Riv. Mat. Univ. Parma, Vol. 4 (2013), 61-69

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Conditional results about primes between consecutive powers

Abstract. A well known conjecture about the distribution of primes asserts that all intervals of type $[n^2, (n+1)^2]$ contain at least one prime. The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. In a previous paper the author, assuming the Lindelöf hypothesis, proved that each of the interval $[n^\alpha, (n+1)^\alpha]$ contains the expected number of primes for $\alpha > 2$ and $n \to \infty$. In this paper we prove the same result assuming in turn two different heuristic hypotheses. It must be stressed that both the hypotheses are implied by the Lindelöf hypothesis.

Keywords. Prime numbers between powers, primes in short intervals.

Mathematics Subject Classification (2010): 11NO5.

1 - Introduction

A well known conjecture about the distribution of primes asserts that all intervals of type $[n^2, (n+1)^2]$ contain at least one prime. The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. D. A. Goldston proved the conjecture assuming a strong form of Montgomery Conjecture, see [**6**]. The author improved this result by proving that all intervals of type $[n^2, (n+1)^2]$ contain the expected number of primes, for $n \to \infty$, assuming a weaker hypothesis about the behavior of Selberg's integral in short intervals, see D. Bazzanella [**2**].

This paper is concerned with the distribution of prime numbers between two consecutive powers of integers, as a natural generalization of the above conjecture.

Received: March 14, 2012; accepted: June 29, 2012.

In a previous paper the author, assuming the Lindelöf hypothesis, proved that each of the interval $[n^\alpha, (n+1)^\alpha]$ contains the expected number of primes for $\alpha > 2$ and $n \to \infty$, see [**4**, Theorem 1].

In this paper we prove the same result assuming in turn two different heuristic hypotheses. It must be stressed that both the hypotheses are implied by the Lindelöf hypothesis.

The first new hypothesis is a weakened version of the hypothesis stated in D. Bazzanella [**3**].

Hypothesis 1. *There exist a constant X_0 and a function $\Delta(y,T)$ such that, for every $5/12 < \beta < 1/2$ and $\varepsilon > 0$, we have*

$$\int_X^{2X} |\psi(y + y/T) - \psi(y) - y/T + \Delta(y,T)|^{2k} \mathrm{d}y \ll X^{2k+\varepsilon} T^{1-2k}$$

and

$$\Delta(y,T) \ll y/(T \log y)$$

for at least one integer $k \geq 1$, uniformly for $X \geq X_0$, $X^{5/12} \leq T \leq X^\beta$ and $X \leq y \leq 2X$.

To state the second new hypothesis we need to use the counting functions $N(\sigma,T)$ and $N^{(k)}(\sigma,T)$. The former is defined as the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function which satisfy $\sigma \leq \beta \leq 1$ and $|\gamma| \leq T$, while $N^{(k)}(\sigma,T)$ is defined as the number of ordered sets of zeros $\rho_j = \beta_j + i\gamma_j$ $(1 \leq j \leq 2k)$, each counted by $N(\sigma,T)$, for which $|\gamma_1 + \cdots + \gamma_k - \gamma_{k+1} - \cdots - \gamma_{2k}| \leq 1$.

We start to observe that D. Bazzanella and A. Perelli [**1**] made the heuristic assumption that there exists a constant T_0 such that

$$N^{(2)}(\sigma,T) \ll \frac{N(\sigma,T)^4}{T} T^\varepsilon \tag{1}$$

for every $T \geq T_0$ and arbitrarily small $\varepsilon > 0$, which is close to being the best possible, in view of the trivial estimate

$$N^{(2)}(\sigma,T) \gg \frac{N(\sigma,T)^4}{T}.$$

The above may be generalized and weakened to

$$N^{(k)}(\sigma,T) \ll \frac{N(\sigma,T)^{2k}}{T} T^\varepsilon \qquad (1/2 \leq \sigma \leq \overline{\sigma}),$$

with suitable $\overline{\sigma} < 1$ and arbitrarily small $\varepsilon > 0$. We now observe that the Lindelöf hypothesis implies that for every $\eta > 0$ we have

$$N(\sigma, T) \ll T^{2(1-\sigma)+\eta} \quad (1/2 \leq \sigma \leq 1),$$

see A. E. Ingham [**10**], and then we are led to claim the following.

Hypothesis 2. *For every $0 \leq \eta < 1/6$ there exists an integer $k \geq 2$ such that*

$$N^{(k)}(\sigma, T) \ll T^{4k(1-\sigma)-1+\eta} \qquad (1/2 \leq \sigma \leq 5/6 + \eta).$$

We note that Hypotheses 1 and 2 are weaker than the Lindelöf hypothesis, see G. Yu [**13**, Lemma B] and D. R. Heath-Brown [8, Lemma 1] respectively.

