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Global existence for an Oldroyd-type model for viscoelastic fluids

Abstract. In this paper we show global existence of weak solutions for a class of integrodifferential equations generalizing the Oldroyd model for viscoelastic fluids.

Keywords. Integrodifferential equation, viscoelastic fluid, weak solution.

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1 - Introduction

In this paper we are interested in the following equation.

$$\frac{d}{dt}u(t)+u(t)\cdot\nabla u(t)=-\nabla p(t)+\operatorname{div}F(\nabla u(t))+\int_0^t G(t-s)\operatorname{div}H(\nabla u(s))\mathrm{d}s+f(t), \tag{1}$$

in a bounded domain Ω in 2D and 3D with C^2 boundary. We assume $\operatorname{div}u=0$, Dirichlet boundary condition and initial condition u_0. We show that we can get the same results as for the equations without integral term.

To be more precise, we assume power-like type nonlinearities F and H. If the integral term is missing and F is monotone and satisfies $\|F(\nabla u)\|_{p'}\leq C\|\nabla u\|_p^{p-1}$ then existence of global solution (and uniqueness in 2-dimensional case) was shown by Ladyzhenskaya (see [7], [8], [9]) for $p\geq 11/5$. This result was extended to $p\geq 2$

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by Málek, Nečas and Růžička (see [**10**]) under additional assumptions on F and then by Wolf (see [**13**]) to $p > 8/5$ without any additional assumptions. Finally, it was solved by Diening, Růžička and Wolf in [**3**] for $p > 6/5$. In our paper we show Ladyzhenskaya's result for the integrodifferential equation.

Equations similar to (1) were studied by Agranovich and Sobolevskii (see [**1**], [**2**]). In [**2**] a local in time solution is obtained for dimension $N = 3$ (more regular than weak solution) under more restrictive assumptions in many aspects (the nonlinearities F and G are local in space, they depend on the second invariant of symmetric gradient of u and satisfy some boundedness conditions, but do not need so strong monotonicity). Let us mention that there are weaker assumptions on G in [**2**], however our assumptions can also be weakened a lot (in fact, only local integrability of $(t,s) \mapsto G(t,s)$ with respect to the second variable uniform with respect to the first variable is needed).

The main problem of [**2**] is that the integral term depends on the past values of u in the same space point x. However, if the equation describes the behaviour of a viscoelastic fluid, it should depend on u in the same material point X, i.e. in a different space point $x(s)$ (see [**4**], [**5**], [**12**]). In our equation H depends on the whole function $u(s,\cdot)$, so the exact space point is not specified. Hence, our result can be a good tool to study more realistic models presented in [**4**], [**5**], [**12**], where global existence of weak solutions has been proved in linear case.

2 - Definitions and Main Results

For $p \in [1,\infty)$, p' will be always the number satisfying $1/p + 1/p' = 1$. Let $\Omega \subset R^N$ be open, bounded with C^2 boundary and $N = 2$ or $N = 3$. Define $W^{1,p}_{0,div}(\Omega)$ (resp. $L^2_{0,div}(\Omega)$) as the closure in $W^{1,p}$-norm (resp. in L^2-norm) of all C^∞ functions with compact support in Ω and zero divergence. We consider solutions in a weak sense according to the following definition. If $u_0 \in L^2_{0,div}(\Omega)$ and $f \in L^{p'}(0,T;(W^{1,p'}_{0,div}(\Omega))')$ we say that a function $u \in L^p_{loc}(0,T;W^{1,p}_{0,div}(\Omega)) \cap L^\infty_{loc}(0,T;L^2(\Omega))$ with $u_t \in L^{p'}_{loc}(0,T;(W^{1,p}_{0,div}(\Omega))')$ is a *weak solution* of (1), if

$$(2) \quad \langle u_t, \varphi\rangle + \int_\Omega (u\cdot\nabla u)\varphi + \int_\Omega F(\nabla u) : \nabla\varphi + \int_\Omega \int_0^t G(t-s)H(\nabla u(s)) : \nabla\varphi(t)\mathrm{d}s = \langle f, \varphi\rangle$$

holds for all $\varphi \in W^{1,p'}_{0,div}(\Omega)$ and a.e. $t \in [0,T]$ and

$$\lim_{t\to 0} \|u(t) - u_0\|_2 = 0.$$

We say that u is *bounded weak solution* if in addition $u \in L^p(0,T;W^{1,p}_{0,div}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Let $\infty > p \geq 2 \geq q > 1$. We introduce the assumptions on F, G, and H. We assume $F : L^p(\Omega)^{N\times N} \to L^{p'}(\Omega)^{N\times N}$, $H : L^q(\Omega)^{N\times N} \to L^{q'}(\Omega)^{N\times N}$ with $F(0) = H(0) = 0$ and there exist positive constants μ_F, C_F, λ_F, C_H, λ_H such that

(Fmon) $$\langle F(x) - F(y), x - y\rangle \geq \mu_F(\|x - y\|_2^2 + \|x - y\|_p^p), \quad x, y \in L^p(\Omega)^{N\times N},$$

(Fbdd) $$\|F(x)\|_{p'} \leq C_F(1 + \|x\|_p^{p-1}), \quad x \in L^p(\Omega)^{N\times N},$$

(Flip) $$\|F(x) - F(y)\|_{p'} \leq \lambda_F\|x - y\|_p, \quad x, y \in L^p(\Omega)^{N\times N} \cap B,$$

(Hbdd) $$\|H(x)\|_{q'} \leq C_H\|x\|_q^{q-1}, \quad x \in L^q(\Omega)^{N\times N},$$

(Hlip) $$\|H(x) - H(y)\|_{q'} \leq \lambda_H\|x - y\|_q, \quad x, y \in L^q(\Omega)^{N\times N},$$

where B is any ball in L^p and λ_F may depend on its radius. Moreover, we assume

(G) $$G \in L^1_{loc}(\mathbb{R}_+).$$

For example, F can be a pointwise mapping $F : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ that is monotone, locally Lipschitz continuous and satisfies

$$\|F(x)\| \leq c(\|x\| + \|x\|^{p-1}), \quad x \in \mathbb{R}^{N\times N}.$$

$$\|F(x) - F(y)\| \geq c_2\|x - y\|, \quad x, y \in \mathbb{R}^{N\times N}.$$

Then F satisfies the above assumptions.

