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ROBERTO MOSSA

A bounded homogeneous domain and a projective manifold are not relatives

Abstract. Let M_1 and M_2 be two Kähler manifolds. Following [**4**] one says that M_1 and M_2 are *relatives* if they share a non-trivial Kähler submanifold S, namely, if there exist two holomorphic and isometric immersions (Kähler immersions) $h_1 : S \to M_1$ and $h_2 : S \to M_2$. Our main results in this paper is Theorem 1.2 where we show that a bounded homogeneous domain with a homogeneous Kähler metric and a projective Kähler manifold (i.e. a projective manifold endowed with the restriction of the Fubini–Study metric) are not relatives. Our result is a generalization of the result obtained in [**4**] for the Bergman metrics.

Keywords. Diastasis, homogeneous bounded domains, Kähler metric, relatives.

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1 - Introduction

The study of Kähler immersions (holomorphic and isometric immersions) started with E. Calabi in his seminal paper [**1**] where he gave necessary and sufficient conditions for the existence of a Kähler immersion of a finite dimensional Kähler manifold into a complex space form. In particular he proved that two complex space forms with curvature of different sign cannot be Kähler immersed one into another. Moreover, he proved that for complex space forms of the same type, just projective spaces can be embedded between themselves in a non trivial way by using Veronese mappings. Almost 25 years later M. Umehara [**10**] proved that two finite dimensional

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complex space forms with holomorphic sectional curvatures of different signs cannot be relatives. Recall the definition of relatives:

Definition 1.1 ([**4**, Definition 1.1]). Let $r \geq 1$ be an integer. Two Kähler manifolds M_1 and M_2 are said to be *r-relatives* if they have in common a complex r-dimensional Kähler submanifold S, i.e. there exist two Kähler immersions $h_1 : S \to M_1$ and $h_2 : S \to M_2$. Otherwise, we say that M_1 and M_2 are not relatives.

Recently, A. J. Di Scala and A. Loi [**4**] proved the following:

Theorem 1.1 ([**4**, Theorem 1.2]). *A bounded domain $D \subset \mathbb{C}^n$ endowed with its Bergman metric and a projective Kähler manifold endowed with the restriction of the Fubini–Study metric are not relatives.*

Our main result is the following theorem which generalizes Theorem 1.1 when the Bergman metric is homogeneous. Recall that a n-dimensional bounded homogeneous domain (Ω, g) is a bounded domain of $\mathbb{C}^n$ endowed with the Kähler metric g such that the group $G = \mathrm{Aut}(\Omega) \cap \mathrm{Isom}(\Omega, g)$ act transitively on it. Here $\mathrm{Aut}(\Omega)$ denotes the group of invertible holomorphic maps of Ω and $\mathrm{Isom}(\Omega, g)$ the group of isometries of (Ω, g).

Theorem 1.2. *A bounded homogeneous domain (Ω, g) and a projective Kähler manifold endowed with the restriction of the Fubini–Study metric are not relatives.*

The proof of this theorem is based on a recent result [8] obtained by the author jointly with A. Loi. We point out that our result is of local nature, i.e. no assumptions are used about the compactness or completeness of the manifolds involved.

2 - Proof of Theorem 1.2

In order to prove Theorem 1.2 it is enough to show that a bounded homogeneous domain and the complex projective space $\mathbb{C}P^m$ are not relatives. Let ω be the Kähler form associated to g. It is well-known that there exists a globally defined Kähler potential Φ for g i.e. $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$. Indeed, Ω is pseudoconvex being biholomorphically equivalent to a Siegel domain (see, e.g. [**11**] for a proof) and so the existence of a global potential follows by a classical result of Hormander (see [**2**]) asserting that the equation $\bar{\partial} u = f$ with f $\bar{\partial}$-closed form has a global solution on pseudoconvex domains (see also the proof of Theorem 4 in [**5**], for an explicit construction of the potential Φ following the ideas developed in [**6**]). Consider the associated weighted Hilbert space $\mathcal{H}_{\lambda\Phi}$ of square integrable holomorphic functions on Ω, with weight $e^{-\lambda\Phi}$, namely

$$\mathcal{H}_{\lambda\Phi} = \left\{ f \in \mathrm{Hol}(\Omega) \;\middle|\; \int_{\Omega} e^{-\lambda\Phi} |f|^2 \, \frac{\omega^n}{n!} < \infty \right\}. \tag{1}$$