We are now able to state our main theorems.

Theorem 1.1. *Let $\alpha > 2$ and assume Hypothesis 1, then each of the interval $[n^\alpha, (n+1)^\alpha]$ contains the expected number of primes for $n \to \infty$.*

Theorem 1.2. *Let $\alpha > 2$ and assume Hypothesis 2, then each of the interval $[n^\alpha, (n+1)^\alpha]$ contains the expected number of primes for $n \to \infty$.*

Note that despite Hypothesis 1 and 2 are implied by the Lindelöf hypothesis we obtain the same expected distribution of primes between consecutive powers and then the two theorems are stronger than Theorem 1 of [**4**].

2 - Definitions and basic lemma

The basic lemma is a result about the structure of the exceptional set for the asymptotic formula

$$\psi(x + h(x)) - \psi(x) \sim h(x) \quad \text{as} \quad x \to \infty. \tag{2}$$

Let X be a large positive number, $\delta > 0$ and let $|\ |$ denote the modulus of a complex number or the Lebesgue measure of a set. Let $h(x)$ be an increasing function such that $x^\varepsilon \leq h(x) \leq x$ for some $\varepsilon > 0$ and

$$E_\delta(X, h) = \{X \leq x \leq 2X : |\psi(x + h(x)) - \psi(x) - h(x)| \geq \delta h(x)\}.$$

It is clear that (2) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, h) = \emptyset$ for $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞ and $h(x)$ suitably small with respect to x, the set $E_\delta(X, h)$ contains the exceptions, if any, to the asymptotic formula (2). We will consider increasing functions $h(x)$ of the form

$h(x) = x^{\theta+\varepsilon(x)}$, with some $0 < \theta < 1$ and a function $\varepsilon(x)$ such that $|\varepsilon(x)|$ is decreasing,

$$\varepsilon(x) = o(1) \quad \text{and} \quad \varepsilon(x+y) = \varepsilon(x) + O\left(\frac{|y|}{x \log x}\right).$$

A function satisfying these requirements will be called of *type* θ.

Lemma . *Let* $0 < \theta < 1$, $h(x)$ *be of type* θ, X *be sufficiently large depending on the function* $h(x)$ *and* $0 < \delta' < \delta$ *with* $\delta - \delta' \geq \exp(-\sqrt{\log X})$. *If* $x_0 \in E_\delta(X, h)$ *then* $E_{\delta'}(X, h)$ *contains the interval* $[x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, *where* $c = (\delta - \delta')\theta/5$. *In particular, if* $E_\delta(X, h) \neq \emptyset$ *then*

$$|E_{\delta'}(X, h)| \gg_\theta (\delta - \delta')h(X).$$

The lemma essentially says that if we have a single exception in $E_\delta(X, h)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta'}(X, h)$, with δ' little smaller than δ. The interesting consequence of this lemma is that we can use an average estimate to prove the non-existence of the exceptions.

The above lemma is part (i) of Theorem 1 of D. Bazzanella and A. Perelli, see [**1**].

3 - Proof of the Theorems

The theorems assert that

$$\psi((n+1)^\alpha) - \psi(n^\alpha) \sim (n+1)^\alpha - n^\alpha \qquad \text{as} \qquad n \to \infty, \tag{3}$$

assuming a suitable heuristic hypothesis. In order to prove the theorems we assume that (3) does not hold. Then there exists $\delta > 0$ and a sequence $n_j \to \infty$ such that

$$\left|\psi((n_j+1)^\alpha) - \psi(n_j^\alpha) - [(n_j+1)^\alpha - n_j^\alpha)]\right| \geq \delta[(n_j+1)^\alpha - n_j^\alpha)].$$

In the remainder of the proof we will always assume that n_j is sufficiently large as prescribed by the various statements. Putting $x_j = n_j^\alpha$ and $h(x) = (x^{1/\alpha} + 1)^\alpha - x$, we then have

$$|\psi(x_j + h(x_j)) - \psi(x_j) - h(x_j)| \geq \delta h(x_j)$$

and hence we have $x_j \in E_\delta(x_j, h)$. The use of the lemma leads to

$$|E_{\delta/2}(x_j, h)| \gg h(x_j) \gg x_j^{1-1/\alpha}. \tag{4}$$

On the other hand we can bound $|E_{\delta/2}(x_j, h)|$ and find a contradiction with (4). For any $y \in E_{\delta/2}(x_j, h)$ we can write

$$|\psi(y + h(y)) - \psi(y) - h(y)| \geq \frac{\delta}{2}h(y) \gg x_j^{1-1/\alpha}, \tag{5}$$