We start with a lemma.

Lemma 2.1. *Let F, H satisfy* (Fbdd), (Hbdd) *on* $[0, T]$, $T < +\infty$ *and* $G \in L^1$. *Then there exists $\tilde{C}_F$ depending on T such that*

1. $\|F(v)\|_{L^{p'}(0,T,L^{p'})} \leq \tilde{C}_F(1 + \|v\|^{p-1}_{L^p(0,T,L^p)})$ *for all* $v \in L^p(L^p)$
2. $\|G * H(v)\|_{L^{q'}(L^{q'})} \leq C_H\|G\|_{L^1}\|v\|^{q-1}_{L^q(L^q)}$ *for all* $v \in L^q(L^q)$.

Proof. We have

$$\|F(v)\|_{L^{p'}(L^{p'})} = \left(\int_0^T \|F(v)\|_{p'}^{p'}\right)^{1/p'} \leq \left(\int_0^T C_F^{p'}(1 + \|v\|_p^{(p-1)})^{p'}\right)^{1/p'} \leq \tilde{C}_F\Big(1 + \|v\|^{p-1}_{L^p(L^p)}\Big),$$

since $p'(p - 1) = p$. The first assertion then follows since $p/p' = p - 1$. The second assertion follows from the inequality

$$\|G * H(v)\|_{L^{q'}(L^{q'})} \leq \|G\|_{L^1}\|H(v)\|_{L^{q'}(L^{q'})}$$

and the same computation as in the first assertion. □

We say that G is *of positive type* if

$$\int_0^t \int_0^s G(s-\tau)\langle v(\tau), v(s)\rangle \mathrm{d}\tau \mathrm{d}s \geq 0$$

for all $v \in L^1_{loc}(\mathbb{R}_+)$ and $t \in [0, +\infty)$.

Now we are ready to formulate the main results of this paper.

Theorem 2.2 (2D existence and uniqueness). *Let $N = 2$, $p \geq 2$, $q \leq 2$. Then for every $0 < T < +\infty$ there exists a unique bounded weak solution u to* (1) *on* $[0, T]$. *There exists a weak solution on $\mathbb{R}_+$.*

Theorem 2.3 (2D boundedness). *Let $N = 2$ and $p \geq 2$, $q \leq 2$. Then the weak solution u to* (1) *is bounded on $\mathbb{R}_+$ if one of the following conditions hold.*

- $G \in L^1(\mathbb{R}_+)$.
- $G \in L^1_{loc}(\mathbb{R}_+)$ *is of positive type and H is linear.*

Theorem 2.4 (3D existence). *Let $N = 3$, $p \geq 11/5$, $q \leq 2$. Then for every $0 < T < +\infty$ there exists a bounded weak solution u to* (1) *on* $[0, T]$. *There exists a weak solution u to* (1) *on* $\mathbb{R}_+$.

Theorem 2.5 (3D boundedness). *Let $N = 3$ and $p \geq 11/5$, $q \leq 2$. Then there exists a bounded solution on $\mathbb{R}_+$ if one of the following conditions hold.*

- $G \in L^1(\mathbb{R}_+)$.
- $G \in L^1_{loc}(\mathbb{R}_+)$ *is of positive type and H is linear.*

3 - Local existence of Galerkin approximations

We use the standard Galerkin method. Let $\{w^n\}_{n=1}^{\infty}$ be the basis of $H := W^{1,p}_{0,div}(\Omega)$ consisting of the eigenfunctions of the Stokes operator. Denote $V_n := span\{w^1, \ldots, w^n\}$.

Proposition 3.1. *There exist $T_{max} > 0$ and noncontinuable functions $c_j \in W^{1,2}_{loc}(0, T_{max})$, $j = 1, \ldots, n$, such that $u^n(t,x) := \sum_{j=1}^{n} c_j(t) w^j(x)$ satisfies*

$$\int_\Omega u^n_t w + \int_\Omega (u^n \cdot \nabla u^n) w = \langle F(\nabla u^n), \nabla w\rangle + \left\langle \int_0^t G(t-s) H(\nabla u^n(s)) \mathrm{d}s, \nabla w \right\rangle + \langle f, w\rangle \tag{3}$$

for all $w \in V_n$ $\big($*here* $\langle f, g\rangle = \int_\Omega fg\big)$ *and*

$$u^n(0) = u_0^n := \sum_{j=1}^n a_j w^j, \quad a_j := \int_\Omega u_0 w^j.$$

Moreover, if $T_{max} < +\infty$ *then* c_j *is unbounded for some* j.

Proof. Since $(w^j, w^k)_2 = \delta_{jk}$ we obtain

$$c_k'(t) = -\sum_{j,l=1}^n c_j(t)c_l(t)\int_\Omega (w^j\cdot\nabla w^l)w^k + \int_\Omega F\left(\sum_{j=1}^n c_j(t)w^j(x)\right) : \nabla w^k(x)\mathrm{d}x$$
$$+\int_\Omega\int_0^t G(t-s)H\left(\sum_{j=1}^n c_j(s)w^j(x)\right) : \nabla w^k(x)\mathrm{d}s\mathrm{d}x + \langle f(t), w^k\rangle, \tag{4}$$

with initial conditions $c_k(0) = a_k^n$. For $c := (c_1, \dots c_n) \in \mathbb{R}^n$ and $w = (w^1, \dots, w^n) \in (C^\infty(\Omega))^n$ we denote

