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Absolutes of Hausdorff spaces and cardinal invariants F_θ and t_θ

Abstract. This article extends the recent study of the cardinal functions F_θ and t_θ for H-closed Urysohn spaces and the research of I. Bandlov and V.I. Ponomarev on tightness type of absolutes. In particular, some results are obtained and used to study the relationships among the cardinal functions t, t_θ, F and F_θ in the context of Iliadis and Banaschewski absolutes of Hausdorff spaces.

Keywords. Closure, θ-closure, free sequence, θ-free sequence, $t(X)$, $t_\theta(X)$, $F(X)$, $F_\theta(X)$, compact spaces, H-closed spaces, Urysohn spaces, absolutes of Hausdorff spaces.

Mathematics Subject Classification (2010): 54A25, 54D10, 54D25, 54D35, 54G20.

1 - Introduction

Absolutes can be traced to the fundamental papers by M. H. Stone [**19, 20**]. Absolutes received a major boost in research by A. M. Gleason [**11**] in 1958 quickly followed by the major studies of B. Banaschewski, J. Flachsmeyer, S. Iliadis, J. Mioduszewski, V. I. Ponomarev, L. Rudolf and L. B. Shapiro [**5, 10, 13, 15, 16, 17**]. The tightness type of cardinal invariants were investigated for absolutes by I. Bandlov and V.I. Ponomarev [**6**] in 1980.

This paper continues this line of research for absolutes using two relatively new cardinal functions F_θ and t_θ. In particular, we obtain some results connecting t, t_θ, F, and F_θ in the context of Iliadis and Banaschewski absolutes of Hausdorff spaces.

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2 - Notations, terminologies and basic properties

Throughout this paper X will denote a Hausdorff space and $\tau(X)$ the topology on X. Our notation and terminology are mainly as in [**9**] (for general topological notions), [**2**], [**12**], [**14**] (for cardinal functions), [**18**] [**21**] (for H-closed spaces, H-closed extensions and absolutes of Hausdorff spaces) and finally in [**7**].

Here are a few basic definitions:

- With $\alpha, \beta, \gamma, \ldots$ are denoted the infinite ordinal numbers and with $\kappa, \lambda, \mu, \ldots$ are denoted the infinite cardinal numbers. With $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{J}$, $\mathbb{R}$ we respectively denote the sets of positive integer, rational, irrational and real numbers with the usual topology. Also, by $\mathbb{I}^\kappa$ and $\mathbb{D}^\kappa$ we respectively denote the *Tychonoff cube* and the *Cantor cube* of weight κ.
- For a space X, recall that $\tau(X)(s)$ is the topology generated by the base $RO(X) = \{U \in \tau(X) : U = int_X(cl_X(U))\}$ (semiregularization of X). A space X is *semiregular* if its topology $\tau(X)$ coincides with the topology $\tau(X)(s)$ and we denote it by $X(s)$ (or X_s).
  Clearly, *every T_3-space X is semiregular* (the converse is not true).
- A function $f : X \to Y$ is *θ-continuous* if for each $x \in X$ and open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f(cl_X(U)) \subseteq cl_Y(V)$. It easy to see that *every continuous function is θ-continuous* (the converse is not true).
- A surjection $f : X \to Y$ is *irreducible* if for each closed set $A \subseteq X$, if $A \neq X$, then $f(A) \neq Y$. Equivalently, f is *irreducible* iff for each nonempty open set $U \in \tau(X)$, there is $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U$.
- A space X is *H-closed* if X is closed in every Hausdorff space containing X as a subspace. Equivalently, X is *H-closed* if every open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{V}$ whose union is dense in X $\left(\text{i.e. } X \subseteq cl_X\left(\bigcup_{V \in \mathcal{V}} V\right)\right)$.
  We need this well-known result (see in [**18**]):

  X is H-closed Urysohn iff X_s is compact Hausdorff.