for every $x_j \le y \le 2x_j$. We divide the interval $[x_j, 2x_j]$ into $O(\ln^2 x_j)$ subintervals $J_i = [a_i, a_{i+1}]$, with

$$a_i = x_j + i\,\frac{x_j}{\log^2 x_j} \tag{6}$$

and define

$$E^i_{\delta/2}(x_j, h) = E_{\delta/2}(x_j, h) \cap J_i.$$

We let

$$T_i = a_i^{1/\alpha}/\alpha \tag{7}$$

and observe that Hypothesis 1 implies that there exist an integer $k \ge 1$, a constant X_0 and a function $\Delta(y, T)$ such that, for every i, we have

$$\int_{x_j}^{2x_j} |\psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i)|^{2k} \mathrm{d}y \ll x_j^{2k+\varepsilon} T_i^{1-2k} \tag{8}$$

and

$$\Delta(y, T_i) \ll y/(T_i \log y), \tag{9}$$

uniformly for $x_j \ge X_0$ and $x_j \le y \le 2x_j$. From the Brun–Titchmarsh theorem, see H. L. Montgomery and R. C. Vaughan [**12**], we can deduce that for every i we have

$$\psi(y + h(y)) - \psi(y) - h(y) = \psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i) + O\left(\frac{x_j^{1-1/\alpha}}{\log x_j}\right),$$

for every $y \in J_i$. The above bound and (5) imply that

$$|\psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i)| \gg x_j^{1-1/\alpha},$$

for every $y \in E^i_{\delta/2}(x_j, h)$. Thus we obtain

$$\begin{aligned} |E_{\delta/2}(x_j, h)| &\ll x_j^{-2k(1-1/\alpha)} \sum_i \int_{E^i_{\delta/2}(x_j,h)} |\psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i)|^{2k} \,\mathrm{d}y \\ &\ll x_j^{-2k(1-1/\alpha)} \sum_i \int_{x_j}^{2x_j} |\psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i)|^{2k} \,\mathrm{d}y. \end{aligned}$$

By (8) we conclude that

$$|E_{\delta/2}(x_j, h)| \ll x_j^{-2k(1-1/\alpha)} \sum_i x_j^{2k+\varepsilon} T_i^{1-2k} \ll x_j^{1/\alpha+\varepsilon}. \tag{10}$$

For $\alpha > 2$, when ε is sufficiently small and x_j is sufficiently large we have a contradiction between (10) and (4), and this completes the proof of Theorem 1.

To prove Theorem 2 we use the classical explicit formula, see H. Davenport [**5**, Chapter 17], to write

$$\psi(y+y/T_i)-\psi(y)-y/T_i=-\sum_{|\gamma|\le R_i} y^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}+O\left(\frac{x_j\log^2 x_j}{R_i}\right), \tag{11}$$

uniformly for $x_j \le y \le 2x_j$, where $\delta_i=\log(1+T_i^{-1})$, $10\le R_i\le x_j$ and $\rho=\beta+i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. If we choose $R_i=T_i\log^3 x_j$ and recall (7) and (6) we have

$$x_j^{1/\alpha}\log^3 x_j \ll R_i \ll x_j^{1/\alpha}\log^3 x_j$$

and

$$\psi(y+y/T_i)-\psi(y)-y/T_i=-\sum_{|\gamma|\le R_i} y^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}+O\left(\frac{x_j^{1-1/\alpha}}{\log x_j}\right).$$

We note also that

$$\left|\frac{e^{\delta_i\rho}-1}{\rho}\right|=\left|\int_0^{\delta_i} e^{t\rho}\,\mathrm{d}t\right|\le\int_0^{\delta_i} e^{t\beta}\,\mathrm{d}t\le e\delta_i\ll\frac{1}{T_i}. \tag{12}$$

Follow the method of D. R. Heath-Brown we can prove that for $\alpha > 2$ and every fixed $u > 5/6$ we have

$$\sum_{|\gamma|\le R_i,\ \beta>u} y^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}\ll\frac{x_j^{1-1/\alpha}}{\log x_j},$$

see (12.79) in [**11**]. Thus we obtain

$$\psi(y+y/T_i)-\psi(y)-y/T_i=-\sum_{|\gamma|\le R_i,\ \beta\le u} y^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}+O\left(\frac{x_j^{1-1/\alpha}}{\log x_j}\right),$$

for every i and $y\in J_i$. As before we observe that for every $y\in J_i$ we have

$$\psi(y+h(y))-\psi(y)-h(y)=\psi(y+y/T_i)-\psi(y)-y/T_i+O\left(\frac{x_j^{1-1/\alpha}}{\log x_j}\right)$$

and then

$$\psi(y+h(y))-\psi(y)-h(y)=-\sum_{|\gamma|\le R_i,\ \beta\le u} y^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}+O\left(\frac{x_j^{1-1/\alpha}}{\log x_j}\right),$$

for every i and $y \in J_i$. This implies that

$$|E_{\delta/2}(x_j,h)| \ll x_j^{-2k(1-1/\alpha)} \sum_i \int_{x_j}^{2x_j} \left| \sum_{|\gamma|\le R_i,\, \beta\le u} y^{\rho}\, \frac{e^{\delta_i\rho}-1}{\rho} \right|^{2k} \mathrm{d}y. \tag{13}$$