In [**8**] it is proven that for $\lambda > 0$ large enough $\mathcal{H}_{\lambda\Phi} \neq \{0\}$. Fixed such λ, let $K_{\lambda\Phi}(z, w)$ be its reproducing kernel. We can define the ε-function:

$$\varepsilon_{\lambda g}(z) = e^{-\lambda\Phi(z)} K_{\lambda\Phi}(z, z), \tag{2}$$

this function does not depend on the potential Φ, it depends only on the constant λ and on the metric g (see [**8**] for details). Moreover, it is invariant with respect to the action of the Lie group G. Since G acts transitively on Ω, it follows that $\varepsilon_{\lambda g} = C$ is constant (for λ large enough) and $\log K_{\lambda\Phi}(z, z)$ is a Kähler potential for the metric λg. By analytic continuation we have

$$\varepsilon_{\lambda g}(z, w) = e^{-\lambda\Phi(z,w)} K_{\lambda\Phi}(z, w) = C > 0, \tag{3}$$

and so $K_{\lambda\Phi}(z, w)$ never vanishes. Then, fixed a point z_0, the function

$$\psi(z, w) = \frac{K_{\lambda\Phi}(z, w) K_{\lambda\Phi}(z_0, z_0)}{K_{\lambda\Phi}(z, z_0) K_{\lambda\Phi}(z_0, w)},$$

is well defined. Observe that $\psi(z_0, w) = \psi(z, z_0) = 1$ and that

$$\mathcal{D}(z) = \log \psi(z, z)$$

is a globally defined Kähler potential for g (actually $\mathcal{D}(z)$ is the *diastasis* centered in z_0, see Calabi in [**1**] for details and further property about the diastasis function). We can now consider the Hilbert space $\mathcal{H}_{\lambda\mathcal{D}}$ given by:

$$\mathcal{H}_{\lambda\mathcal{D}} = \left\{ f \in \mathrm{Hol}(\Omega) \;\middle|\; \int_{\Omega} e^{-\lambda\mathcal{D}} |f|^2 \, \frac{\omega^n}{n!} < \infty \right\}.$$

Let us denote $K_{\lambda\mathcal{D}}(z, w)$ its reproducing kernel, as the ε-function does not depend on the Kähler potential, by (3) we have

$$\varepsilon_{\lambda g}(z, w) = e^{-\lambda\mathcal{D}(z,w)} K_{\lambda\mathcal{D}}(z, w) = C, \tag{4}$$

where $\mathcal{D}(z, w)$ is the analytic continuation of $\mathcal{D}(z)$. In particular

$$K_{\lambda\mathcal{D}}(z_0, w) = K_{\lambda\mathcal{D}}(z, z_0) = C$$

and so $\mathcal{H}_{\lambda D}$ contains the constant functions and by boundedness of Ω all polynomials belong to $\mathcal{H}_{\lambda D}$. In particular $\mathcal{H}_{\lambda D}$ contains the sequence $\{z_1^k\}_{k\in\mathbb{N}}$, by applying the Gram-Schmidt orthonormalization procedure we get a sequence $P = \{P_k\}_{k\in\mathbb{N}}$ of orthonormal polynomials in the variable z_1.

Consider now the *coherent states map* (see [8]) $\varphi : \Omega \to \mathbb{C}P^\infty$ from Ω into the infinite dimensional complex projective space $\mathbb{C}P^\infty$ given by

$$\varphi : \Omega \to \mathbb{C}P^\infty,\ z \mapsto [P_0(z_1), P_1(z_1), \dots, F_0(z), F_1(z), \dots], \tag{5}$$

where $\{P_0(z_1), \dots, F_0(z), \dots\}$ is an orthonormal basis of $\mathcal{H}_{\lambda D}$ obtained completing P to an orthonormal basis. Since $K_{\lambda D}(z, z) = \sum_{k=0}^{\infty} |P_k(z_1)|^2 + |F_k(z)|^2$ by (4) we see that the map (5) is well-defined. Moreover, the constancy of ε also implies that $\varphi^* g_{FS} = \lambda g$, where g_{FS} is the Fubini–Study metric on $\mathbb{C}P^\infty$ (see [9] for a proof). In other words, the metric λg is projectively induced via the coherent states map.