- A space X is *extremally disconnected* (or *ED* for short) if the closure of every open set is open or, equivalently, if the closure of every open subset is clopen in X, i.e., in symbol $CLOP(X) = RO(X)$.
  It is easy to verify that

  X is ED iff X_s is ED and semiregular.

- [**9**] Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be two collections of spaces, $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$ their product spaces and let $\{f_i\}_{i \in I}$ be a family of functions

$f_i \in F(X_i, Y_i)$. The *product function* $f = \prod_{i\in I} f_i$ is defined by $\Pi_i^Y \circ f = f_i \circ \Pi_i^X$ for each $i \in I$ (where $\Pi_i^X : X \to X_i$ and $\Pi_i^Y : Y \to Y_i$ are respectively the i-th projection functions of X onto X_i and Y onto Y_i).

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \Pi_i^X \big\downarrow & & \big\downarrow \Pi_i^Y \\ X_i & \xrightarrow{\ f_i\ } & Y_i \end{array}$$

Then, the product function is explicitely defined by:

$$f\left(\langle x_i\rangle_{i\in I}\right) = \left(\prod_{i\in I} f_i\right)\left(\langle x_i\rangle_{i\in I}\right) = \langle f_i(x_i)\rangle_{i\in I}$$

for each $x = \langle x_i\rangle_{i\in I} \in X$.

It is well known that if for each $i \in I, f_i$ is continuous (or θ-continuous), then $f = \prod_{i\in I} f_i$ is continuous (or θ-continuous) and conversely.

– [18] For a space X, let $X^* = X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$. Let κX be the set X^* with the topology generated by the base $\tau(X) \cup \{U \cup \{\mathcal{U}\} : U \in \mathcal{U} \in X^* \setminus X\}$, and σX be the set X^* with the topology generated by the base $\{o(U) : U \in \tau(X)\}$ where $o(U) = U \cup \{\mathcal{U} \in X^* \setminus X : U \in \mathcal{U}\}$. Both spaces κX and σX are H-closed extensions of X. κX is called the *Katětov H-closed extension of* X and σX is said the *Fomin H-closed extension of* X. The identity function $id : \kappa X \to \sigma X$ is continuous. The remainder of κX $(= \kappa X \setminus X)$ is *discrete* and *closed* in κX, and the remainder of σX $(= \sigma X \setminus X)$ is a *zero-dimensional subspace* of σX. If X is a Tychonoff space, then $\kappa X \geq_X \sigma X \geq_X \beta X$ where βX denote the Stone-Čech compactification of X. When X is Tychonoff, $\kappa X = \beta X$ iff X is compact and $\sigma X = \beta X$ iff every closed nowhere dense subset of X is compact. Also, we have that $(\kappa X)_s = (\sigma X)_s = \beta X$.

– [18] Let X be a space and θX (called *the Stone space generated by* $RO(X)$ or *the Gleason cover of* X) denote the set of all open ultrafilters on X. For $U \in \tau(X)$ let $oU = \{\mathcal{U} \in \theta X : U \in \mathcal{U}\}$ and the topology on θX generated by $\{oU : U \in \tau(X)\}$ is ED and compact Hausdorff. The subspace $EX = \{\mathcal{U} \in \theta X : a(\mathcal{U}) \neq \emptyset\}$ (called *the Iliadis absolute of* X) is dense, ED and T_3 (hence 0-dimensional). We define $k_X : EX \to X$ by $k_X(\mathcal{U}) = p$ where $a(\mathcal{U}) = \{p\}$. The function k_X is *onto*, *perfect*, *irreducible* and *θ-continuous*. Also, the function k_X is continuous if and only if X is T_3. Note that $EX = \bigcup_{p\in X} k_X^{\leftarrow}(p)$.

In general, $\{oU \cap k_X^{\leftarrow}(V) : U, V \in \tau(X)\}$ is a base for a topology on EX (finer than $\tau(EX)$). The set EX with this finer topology is denoted by PX (called *the*

Banaschewski absolute of X). The map $\Pi_X : PX \to X$ defined by $\Pi_X(\mathcal{U}) = k_X(\mathcal{U})$ is *onto*, *perfect*, *irreducible* and *continuous*. The space PX is ED but may not be T_3 (hence not 0-dimensional). Also, $\tau(PX)(s) = \tau(EX)$ and when X is T_3, $PX = EX$.