$$\begin{aligned}
\tilde{A}(c) &:= -\sum_{j,l=1}^n c_j c_l \int_\Omega (w^j(x)\cdot\nabla w^l(x))w(x)\mathrm{d}x\\
\tilde{F}(c) &:= \int_\Omega F\left(\sum_{j=1}^n c_j w^j(x)\right) : \nabla w(x)\mathrm{d}x\\
\tilde{H}(c) &:= \int_\Omega H\left(\sum_{j=1}^n c_j w^j(x)\right) : \nabla w(x)\mathrm{d}x\\
\tilde{G}(t) &:= \int_0^t G(s)\mathrm{d}s\\
\tilde{f}(t) &:= \int_0^t \langle f(s)w\rangle \mathrm{d}s + a^n
\end{aligned}$$

and after integration from 0 to t, (4) yields

$$c(t) = \int_0^t \tilde{A}(c(s)) + \tilde{F}(c(s)) + \tilde{G}(t-s)\tilde{H}(c(s))\mathrm{d}s + \tilde{f}(t). \tag{5}$$

Since $g(t,s,z) := \tilde{A}(z) + \tilde{F}(z) + \tilde{G}(t-s)\tilde{H}(z)$ satisfies the assumptions of Theorem 12.2.6 in [**6**] (here we need local Lipschitz continuity of F and H and local integrability of G), this theorem gives us a unique continuous noncontinuable solution c to (5), which is defined on $\mathbb{R}_+$ or blows up in finite time. Since the right-hand side of (5) is in $W^{1,p}_{loc}(0,T;\mathbb{R}^n)$, we have $c \in W^{1,p}_{loc}(0,T;\mathbb{R}^n)$ and c' solves (4) for a.e. $t \in [0, T_{max})$. $\square$

4 - Energy estimates

In this section we prove some estimates on ∇u and u_t.

Proposition 4.1. *There exists a constant K depending only on Ω, μ_F and C_H a constant C depending on $\|f\|_{L^{p'}((W^{1,p}_{0,div})')}$ and $\|u_0\|_2$ but independent of n such that if T satisfies $\int_0^T |G|\mathrm{d}s < K$ then any solution of* (3) *on* $[0,T)$ *satisfies*

$$\|u^n(t)\|_2^2 + \|\nabla u^n\|^p_{L^p(L^p)} \le C. \tag{6}$$

Proof. Let us take $w = u^n(t)$ in (3). Since

$$\int_\Omega (u^n \cdot \nabla u^n)u^n = 0$$

we obtain

$$\int_\Omega \frac{d}{dt}(u^n)u^n = -\langle F(\nabla u^n), \nabla u^n\rangle - \int_0^t G(t-s)\langle H(\nabla u^n(s)), \nabla u^n(t)\rangle \mathrm{d}s + \langle f, u^n\rangle. \tag{7}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|u^n(t)\|_2^2 + \mu_F\|\nabla u^n(t)\|_p^p \le \left|\int_0^t G(t-s)\langle H(\nabla u^n(s)), \nabla u^n(t)\rangle \mathrm{d}s\right| + \|f(t)\|_{(W^{1,p}_{0,div})'}\|u^n(t)\|_{W^{1,p}_{0,div}},$$

where we used (Fmon) and $F(0)=0$. Integrating this inequality from 0 to t and applying Young inequality and Poincaré inequality to the last term we obtain

$$\|f\|_{L^{p'}((W^{1,p}_{0,div})')}\|u^n\|_{L^p(W^{1,p}_{0,div})} \le C(\varepsilon)\|f\|^{p'}_{L^{p'}((W^{1,p}_{0,div})')} + \varepsilon\|\nabla u^n\|^p_{L^p(L^p)},$$

hence,

$$\frac{1}{2}\|u^n(t)\|_2^2 + (\mu_F - \varepsilon)\|\nabla u^n\|^p_{L^p(L^p)}\mathrm{d}s \le \int_0^t \left|\int_0^\tau G(\tau-s)\langle H(\nabla u^n(s)), \nabla u^n(\tau)\rangle \mathrm{d}s\right| \mathrm{d}\tau + C(\varepsilon)\|f\|_{L^p((W^{1,p}_{0,div})')} + \frac{1}{2}\|u_0\|_2^2. \tag{8}$$

The integral term in (8) can be estimated using Hölder inequality and Lemma 2.1 by

$$\|G * H(\nabla u^n)\|_{L^{q'}(L^{q'})}\|\nabla u^n\|_{L^q(L^q)} \le C_H\|G\|_{L^1}\|\nabla u^n\|^q_{L^q(L^q)}. \tag{9}$$

Since for $p > q$ we have

$$\|v\|_q^q \le C \cdot \|v\|_p^q \le C \cdot (\|v\|_p^p + 1)$$

(with C depending only on Ω) then taking t small enough (so, $CC_H\|G\|_{L^1(0,t)} < \varepsilon$ is small) we obtain

$$\frac{1}{2}\|u^n(t)\|_2^2 + (\mu_F - 2\varepsilon)\|\nabla u^n\|_{L^p(L^p)}^p \le C(\varepsilon)\|f\|_{L^{p'}((W^{1,p}_{0,div})')}^{p'} + \frac{1}{2}\|u_0\|_2^2 + \varepsilon. \tag{10}$$

This proves the assertion. □

Proposition 4.2. *If H is linear and G of positive type, then*

$$\|u^n(t)\|_2^2 + \|\nabla u^n\|_{L^p(L^p)}^p \le C \tag{11}$$

holds on $[0, +\infty)$.

Proof. The proof is similar to the previous one. We integrate (7) from 0 to t and estimate the convolution term by 0 from below (according to the definition of positive-type functions). The remaining terms we estimate as in the proof above. □

For $T < K$ from Proposition 4.1 (resp. for all T in linear case) we already know that the sequence u^n is bounded in $L^\infty(0,T;L^2)$ and $L^p(0,T;W^{1,p}_{0,div})$. We want to show boundedness of u_t^n. It will be needed to show convergence of u^n to a solution u.

