2. We observed that it is sufficient to do the "only if" part. Assume $h^1(\mathcal{I}_{A \cup B}(m)) > 0$. Set $A_0 := A$ and $B_0 := B$. Let $H_1 \subset \mathbb{P}^r$ be a hyperplane such that $b_1 := \sharp(H_1 \cap B)$ is maximal and, among the hyperplanes H with $\sharp(B \cap H) = b_1$, with $b_1 := A \cap H_1$ maximal. Set $A_1 := A_0 \setminus A_0 \cap H_1$, $a_i := \sharp(A_0 \cap H_1)$ and $B_1 := B_0 \setminus B_0 \cap H_1$. Since A is in linearly general position, we have $0 \leq a_i \leq r$ for all i. The maximality property of the integer b_i implies that if $b_i \leq r - 1$, then $B_i = \emptyset$ and that if $b_i \leq r$ and $B_i \neq \emptyset$, then B_{i-1} is in linearly general position in $\mathbb{P}^r$. Set $S_i := A_i \cup B_i$. The exact sequences (2) imply the existence of an integer $t \geq 1$ such that $h^1(H_t, \mathcal{I}_{S_{t-1} \cap H_t, H_t}(m + 1 - t)) > 0$ and we call e the minimal such an integer. As in the proof of Theorem 1 we get $e \leq m + 1$. Since $h^1(\mathcal{I}_{\{P\}}) = 0$ for each $P \in \mathbb{P}^r$, $\sharp(A \cup B) \leq mr + 1$ and $a_i + b_i \geq r$ if $S_i \neq \emptyset$, we get $e \leq m$. Let U denote the linear span of $S_{e-1} \cap H_e$. Set $\alpha := \dim(U)$. Since A is in linearly general position, we have $a_e \leq \alpha + 1$.

First assume $e = 1$. Apply [**1**], Theorem 1, to $S \cap H_1$. We get that we are in one of the cases (a), ..., (e). Hence we may assume $e \geq 2$. Since A is in linearly general position, we have $h^1(H, \mathcal{I}_{A\cap H}(1)) = 0$ for every hyperplane H. Since $e \leq m$, we get $b_e > 0$. Hence $b_i \geq r$ for all $i < e$. If $e = m$, then we get $\sharp(B) \geq (m-1)r + 1$, a contradiction. Now assume $e \leq m - 1$. Since $m \geq 6$, we have $(m/2 - 1)r + 1 \geq 1 + (m+1)(r + 1 - \lfloor (r+2)/2 \rfloor)/2$. Since $\sharp(B) < (m/2 - 1)r + 1$, we get $e < m/2$.

First assume $a_e + b_e \geq 4(m - e + 1) + \alpha - 4$. Hence $b_e \geq -1 + 4(m - e)$. Hence $\sharp(B) \geq 4e(m - e) - e$. Set $\psi(t) := 4t(m - t) - t$. The function ψ is non-decreasing in the interval $2 \leq t \leq (m-1)/2$. Since $2 \leq e < m/2$ and $\psi(2) = 8m - 18$, we get $\sharp(B) \geq 8m - 18$, a contradiction.

Now assume $a_e + b_e \leq 4(m - e + 1) + \alpha - 5$. By [**1**], Theorem 1, applied to the integer $m - e + 1$ there are an integer $j \in \{1, 2, 3\}, j \leq \alpha$, and a j-dimensional linear subspace $N_1 \subseteq U$ such that $\sharp(S_{e-1} \cap N_1) \geq j(m - e + 1) + 2$. Notice that $\sharp(A \cap N_1) \leq j + 1$ and hence $\sharp(B_{e-1} \cap N_1) \geq j(m - e) + 1$. Iterating we get an integer $s \geq 2$ and integers $e(i)$, $1 \leq i \leq s$. By the cases just done we get a string of non-negative integers $a_{i,j}$ and $b_{i,j}$, $1 \leq i \leq s, 1 \leq j \leq e(i)$, such that $a_{i,j} \leq r$ for all i, j, each sequence $\{b_{i,j}\}_{j \geq 2}$ is non-decreasing and $b_{i,j} \geq r$ if $j < e(i)$. Since $\sharp(B) < (m/2 - 1)r + 1$, we get $e(i) < m/2$ for all i. As in step (d) of the proof of Theorem 1 we get an integer $s > 0$, linear spaces N_i, $1 \leq i \leq s$, of dimension $r_i \in \{1, 2, 3\}$, with $r_1 + \cdots + r_{s-1} + s - 2 \leq r - 1 < r_1 + \cdots + r_s + s - 1$ and $\sharp(N_i \cap (B \setminus B \cap (N_1 \cup \cdots \cup N_{i-1})) \geq r_i(m - e(i)) + 1$ for all $i \in \{2, \ldots, s\}$. Set $\rho := \sum_{i=1}^{s-1} r_i$. Since $m - e(i) \geq (m+1)/2$, we get $b_{s,1} \geq s - 1 + \sum_{i=1}^{s-i} r_i(m - e(i)) \geq s - 1 + \rho(m+1)/2$. Instead of (6) we get the inequality

$$\sharp(B) \geq (e(s) - 2)((s-1) + \sum_{i=1}^{s-1} (m - e(i))) + 1 + r_s(m - e(s)) + b_{s,1}. \tag{8}$$

Since $e(i) < m/2$ and $b_{s,1} \geq s - 1 + \rho(m+1)/2$, we get $\sharp(B) \geq \rho(m+1)/2 + 1 + r_s(m+1)/2$. We have $\rho \geq r - s + 1 - r_s$ and $s \leq \lfloor (r+2)/2 \rfloor$. Hence $\sharp(B) \geq 1 + (m+1)(r + 1 - \lfloor (r+2)/2 \rfloor)/2$, a contradiction. □

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