This following fact is well-known:

X is H-closed iff EX is compact iff PX is H-closed Urysohn.

For the *Katětov H-closed extension* $\kappa\omega$ *of* ω, note that $P(\kappa\omega) = \kappa\omega$ and $E(\kappa\omega) = (P(\kappa\omega))_s = (\kappa\omega)_s = \beta\omega$.

For the *Fomin H-closed extension* $\sigma\omega$ *of* ω, note that $P(\sigma\omega) = \sigma\omega = P(\beta\omega) = \beta\omega$ and $E(\sigma\omega) = (P(\sigma\omega))_s = (\beta\omega)_s = \beta\omega$.

Definition 2.1. For $x \in X$, $t(x, X) = \min\{\kappa : \forall\, A \subset X$ with $x \in \overline{A}\ \exists\, B \subset A$ s.t. $|B| \leq \kappa$ and $x \in \overline{B}\}$ is called the *tightness of* X *at* x.

$t(X) = \sup_{x \in X} \{t(x, X)\} + \omega$ is called the *tightness of* X.

$t_\theta(x, X) = \min\{\kappa : \forall\, A \subset X$ with $x \in cl_\theta(A)\ \exists\, B \subset A$ s.t. $|B| \leq \kappa$ and $x \in cl_\theta(B)\}$ is called the *θ-tightness of* X *at* x.

$t_\theta(X) = \sup_{x \in X} \{t_\theta(x, X)\} + \omega$ is called the *θ-tightness of* X.

Definition 2.2. A sequence $(x_\alpha : \alpha \in \mu)$ in a space X is called a *free sequence of length* μ if for every $\alpha \in \mu$ we have

$$cl_X\{x_\beta : \beta < \alpha\} \cap cl_X\{x_\beta : \beta \geq \alpha\} = \emptyset.$$

A sequence $(x_\alpha : \alpha \in \mu)$ in a space X is called a *θ-free sequence of length* μ if for every $\alpha \in \mu$ we have

$$cl_\theta\{x_\beta : \beta < \alpha\} \cap cl_\theta\{x_\beta : \beta \geq \alpha\} = \emptyset.$$

We define:

$F(X) = \sup\{\mu :$ there is a free sequence of length μ in X$\} + \omega$.
$F_\theta(X) = \sup\{\mu :$ there is a θ-free sequence of length μ in X$\} + \omega$.

Here are some basic results that we will use throughout the paper:

Proposition 2.1. *Let X be a space and $Y \subseteq X$ as subspace.*

(a) *$t_\theta(X) = t_\theta(X_s)$ and $F_\theta(X) \leq F(X)$;*
(b) *If σ is a topology in X such that $\sigma \supseteq \tau(X)$, then $F(X) \leq F(X, \sigma)$ (in particular $F(X_s) \leq F(X)$ and $F(EX) \leq F(PX)$);*

(c) $t(Y) \le t(X)$;
(d) *If Y is closed in X, then $F(Y) \le F(X)$;*
(e) *If X is T_3, then $t_\theta(X) = t(X)$ and $F_\theta(X) = F(X)$.*

Recall the following result:

Theorem 2.1.

(a) [**1**] *If X is compact Hausdorff, then $F(X) = t(X)$;*
(b) [**7**] *If X is H-closed Urysohn, then*

$$F_\theta(X) = F_\theta(X_s) = F(X_s) = t_\theta(X) = t_\theta(X_s) = t(X_s).$$

Note 2.1. In [**7**], we constructed an H-closed space H for which $F_\theta(H) < t_\theta(H)$.