Assume now, by contradiction, that (Ω, g) is r-relative (for some positive integer $r \geq 1$) to the complex projective space $\mathbb{C}P^m$ (the Kähler metric on M is induced by the Fubini-Study metric through the immersion j). Then we can assume $r = 1$ and that there exists an open subset S of $\mathbb{C}$ through the origin and two Kähler immersions $f : S \to D$ and $h : S \to \mathbb{C}P^m$.

Consider the Kähler map $\varphi \circ f : S \to \mathbb{C}P^\infty$

$$\varphi \circ f(\xi) = [P_0(f_1(\xi)), P_1(f_1(\xi)), \dots, F_0(f(\xi)), F_1(f(\xi)), \dots]$$

were f_1 is the first component of f. Without loss of generality assume $\frac{\partial f_1}{\partial \xi}(0) \neq 0$. We claim that $P_0(f_1(\cdot)), P_1(f_1(\cdot)), \dots$ are linearly independent functions on S. Let $a_0, \dots, a_q$ be complex numbers such that

$$a_0 P_{k_0}(f_1(\xi)) + \dots + a_q P_{k_q}(f_1(\xi)) = 0.$$

By the assumption on f_1 it follows that $f_1(S)$ contains an open set of $\mathbb{C}$, therefore $a_0 P_{k_0}(z) + \dots + a_q P_{k_q}(z) = 0$ for every $z \in \mathbb{C}$. Since $P_{k_0}, \dots, P_{k_q}$ are linearly independent all the a_k must be zero, proving our claim.

On the other hand, if $i : \mathbb{C}P^m \to \mathbb{C}P^\infty$ is the standard totally geodesic embedding, then $i \circ h : S \to \mathbb{C}P^\infty$ is a Kähler immersion. Thus the smallest subspace containing $i \circ h(S)$ is infinite dimensional, while $\varphi \circ f$ is contained in a m-dimensional subspace, this is in contrast with the Calabi's rigidity theorem (see [1]), as wished. □

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References

[1] E. CALABI, *Isometric imbeddings of complex manifolds*, Ann. of Math. **58** (1953), 1-23.

[2] S.-C. CHEN and M.-C. SHAW, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, **19**. American Mathematical Society, Providence, RI; International Press, Boston, MA 2001, xii+380 pp.

[3] J. E. D'ATRI, *Holomorphic sectional curvatures of bounded homogeneous domains and related questions*, Trans. Amer. Math. Soc. **256** (1979), 405-413.

[4] A. J. DI SCALA and A. LOI, *Kähler manifolds and their relatives*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), no. 3, 495-501.

[5] A. J. DI SCALA, H. HISHI and A. LOI, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, Asian J. Math. **16** (2012), no. 3, 479-488.

[6] J. DORFMEISTER, *Simply transitive groups and Kähler structures on homogeneous Siegel domains*, Trans. Amer. Math. Soc. **288** (1985), 293-305.

[7] D. HULIN, *Kähler-Einstein metrics and projective embeddings*, J. Geom. Anal. **10** (2000), no. 3, 525-528.

[8] A. LOI and R. MOSSA, *Berezin quantization of homogeneous bounded domains*, Geom. Dedicata, **161** (2012), no. 1, 119-128.

[9] J. H. RAWNSLEY, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford Ser. (2) **28** (1977), 403-415.

[10] M. UMEHARA, *Kaehler submanifolds of complex space forms*, Tokyo J. Math. **10** (1987), 203-214.

[11] E. B. VINBERG, S. G. GINDIKIN and I. I. PJATECKIĬ-ŠAPIRO, *Classification and canonical realization of complex bounded homogeneous domains*, Trans. Moscow Math. Soc. **12** (1963), 404-437.

ROBERTO MOSSA
Laboratoire de Mathèmatiques Jean Leray (UMR 6629) CNRS
2 rue de la Houssinire (B.P. 92208)
44322 Nantes Cedex 3, France
e-mail: roberto.mossa@gmail.com