Lemma 4.3. *Let $v \in L^\infty(L^2) \cap L^p(W^{1,p}_{0,div})$.*

1. *If $N = 3$ then $v \in L^{\frac{5}{3}p}(L^{\frac{5}{3}p})$ and $\|v\|_{L^{\frac{5}{3}p}(L^{\frac{5}{3}p})} \le C(\|v\|_{L^\infty(L^2)}, \|\nabla v\|_{L^p(L^p)})$.*
2. *If $N = 2$ then $v \in L^{2p}(L^{2p})$ and $\|v\|_{L^{2p}(L^{2p})} \le C(\|v\|_{L^\infty(L^2)}, \|\nabla v\|_{L^p(L^p)})$.*

Proof. This lemma follows easily from Hölder's inequality and Sobolev imbeddings. □

Proposition 4.4. *Let $p \ge 11/5$ if $N = 3$ and $p \ge 2$ if $N = 2$. For any finite T from Proposition* 4.1 *there exists $C(f, u_0) > 0$ (independent of n) such that any solution of* (3) *on* $[0, T)$ *satisfies*

$$\|u_t^n\|_{L^{p'}(0,T;(W^{1,p}_{0,div})')} \le C(f, u_0). \tag{12}$$

Proof. We have

$$\|u_t^n\|_{L^{p'}(0,T;(W^{1,p}_{0,div})')} = \sup_{\varphi \in L^p(0,T;W^{1,p}_{0,div}), \|\varphi\| \le 1} \int_0^T \langle u_t^n, \varphi \rangle \mathrm{d}t.$$

Moreover, we have

$$\int_0^T \langle u_t^n, \varphi\rangle \mathrm{d}t \le \left|\int_0^T\int_\Omega u_t^n \varphi^n \mathrm{d}x\mathrm{d}t\right| \le \left|\int_0^T\int_\Omega (u^n\cdot\nabla u^n)\varphi^n\right| + \left|\int_0^T \langle F(\nabla u^n), \nabla\varphi^n\rangle\right|$$
$$+\left|\int_0^T\int_0^t G(t-s)\langle H(\nabla u^n(s)), \nabla\varphi^n\rangle \mathrm{d}s\right| + \left|\int_0^T \langle f, \varphi^n\rangle\right|.$$

Using Lemma 2.1 and Proposition 4.1 the second, third, and forth term on the right-hand side can be easily estimated by

$$\|\nabla\varphi^n\|_{L^p(L^p)}\cdot(\|f\|_{L^{p'}(0,T;(W^{1,p}_{0,div})')} + \tilde{C}_F(\|\nabla u^n\|^{p-1}_{L^p(L^p)}+1)$$
$$+ C_H\|G\|_{L^1}\|\nabla u^n\|^{q-1}_{L^q(L^q)}) \le C\|\nabla\varphi^n\|_{L^p(L^p)}.$$

The convective term can be estimated as usually. In fact, we have

$$\left|\int_0^T\int_\Omega (u^n\cdot\nabla u^n)\varphi^n\right| \le C\int_0^T\int_\Omega |u^n|^2|\nabla\varphi^n| \le C\|\nabla\varphi^n\|_{L^p(L^p)}\||u^n|^2\|_{L^{p'}(L^{p'})} \tag{13}$$
$$= C\|\nabla\varphi^n\|_{L^p(L^p)}\|u^n\|^2_{L^{2p'}(L^{2p'})} = C\|\nabla\varphi^n\|_{L^p(L^p)}\|u^n\|^2_{L^{\frac{2p}{p-1}}\left(L^{\frac{2p}{p-1}}\right)}.$$

If $N=3$, $p\ge\frac{11}{5}$ then $\frac{2p}{p-1}\le\frac{5}{3}p$. If $N=2$, $p\ge 2$ then $\frac{2p}{p-1}\le 2p$. In both these cases, by Lemma 4.3, the right-hand side of (13) is bounded by $C\|\nabla\varphi^n\|_{L^p(L^p)}$. The proof is complete. □

5 - Convergence of approximations

In this subsection we show that a subsequence of u^n converge to a candidate function u and that the function u is a solution to the original problem. The method will be standard using Aubin-Lions Lemma and interpolations to obtain as good convergence as possible and then using Minty trick to pass to the limit in the non-linear terms.

Lemma 5.1. *There exists a function $u:[0,T)\to L^2$ and a subsequence of the functions u^n from Proposition* 3.1 *such that*

1. *$u^n\to u$ weakly* in $L^\infty(0,T;L^2)$*
2. *$\nabla u^n\to\nabla u$ weakly in $L^p(0,T;L^p)$*

3. $u^n \to u$ *weakly in* $L^{5p/3}(0, T; L^{5p/3})$ *resp.* $L^{2p}(0, T; L^{2p})$
4. $u_t^n \to u_t$ *weakly in* $L^{p'}(0, T; (W^{1,p}_{0,div})')$
5. $u^n \to u$ *strongly in* $L^p(0, T; L^p)$
6. $u^n \to u$ *strongly in* $L^r(0, T; L^p)$, $r < +\infty$
7. $u^n \to u$ *strongly in* $L^p(0, T; L^r)$, $r < \dfrac{3p}{3-p}$ *resp.* $r < +\infty$
8. $\|u^n(t)\|_2 \to \|u(t)\|_2$ *for a.e.* $t \in [0, T]$

for $N = 3$, *resp. for* $N = 2$. *The assertion* 7. *for* $N = 3$ *holds if* $p < 3$, *if* $p \geq 3$, *then it holds for all* $r < +\infty$.

Proof. The 1. and 2. assertions follow from Proposition 4.1, the 3. from Lemma 4.3, the 4. from Proposition 4.4, 5. from Aubin-Lions lemma and 6. and 7. from interpolation and compact Sobolev imbeddings.