Now, we start with these straightforward results:

Lemma 2.1 ([**18**]). *Let X and Y be spaces, $A \subseteq X$ and $f : X \to Y$ θ-continuous. Then, $f(cl_\theta(A)) \subseteq cl_\theta(f(A))$.*

Lemma 2.2. *Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two collections of spaces, $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. Suppose $f = \prod_{i\in I} f_i : X \to Y$ where $f_i : X_i \to Y_i$ (for each $i \in I$) are surjections. Then, f is irreducible if and only if f_i is irreducible for each $i \in I$.*

Proof. Let $f : X \to Y$ be irreducible and, fixed $i \in I$, we want to show that $f_i : X_i \to Y_i$ is irreducible. So, let $i \in I$ and a nonempty open set U_i in X_i. Then, $U = U_i \times \prod_{j\neq i} X_j$ is a nonempty open set in X. By hypothesis, there exists $y = \langle y_i\rangle_{i\in I} \in Y$ such that $f^{\leftarrow}(y) \subseteq U$. So, for each $i \in I$, $f_i^{\leftarrow}(y_i) \subseteq U_i$. Thus, $f_i : X_i \to Y_i$ is irreducible for each $i \in I$.

Conversely, suppose $f_i : X_i \to Y_i$ is irreducible for each $i \in I$ and we want to show that $f : X \to Y$ is irreducible. So, let U be a nonempty open set in X. We want to find $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U$. Now, there are a finite set $J \subseteq I$ and a nonempty open set U_i in X_i for each $i \in I$ such that $\emptyset \neq \prod_{i\in J} U_i \times \prod_{i\in I\setminus J} X_i \subseteq U$. By hypothesis, for each $i \in I$, there is $y_i \in Y_i$ such that $f_i^{\leftarrow}(y_i) \subseteq U_i$ and for $i \notin J$ select a point $y_i \in Y_i$. For $y = \langle y_i\rangle_{i\in I}$, $f^{\leftarrow}(y) = \prod_{i\in I} f_i^{\leftarrow}(y_i) \subseteq \prod_{i\in J} U_i \times \prod_{i\in I\setminus J} X_i \subseteq U$. Thus, $f : X \to Y$ is irreducible. □

Lemma 2.3. *Let X and Y be spaces, $p \in X$ and $f : X \to Y$ be a perfect, irreducible, θ-continuous surjection. Then, p is isolated in X if and only if $f(p)$ is isolated in Y.*

Proof. Suppose p is isolated in Y and, since f is irreducible, it follows that $f^{\leftarrow}(f(p)) = \{p\}$. As, $X \setminus \{p\}$ is closed, $f(X \setminus \{p\}) = f(X \setminus f^{\leftarrow}(f(p))) = f(X) \setminus \{f(p)\} = Y \setminus \{f(p)\}$ is closed. So, $\{f(p)\}$ is open and $f(p)$ is isolated in Y. Conversely, suppose $f(p)$ is isolated in Y and, since f is θ-continuous, there is an open set $U \in \tau(X)$ such that $p \in U$ and $f(cl_X U) \subseteq cl_Y\{f(p)\} = \{f(p)\}$. So, $U \setminus \{p\}$ is open and if $U \setminus \{p\} \neq \emptyset$, there is $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U \setminus \{p\}$. As $f(U) = \{f(p)\}$, $y = f(p)$ implying $p \in f^{\leftarrow}(f(p)) \subseteq U \setminus \{p\}$: a contradiction. Thus, $U = \{p\}$ is open and p is isolated in X. □

Note 2.2.

(a) Example 15 in [7] shows that there is an H-closed and Urysohn space X such that $F_\theta(X) < F(X)$.

(b) In general, neither $t(X) \leq t_\theta(X)$ nor $t_\theta(X) \leq t(X)$. In fact, in [7], the space in Example 11 shows that $t_\theta(X) < t(X)$ and the space in Example 12 shows that $t(X) < t_\theta(X)$.