To estimate the $2k$-power integral we divide the interval $[0,u]$ into $O(\ln x_j)$ subintervals I_r of the form

$$I_r = \left[\frac{r}{\log x_j}, \frac{r+1}{\log x_j} \right].$$

By Hölder inequality we obtain

$$\left| \sum_{|\gamma|\le R_i,\, \beta\le u} y^{\rho}\, \frac{e^{\delta_i\rho}-1}{\rho} \right|^{2k} \ll (\ln x_j)^{2k-1} \sum_r \left| \sum_{|\gamma|\le R_i,\, \beta\in I_r} y^{\rho}\, \frac{e^{\delta_i\rho}-1}{\rho} \right|^{2k}.$$

Following again the method of D. R. Heath-Brown, we write

$$\int_{x_j}^{2x_j} \left| \sum_{|\gamma|\le R_i,\, \beta\in I_r} y^{\rho}\, \frac{e^{\delta_i\rho}-1}{\rho} \right|^{2k} \mathrm{d}y \ll$$

$$\sum_{\substack{\beta_1,\dots,\beta_{2k}\in I_r \\ |\gamma_1|\le R_i,\dots,|\gamma_{2k}|\le R_i}} \frac{(2x_j)^{\rho_1+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1} - x_j^{\rho_1+\rho_2+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1}}{\rho_1\cdots\rho_{2k}\,(\rho_1+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1)}$$

$$\times(e^{\delta_i\rho_1}-1)\cdots(e^{\delta_i\rho_k}-1)(e^{\delta_i\overline{\rho_{k+1}}}-1)\dots(e^{\delta_i\overline{\rho_{2k}}}-1)$$

$$\ll \frac{1}{T_i^{2k}}\; x_j^{1+2kr/\log x_j} \sum_{\substack{\beta_1,\dots,\beta_{2k}\ge r/\log x_j \\ |\gamma_1|\le R_i,\dots,|\gamma_{2k}|\le R_i}} \frac{1}{|\rho_1+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1|}.$$

This implies

$$\int_{x_j}^{2x_j} \left| \sum_{|\gamma|\le R_i,\, \beta\le u} y^{\rho}\, \frac{e^{\delta_i\rho}-1}{\rho} \right|^{2k} \mathrm{d}y \ll \frac{1}{T_i^{2k}} \max_{\sigma\le u} x_j^{2k\sigma+1+\varepsilon} M_k(\sigma,R_i), \tag{14}$$

where

$$M_k(\sigma,R_i) = \sum_{\substack{\beta_1,\dots,\beta_{2k}\ge\sigma \\ |\gamma_1|\le R_i,\dots,|\gamma_{2k}|\le R_i}} \frac{1}{1+|\gamma_1+\cdots+\gamma_k-\gamma_{k+1}-\cdots-\gamma_{2k}|}$$

and

$$M_k(\sigma, R_i) \ll N^{(k)}(\sigma, R_i) \log x_j \,, \tag{15}$$

see [**11**, p. 336]. From (13), (14) and (15) we have

$$|E_{\delta/2}(x_j, h)| \ll x_j^{1-2k+\varepsilon} \sum_i \max_{\sigma \le u} x_j^{2k\sigma} N^{(k)}(\sigma, R_i) \,. \tag{16}$$

We now consider an arbitrarily small constant $\eta > 0$, let $u = 5/6 + \eta$ and assume Hypothesis 2. Thus for every $1/2 \le \sigma \le u$ we have

$$x_j^{2k\sigma} N^{(k)}(\sigma, R_i) \ll x_j^{2k\sigma} R_i^{4k(1-\sigma)-1+\eta} \ll x_j^{2k\sigma+(4k(1-\sigma)-1)/\alpha+\eta}.$$

For $\alpha > 2$ the above upper bound attains its maximum at $\sigma = u$ and then from (16) we obtain

$$|E_{\delta/2}(x_j, h)| \ll x_j^{1-k/3+(2k/3-1)/\alpha+\varepsilon}. \tag{17}$$

For $\alpha > 2$, when ε is sufficiently small and x_j is sufficiently large we have a contradiction between (17) and (4), and this completes the proof of Theorem 2.

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