Let us prove 8. From strong convergence in $L^2(0, T; L^2)$ and Cauchy-Schwarz inequality we have

$$0 = \lim \int_0^{T'} \|u^n - u\|_2^2 \mathrm{d}t \geq \limsup \int_0^{T'} \left| \|u^n\|_2 - \|u\|_2 \right|^2 \mathrm{d}t$$
$$\geq \liminf \int_0^{T'} \left| \|u^n\|_2 - \|u\|_2 \right|^2 \mathrm{d}t \geq 0.$$

So, scalar functions $t \mapsto \|u^n(t)\|_2$ converges to $t \mapsto \|u(t)\|_2$ in $L^2(0, T)$. Hence there is a subsequence converging for a.e. $t \in [0, T]$. □

Proposition 5.2. *For any finite* T *from Proposition* 4.1 (*resp. for any* $0 < T < +\infty$ *in linear case*) *there exists a bounded weak solution* u *to problem* (1) *on* $[0, T)$.

Proof. Writing the weak formulation for $u^n \in V_n$, $\psi \in C_0^\infty(0, T)$ and w^k we obtain

$$\int_0^T \langle (u^n)', w^k \rangle \psi + \int_0^T \int_\Omega (u^n \cdot \nabla u^n) w^k \psi = -\int_0^T \langle F(\nabla u^n), \nabla w^k \rangle \psi$$
$$- \int_0^T \int_0^t G(t-s) \langle H(\nabla u^n(s)), \nabla w^k \rangle \psi(t) \mathrm{d}s + \int_0^T \langle f, w^k \rangle \psi. \tag{14}$$

We pass to the limit for $n \to \infty$. For the linear term we have

$$\lim_{n\to\infty} \int_0^T \langle (u^n)', w^k \rangle \psi = \int_0^T \langle u', w^k \rangle \psi.$$

For the convective term we have

$$(15)\quad \left| \int_0^T \int_\Omega (u^n \cdot \nabla u^n - u \cdot \nabla u) w^k \mathrm{d}x \psi(t) \mathrm{d}t \right|$$
$$\leq \int_0^T \int_\Omega |u^n| \cdot |u^n - u| |\nabla w^k \psi| \mathrm{d}x \mathrm{d}t + \int_0^T \int_\Omega |u^n - u| \cdot |u| |\nabla w^k \psi| \mathrm{d}x \mathrm{d}t$$
$$\leq \|u^n - u\|_{L^2(L^3)} \|\nabla w^k\|_{L^2} \|\psi\|_\infty (\|u\|_{L^2(L^6)} + \|u^n\|_{L^2(L^6)}) \to 0.$$

It remains to show convergence for the other two nonlinear terms. Since

$$F(\nabla u^n) + \int_0^t G(t-s) H(\nabla u^n(s)) =: B(\nabla u^n)$$

is by Lemma 2.1 a bounded sequence in $L^{p'}(L^{p'})$, there exists a subsequence of u^n (denoted again by u^n) and a function $\tilde{B} \in L^{p'}(L^{p'})$ such that $B(\nabla u^n) \to \tilde{B}$ weakly in $L^{p'}(L^{p'})$. If we showed that

$$(16)\qquad \langle \tilde{B}, \nabla \varphi \rangle = \langle B(\nabla u), \nabla \varphi \rangle \quad \text{for all } \varphi \in W^{1,p}_{0,div} \text{ and a.e. } t \in (0,T)$$

then u would be a weak solution and the proof would be complete.

In Lemma 5.4 we show that

$$\int_0^t \langle B(\nabla(u + \varepsilon\varphi)) - \tilde{B}, \varepsilon \nabla \varphi \rangle \mathrm{d}s \geq 0$$

for all $\varphi \in L^p(W^{1,p}_{0,div})$, $\varepsilon \in \mathbb{R}$. Since

$$0 \leq \langle B(\nabla(u+\varepsilon\varphi)) - \tilde{B}, \varepsilon\nabla\varphi \rangle = \langle B(\nabla(u+\varepsilon\varphi)) - B(\nabla u), \varepsilon\nabla\varphi \rangle + \langle B(\nabla u) - \tilde{B}, \varepsilon\nabla\varphi \rangle$$
$$\leq \varepsilon^2 \lambda_B \|\nabla\varphi\|_p^p + \varepsilon \langle B(\nabla u) - \tilde{B}, \nabla\varphi \rangle,$$

holds for every $\varepsilon \in (-\delta, +\delta)$ (and for φ as well as for $-\varphi$), we obtain $\langle B(\nabla u) - \tilde{B}, \nabla\varphi \rangle = 0$, i.e., (16) holds and the proof will be finished as soon as we prove Lemma 5.4. □

Lemma 5.3. *Let $u^n \to u$ in the sense as in Lemma 5.1, let $B(\nabla u^n) \to \tilde{B}$ weakly in $L^{p'}(L^{p'})$. Then*

$$\langle \tilde{B}, \nabla\varphi\rangle = \langle f - u_t - u\cdot\nabla u, \varphi\rangle \tag{17}$$

for all $\varphi \in W^{1,p}_{0,div}$ and a.e. $t \in [0,T]$.

Proof. Since u^n satisfies

$$\langle B(u^n), \nabla w^k\rangle = \langle f - u^n_t - u^n\cdot\nabla u^n, \varphi\rangle$$

for all $n \geq k$ and a.e. $t \in [0,T]$, we have

$$\int_0^T \langle B(u^n), \nabla w^k\rangle \psi(s)\mathrm{d}s = \int_0^T \langle f - u^n_t - u^n\cdot\nabla u^n, w^k\rangle\psi(s)\mathrm{d}s$$

for every $\psi \in C^\infty(0,T)$. Taking the limit for $n \to \infty$ we obtain

$$\int_0^T \langle \tilde{B}, \nabla w^k\rangle \psi(s)\mathrm{d}s = \int_0^T \langle f - u_t - u\cdot\nabla u, w^k\rangle\psi(s)\mathrm{d}s$$

since $u^n_t \to u_t$ weakly in $L^{p'}((W^{1,p}_{0,div})')$ by Lemma 5.1 and $u^n\cdot\nabla u^n \to u\cdot\nabla u$ weakly in $L^{p'}((W^{1,p}_{0,div})')$ according to (13). Since this holds for all k, we have

$$\int_0^T \langle \tilde{B}, \nabla\varphi\rangle \psi(s)\mathrm{d}s = \int_0^T \langle f - u_t - u\cdot\nabla u, \varphi\rangle\psi(s)\mathrm{d}s$$

for all $\varphi \in W^{1,p}_{0,div}$ (passing to the limit for $k \to \infty$) and for all $\psi \in C^\infty$. Now, (17) easily follows. □