3 - Some results

First, we examine some of the basic properties concerning cardinal functions F_θ and t_θ on absolutes:

Theorem 3.1. *For a Hausdorff space X we have that*

(a) $F(PX) \geq F_\theta(PX) = F(EX) = F_\theta(EX) \geq F_\theta(X)$;

(b) $t(PX) = t_\theta(PX) = t(EX) = t_\theta(EX) \geq t_\theta(X)$.

Proof. (a) Let $(x_\alpha)_{\alpha\in\mu}$ be a θ-free sequence in X and choose $y_\alpha \in k_X^{\leftarrow}(x_\alpha)$ and $\beta < \mu$.

Then, by Lemma 2.1, we have that

$$k_X\big(cl_\theta\{y_\alpha\}_{\alpha\leq\beta}\big) \subseteq cl_\theta\{k_X(y_\alpha)\}_{\alpha\leq\beta} = cl_\theta\{x_\alpha\}_{\alpha\leq\beta}$$

and also

$$k_X\big(cl_\theta\{y_\alpha\}_{\alpha>\beta}\big) \subseteq cl_\theta\{k_X(y_\alpha)\}_{\alpha>\beta} = cl_\theta\{x_\alpha\}_{\alpha>\beta}\,.$$

Moreover, $cl_\theta\{x_\alpha\}_{\alpha\leq\beta} \cap cl_\theta\{x_\alpha\}_{\alpha>\beta} = \emptyset$ and so $cl_\theta\{y_\alpha\}_{\alpha\leq\beta} \cap cl_\theta\{y_\alpha\}_{\alpha>\beta} = \emptyset$. Thus $F_\theta(X) \leq F_\theta(EX) = F(EX) = F_\theta(PX) \leq F(PX)$.

(b) Let $A \subseteq X$ and $p \in cl_\theta(A)$; we have that $k_X^{\leftarrow}(p)$ is compact and assume $k_X^{\leftarrow}(p) \cap cl_{EX} k_X^{\leftarrow}(A) = \emptyset$ and this means that there exists a clopen set U such that

$k_X^{\leftarrow}(p) \subseteq U$ and $U \cap k_X^{\leftarrow}(A) = \emptyset$; then $k_X(U) \cap A = \emptyset$ and so $k_X(U) = cl_X(V)$ with $p \in V$. Thus $p \notin cl_\theta(A)$ and this means that there exists $q \in k_X^{\leftarrow}(p) \cap cl_{EX} k_X^{\leftarrow}(A)$ and then $q \in cl_{EX} k_X^{\leftarrow}(A)$. So, there is $B \subset k_X^{\leftarrow}(A)$ such that $|B| \leq t(EX) = t_\theta(EX)$ and $q \in cl_{EX}(B) = cl_\theta(B)$.

Now we have $k_X(q) \in cl_\theta(k_X(B))$ where $p = k_X(q)$ and $k_X(B) \subseteq A$; so, $|k_X(B)| \leq |B| \leq t(EX)$.

Thus $t_\theta(X) \leq t_\theta(EX) = t(EX) = t_\theta(PX) = t(PX)$. □

Note 3.1.

(a) [**7**, Ex. 12] $\omega = t(\kappa\omega) < \mathfrak{c} = t_\theta(\kappa\omega) = F_\theta(\kappa\omega) < 2^{\mathfrak{c}} = F(\kappa\omega)$.

(b) [**12**, Ex. 7.22] As $\sigma\omega = \beta\omega$, $\mathfrak{c} = t(\sigma\omega) = F(\sigma\omega) = F_\theta(\sigma\omega) = t_\theta(\sigma\omega)$.

Proposition 3.1. *Let X be a Hausdorff space and Y be ED.*

(a) $|X| \leq |EX| \leq 2^{2^{|X|}}$ *and* $|X| \leq |PX| \leq 2^{2^{|X|}}$;

(b) *EX is ED Tychonoff and therefore semiregular and EY is homeomorphic to Y_s;*

(c) *PX is ED and PY is homeomorphic to Y;*

(d) *If D is a dense set of isolated points in X, then $D \subseteq EX \subseteq \beta D$;*

(e) *There is a discrete space D such that $|D| \leq d(Y_s)$ and Y_s can be embedded in βD;*

(f) *A countable subset A of Y_s is C^*-embedded in Y_s. In particular, if B is an infinite compact subspace of Y_s, then B contains a copy of $\beta\omega$ (i.e. contains a subset $C \simeq \beta\omega$);*

(g) *If $\beta\omega \hookrightarrow Y$, then $t(Y) \geq \mathfrak{c}$ and $F(Y) \geq \mathfrak{c}$;*

(h) *If $p \in X$, then $\tau(EX)|_{k_X^{\leftarrow}(p)} = \tau(PX)|_{\Pi_X^{\leftarrow}(p)}$.*

Proof. (a) As $k_X : EX \to X$ is onto, $|X| \leq |EX| = |PX|$. Also, as $\{\mathcal{U} : \mathcal{U}$ is a fixed open ultrafilter on $X\} \subseteq \mathcal{P}(\mathcal{P}(X))$, then $|PX| = |EX| \leq |\mathcal{P}(\mathcal{P}(X))| = 2^{2^{|X|}}$.

For the facts (b), (c), (d), (e) and (f) we refer the reader to [**18**].

(g) We have that $\mathfrak{c} = t(\beta\omega) \leq t(Y)$ and $\mathfrak{c} = F(\beta\omega) \leq F(Y)$.

(h) Both $k_X^{\leftarrow}(p)$ and $\Pi_X^{\leftarrow}(p)$ are compact subspaces in the same set. As $\tau(EX) \subseteq \tau(PX)$, $\tau(k_X^{\leftarrow}(p)) \subseteq \tau(\Pi_X^{\leftarrow}(p))$. Since compact Hausdorff spaces are minimal Hausdorff, it follows that $\tau(k_X^{\leftarrow}(p)) = \tau(\Pi_X^{\leftarrow}(p))$. □

By Proposition 3.1(a), it is natural to ask whether the following inequalities are true or not:

* If X is Hausdorff, then $F_\theta(X) \leq F_\theta(EX) \leq 2^{2^{F_\theta(X)}}$.
* If X is Hausdorff, then $t_\theta(X) \leq t_\theta(EX) \leq 2^{2^{t_\theta(X)}}$.

Proposition 3.2. *Let E be an ED, semiregular space and D a discrete subspace of E such that $|D| = d(E)$. Then,*

$$t(E) \leq t(\beta E) \leq t(\beta D) \leq w(\beta D) \leq 2^{|D|} = 2^{d(E)}.$$

Proof. By Proposition 3.1(e), $\beta E \hookrightarrow \beta D \setminus D$ and $t(\beta E) \leq t(\beta D)$. As, $E \subseteq \beta E$, then $t(E) \leq t(\beta E)$. It always true that $t(\beta D) \leq w(\beta D)$. Finally, by 3.3(b) in [**12**], $w(\beta D) \leq 2^{d(\beta D)} = 2^{|D|} = 2^{d(E)}$. □

Proposition 3.3. *Let X be a H-closed space with a dense set D of isolated points. Then,*

(a) *$ED = D$ is dense in EX;*

(b) *$EX \equiv_{ED} \beta D$;*

(c) *PX is the set βD with a finer topology σ and $\tau(\beta D) \subseteq \sigma \subseteq \tau(\kappa D)$.*

Proof. For the facts (a) and (b) we refer the reader to [**18**].