Lemma 5.4. *Let $u^n \to u$ as in Lemma 5.1, let $B(\nabla u^n) \to \tilde{B}$ weakly in $L^{p'}(L^{p'})$. Then*

$$\int_0^t \langle B(\nabla v) - \tilde{B}, \nabla v - \nabla u\rangle \mathrm{d}s \geq 0 \tag{18}$$

for all $v \in L^p(W^{1,p}_{0,div})$.

Proof. If we justify all steps in the following computations, the assertion will be proved.

$$\int_0^t \langle B(\nabla v) - \tilde{B}, \nabla v - \nabla u\rangle \mathrm{d}s = \int_0^t \langle B(\nabla v), \nabla v - \nabla u\rangle \mathrm{d}s - \int_0^t \langle \tilde{B}, \nabla v\rangle \mathrm{d}s + \int_0^t \langle \tilde{B}, \nabla u\rangle \mathrm{d}s \tag{19}$$

$$= \lim_{n\to\infty}\left(\int_0^t \langle B(\nabla v), \nabla v - \nabla u^n\rangle \mathrm{d}s - \int_0^t \langle B(\nabla u^n), \nabla v\rangle \mathrm{d}s + \int_0^t \langle \nabla B(u^n), \nabla u^n\rangle \mathrm{d}s\right)$$

$$= \lim_{n\to\infty}\int_0^t \langle B(\nabla v) - B(\nabla u^n), \nabla v - \nabla u^n\rangle \mathrm{d}s \geq 0.$$

The first equality is trivial. To show the second equality we need to justify

$$\lim_{n\to\infty}\left(\int_0^t \langle B(\nabla v), \nabla v - \nabla u^n\rangle \mathrm{d}s - \int_0^t \langle B(\nabla u^n), \nabla v\rangle \mathrm{d}s\right) = \int_0^t \langle B(\nabla v), \nabla v - \nabla u\rangle \mathrm{d}s - \int_0^t \langle \tilde{B}, \nabla v\rangle \mathrm{d}s$$

(this follows immediately from weak $L^{p'}(L^{p'})$ convergence of $\nabla u^n \to \nabla u$ and $B(\nabla u^n) \to \tilde{B}$) and

$$\lim_{n\to\infty}\int_0^t \langle \nabla B(u^n), \nabla u^n\rangle \mathrm{d}s = \int_0^t \langle \tilde{B}, \nabla u\rangle \mathrm{d}s. \tag{20}$$

However, since u^n is a solution of (3) and by the previous lemma, equality (20) can be rewritten as

$$\lim_{n\to\infty}\int_0^t \langle f - u^n_t - u^n\cdot\nabla u^n, u^n\rangle \mathrm{d}s = \int_0^t \langle f - u_t - u\cdot\nabla u, u\rangle \mathrm{d}s.$$

Clearly,

$$\lim_{n\to\infty}\int_0^t \langle f, u^n - u\rangle \mathrm{d}s = 0 \tag{21}$$

by weak $L^p(W^{1,p}_{0,div})$ convergence of $u^n \to u$. The terms

$$\langle u^n\cdot\nabla u^n, u^n\rangle = 0, \quad \langle u\cdot\nabla u, u\rangle = 0$$

vanish for a.e. $t\in[0,T]$, so it remains to show

$$\lim_{n\to\infty}\int_0^t \langle u^n_t, u^n\rangle - \langle u_t, u\rangle \mathrm{d}s = 0.$$

Since $u \in L^p(W^{1,p}_{0,div})$ and u_t is in the dual space (and the same holds for u^n and u^n_t), the left-hand side of (21) is equal to

$$\lim_{n\to\infty} \frac{1}{2}(\|u^n(t)\|_2 - \|u(t)\|_2 + \|u^n(0)\|_2 - \|u(0)\|_2),$$

convergence to 0 follows from Lemma 5.1, part 8. and L^2 convergence of initial values. The second equality in (19) is proved.

The third equality in (19) is trivial and the last inequality follows from

$$(22)\quad \int_0^t \langle B(\nabla v) - B(\nabla u^n), \nabla v - \nabla u^n\rangle \mathrm{d}s = \int_0^t \langle F(\nabla v) - F(\nabla u^n), \nabla v - \nabla u^n\rangle$$
$$+\left\langle \int_0^t G(t-s)(H(\nabla v(s)) - H(\nabla u^n(s))), \nabla v - \nabla u^n \right\rangle \mathrm{d}s$$
$$\geq \mu_F \|\nabla v - \nabla u^n\|^2_{L^2(L^2)} - \|G\|_1 \lambda_H \|\nabla v - \nabla u^n\|^2_{L^2(L^2)} \geq 0$$

since $\mu_F - \lambda_H\|G\| \geq 0$ if $T < K$ from Proposition 4.1. If H is linear and G of positive type, then the estimates in (22) follow easily. The proof is complete. □

6 - Energy inequality and the initial condition

Proposition 6.1. *The weak solution u obtained by the Galerkin method satisfies*

$$(23)\qquad \|u(t)\|^2_2 + c\int_0^t \|\nabla u(s)\|^p_p \mathrm{d}s \leq \int_0^t \langle f(s), u(s)\rangle \mathrm{d}s + \|u_0\|^2_2.$$

Proof. Taking $\varphi = u^n(t)$ in (3) and integrating from 0 to t we obtain

$$\frac{1}{2}\|u^n(t)\|^2_2 + \int_0^t\int_\Omega F(\nabla u^n(s)) : \nabla u^n(s)\mathrm{d}s + \int_0^t\int_\Omega\int_0^\tau G(\tau - s)H(\nabla u^n(s)) : \nabla u^n(t)\mathrm{d}s\mathrm{d}\tau$$
$$= \int_0^t \langle f(s), u^n(s)\rangle \mathrm{d}s + \frac{1}{2}\|u^n(0)\|^2_2.$$

Estimating the second term from below and the third term from above we get

$$\frac{1}{2}\|u^n(t)\|^2_2 + c\int_0^t \|\nabla u^n(s)\|^p_p \mathrm{d}s \leq \int_0^t \langle f(s), u^n(s)\rangle \mathrm{d}s + \frac{1}{2}\|u^n(0)\|^2_2.$$

Multiplying by $\psi \in C_0^\infty(0,T)$, $\psi \geq 0$ and integrating from 0 to T, we get

$$\int_0^T \left[\frac{1}{2}\|u^n(t)\|_2^2 + c\int_0^t \|\nabla u^n(s)\|_p^p \mathrm{d}s - \int_0^t \langle f(s), u^n(s)\rangle \mathrm{d}s - \frac{1}{2}\|u_0^n\|_2^2\right]\psi(t)\mathrm{d}t \leq 0. \tag{24}$$

We have (by strong convergence in $L^2(L^2)$)

$$\int_0^T \|u^n(t)\|_2^2 \psi(t)\mathrm{d}t \to \int_0^T \|u(t)\|_2^2 \psi(t)\mathrm{d}t$$

and (by weak semicontinuity of the norm)

$$\liminf_{n\to\infty} \int_0^T\int_0^t \|\nabla u^n(s)\|_p^p \mathrm{d}s\psi(t)\mathrm{d}t \geq \int_0^T\int_0^t \|\nabla u(s)\|_p^p \mathrm{d}s\psi(t)\mathrm{d}t.$$

Since the other two terms in (24) converge, we obtain

$$\int_0^T \left[\frac{1}{2}\|u(t)\|_2^2 + c\int_0^t \|\nabla u(s)\|_p^p \mathrm{d}s - \int_0^t \langle f(s), u(s)\rangle \mathrm{d}s - \frac{1}{2}\|u_0\|_2^2\right]\psi(t)\mathrm{d}t \leq 0.$$

If ψ is a molifier ω_ε and we let $\varepsilon \to 0$, we obtain (23). □

Proposition 6.2. *The solution from Proposition* 5.2 *satisfies* $\|u(t) - u_0\|_2 \to 0$.

Proof. Multiply (3) by $\psi \in C^\infty[0,T]$, $\psi(T) = 0$, integrate from 0 to T and apply integration by parts to the first term

$$\int_\Omega u^n(0)w\psi(0) - \int_0^T\int_\Omega u^n w\psi' + \int_0^T\int_\Omega (u^n\cdot\nabla u^n)w\psi + \int_0^T \langle F(\nabla u^n), \nabla w\rangle\psi$$
$$+ \int_0^T \left\langle \int_0^t G(t-s)H(\nabla u^n(s))\mathrm{d}s, \nabla w\right\rangle\psi - \int_0^T \langle f, w\rangle\psi = 0.$$

Pass to the limit for $n \to \infty$ and use completeness of $\{w^k\}_{k=1}^\infty$ in H, we have

$$\int_\Omega u_0\varphi\psi(0) - \int_0^T\int_\Omega u\varphi\psi' + \int_0^T\int_\Omega (u\cdot\nabla u)\varphi\psi + \int_0^T \langle F(\nabla u), \nabla\varphi\rangle\psi$$
$$+ \int_0^T \left\langle \int_0^t G(t-s)H(\nabla u(s))\mathrm{d}s, \nabla\varphi\right\rangle\psi - \int_0^T \langle f, \varphi\rangle\psi = 0$$

for all $\varphi \in H$. Using again integration by parts we have

$$\int_0^T \int_\Omega u\varphi\psi' = u(0)\varphi\psi(0) - \int_0^T \int_\Omega u_t.\varphi\psi.$$

Inserting this equality into the previous one and using the fact that u is a weak solution to (1), we obtain

$$\int u(0)\varphi\psi(0) = \int_\Omega u_0\varphi\psi(0).$$

Hence, $u(t)$ to u_0 weakly in L^2 and as a consequence we have $\liminf_{t\to 0+} \|u(t)\| \geq \|u_0\|$. Since u also satisfies $\limsup \|u(t)\|_2 \leq \|u_0\|_2$ by (23), we have $\|u(t)\|_2 \to \|u_0\|_2$. Together with weak convergence and uniform convexity of L^2 this implies the assertion. □

7 - Proofs of the main results

In this section we finish the proofs of Theorems 2.2 - 2.5. If H is linear and G of positive type, then the existence of a bounded solution on any bounded interval follows from Propositions 5.2 and 6.2. For the nonlinear case we have only existence on $[0, T)$ for $T < K$. First we show that we can continue a solution by taking $u(t)$ as a new initial value and pasting the two solutions together. Then we show that after finitely many steps we get a solution on any bounded interval.

Proposition 7.1. *Let u be a solution of* (1) *on* $[0, T)$ *and* $T_1 < T$. *Let $\tilde{u}$ be a solution of* (1) *on* $[0, \tilde{T})$ *with G replaced by* $\tilde{G}(t) := G(t + T_1)$ *and f replaced by* $\tilde{f}(t) := f(t + T_1) + \int_0^{T_1} G(t - s)\,\mathrm{div}\, H(\nabla u(s))\mathrm{d}s$ *and* $\tilde{u}_0 := u(T_1)$. *Then*

$$v(t) = \begin{cases} u(t) & t < T_1 \\ \tilde{u}(t - T_1) & T_1 \leq t < T_1 + \tilde{T} \end{cases}$$

is a solution to (1) *on* $[0, T_1 + \tilde{T})$.