(c) PX is EX (and by (b), $\equiv_{ED} \beta D$) with a finer topology, i.e, $\tau(PX) \supseteq \tau(EX)$, and $\tau(PX)(s) = \tau(EX)$. That is, we can consider PX as βD with a topology σ such that $\sigma \supseteq \tau(\beta D)$ and $\sigma(s) = \tau(\beta D)$. Also, κD is βD with a finer topology such that $\tau(\kappa D) \supseteq \tau(\beta D)$ and $\tau(\kappa D)(s) = \tau(\beta D)$. By 7.7 in [**18**], there is a continuous bijection from κD to $(\beta D, \sigma)(= PX)$ that leaves the points of D fixed. Thus, $\sigma \subseteq \tau(\kappa D)$. □

Example 3.1. The inequalities $F_\theta(EX) \leq 2^{2^{F_\theta(X)}}$ and $t_\theta(EX) \leq 2^{2^{t_\theta(X)}}$ are false. To show this, let D be infinite discrete space of cardinality κ and $X = \alpha D$ be one-point compactification of D. Note that $t_\theta(X) = t(X) = F_\theta(X) = F(X) = \omega$. Also, $EX = E(\alpha D) = \beta D$ and $t_\theta(EX) = t(EX) = F_\theta(EX) = F(EX) \geq \kappa$ as $|D| = \kappa$. We have $2^{2^{F_\theta(X)}} = 2^{2^\omega} = 2^{\mathfrak{c}}$ but $F_\theta(EX) \geq \kappa$ for any cardinal κ.

When $\kappa = (2^{\mathfrak{c}})^+$ we have that

– $F_\theta(EX) \geq (2^{\mathfrak{c}})^+ > 2^{\mathfrak{c}} = 2^{2^{F_\theta(X)}}$.
– $t_\theta(EX) \geq (2^{\mathfrak{c}})^+ > 2^{\mathfrak{c}} = 2^{2^{t_\theta(X)}}$.

Proposition 3.4. *Let X be a space*

(a) *$d(EX) \leq d(X_s) \leq d(X)$;*
(b) *If X is T_3, then $d(EX) = d(X)$;*
(c) *$|EX| \leq 2^{2^{d(X_s)}}$ and $|EX| \leq d(X_s)^{\chi(EX)}$;*
(d) *If X is separable, then EX is separable, $|EX| \leq 2^{\mathfrak{c}}$ and $t(EX) \leq \mathfrak{c}$;*
(e) *$w(EX) \leq 2^{w(X)}$ and $w(PX) \leq 2^{w(X)}$.*

Proof. (a) The inequality $d(X_s) \le d(X)$ is clear. To prove the inequality $d(EX) \le d(X_s)$, let D be a dense subset in X_s such that $d(X_s) = |D|$. For each $d \in D$, select $x_d \in k_X^{\leftarrow}(d)$ and let $D' = \{x_d : d \in D\}$. We have D' is dense in EX and $|D'| = |D| = d(X_s)$. Then, $d(EX) \le |D'| = d(X_s)$.

(b) Suppose X is T_3 and D be a dense subset in EX such that $d(EX) = |D|$. The map $k_X : EX \to X$ is continuous and onto. Then, $k_X(D)$ is dense in X and $d(X) \le |k_X(D)| \le |D| = d(EX)$. Thus, with (a) we have that $d(EX) = d(X)$.

(c) Recall two well-known results by Pospišil [**14**]: "If X is Hausdorff, then $|X| \le 2^{2^{d(X)}}$ and $|X| \le d(X)^{\chi(X)}$". So, $|EX| \le 2^{2^{d(EX)}}$ and, by (a), $|EX| \le 2^{2^{d(X_s)}}$. Also, $|EX| \le d(EX)^{\chi(EX)}$ and, by (a), $|EX| \le d(X_s)^{\chi(EX)}$.

(d) If X is separable, then $d(EX) \le d(X_s) \le d(X) \le \omega$. Thus, by (c), $|EX| \le 2^{2^\omega} = 2^{\mathfrak{c}}$ and moreover EX is separable too. Also, as $EX \subseteq \beta\omega$, then $t(EX) \le t(\beta\omega) = \mathfrak{c}$.

(e) First note that $o(X) = |\tau(X)| \le 2^{w(X)}$. A base for EX is $\{oU : U \in \tau(X)\}$; so, $w(EX) \le o(X)$. Likewise, a base for PX is $\{oU \cap k_X^{\leftarrow}(V) : U, V \in \tau(X)\}$; so, $w(PX) \le o(X)$. □

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