Proof. It is clear that $v \in L^p_{loc}(0, T_1 + \tilde{T}; W^{1,p}_{0,div}(\Omega)) \cap L^\infty_{loc}(0, T_1 + \tilde{T}; L^2(\Omega))$ and attains the initial condition in L^2-sense. It remains to show that $v_t \in L^p_{loc}(0, T_1 + \tilde{T}; (W^{1,p}_{0,div}(\Omega))^*)$ and that it satisfies (1) for a.e. $t \in [T_1, T_1 + \tilde{T})$. Since v_t is in the L^p space on $[0, T_1)$ and also on $[T_1, T_1 + \tilde{T})$, we only need to show that v is continuous in T_1 in the norm of $(W^{1,p}_{0,div}(\Omega))^*$. The continuity from the left follows from the fact that u is a solution on $[0,T)$, $T > T_1$, continuity from the right follows from the fact that $\tilde{u}$ attains its initial value in L^2-norm (hence also in $(W^{1,p}_{0,div}(\Omega))^*$-norm).

We show that v satisfies the equation for a.e. $t \in [T_1, T_1 + \tilde{T})$. For a.e. $t \in [0, \tilde{T})$ we have

$$(25)\quad 0 = \langle \tilde{u}_t, \varphi\rangle + \int_\Omega (\tilde{u}\cdot\nabla\tilde{u})\varphi + \int_\Omega F(\nabla\tilde{u}) : \nabla\varphi + \int_\Omega\int_0^t \tilde{G}(t-s)H(\nabla\tilde{u}(s)) : \nabla\varphi(t)\mathrm{d}s$$

$$- \langle\tilde{f},\varphi\rangle = \langle v_t(t+T_1),\varphi\rangle + \int_\Omega (v(t+T_1)\cdot\nabla v(t+T_1))\varphi + \int_\Omega F(\nabla v(t+T_1)) : \nabla\varphi$$

$$+ \int_\Omega\int_0^t G(t+T_1-s)H(\nabla v(s+T_1)) : \nabla\varphi(t)\mathrm{d}s - \langle f(t+T_1),\varphi\rangle$$

$$+ \int_\Omega\int_0^{T_1} G(t-s)H(\nabla v(s))\mathrm{d}s : \nabla\varphi\mathrm{d}s.$$

If we add the two convolution terms, we can see that v satisfies (1). □

To finish the proof of global existence, let us note that the problem $(\tilde{1})$ from Proposition 7.1 (with $\tilde{G}$, $\tilde{f}$ and $\tilde{u}_0$) satisfies the same assumptions as (1), i.e. $\tilde{u}_0 = u(T_1) \in L^2$ ($u(t) \in L^2$ a.e., so we can take an appropriate T_1), $\tilde{G} \in L^1_{loc}$ and $\tilde{f} \in L^{p'}((W^{1,p}_{0,div})')$ (according to Lemma 2.1, $G * (H \circ \nabla u) \in L^{q'}(L^{q'})$). Hence, by Proposition 5.2 this tilde-problem has also a solution on $[0,\tilde{T})$ for some $\tilde{T}$. Moreover, by Proposition 4.1 $\tilde{T}$ is an arbitrary number satisfying $K > \int_0^{\tilde{T}} |\tilde{G}| = \int_{T_1}^{T_1+\tilde{T}} |G|$. So, if $G \in L^1_{loc}$ then we get the solution on any bounded interval in finitely many steps, hence it is bounded.

Consider the interval $I_n := [T_n, +\infty)$, such that $\|G\|_1 < K$ on I_n or H linear and G of positive type. Then the estimate (6) holds on the whole interval I_n. However, the estimate of u_t holds on bounded subintervals only with the constant C depending on the length of the subinterval. However, we can write I_n as union of countably many subintervals, paste the solutions on these subintervals and the solution that we obtain will be bounded since the estimate (6) holds on the whole I_n.

Now the Theorems 2.2 - 2.5 are almost proved, the only remaining thing is the uniqueness. It can be shown using Gronwall Lemma. Let u_1, u_2 are two weak solutions. Subtracting the corresponding equations we obtain

$$\left\langle \frac{d}{dt}(u_1 - u_2), \varphi \right\rangle + \int_\Omega (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2)\varphi + \int_\Omega (F(\nabla u_1) - F(\nabla u_2)) : \nabla \varphi + \int_\Omega \int_0^t G(t-s)(H(\nabla u_1(s)) - H(\nabla u_2(s))) : \nabla \varphi(t) \mathrm{d}s = 0 \tag{26}$$

for $\varphi \in H$. Since $u_1 - u_2 \in H$, we have

$$\frac{d}{dt}\|(u_1 - u_2)(t)\|_2 + \int_\Omega (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2)(u_1 - u_2) + \int_\Omega (F(\nabla u_1) - F(\nabla u_2)) : \nabla (u_1 - u_2) + \int_\Omega \int_0^t G(t-s)(H(\nabla u_1(s)) - H(\nabla u_2(s))) : \nabla (u_1 - u_2)(t) \mathrm{d}s = 0. \tag{27}$$

We have

$$\int_\Omega (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2)(u_1 - u_2) \leq \|u_1 - u_2\|_2 \|\nabla(u_1 - u_2)\|_2 \|\nabla u_1\|_2.$$

Hence,

$$\frac{d}{dt}\|(u_1 - u_2)(t)\|_2 + \mu_f \|\nabla(u_1 - u_2)(t)\|_2^2 \leq \|u_1 - u_2\|_2 \|\nabla(u_1 - u_2)\|_2 \|\nabla u_1\|_2 + \|G\|_1 \lambda_H \int_0^t \|\nabla(u_1 - u_2)(s)\|_2^2 \mathrm{d}s. \tag{28}$$

Using

$$\|u_1 - u_2\|_2 \|\nabla(u_1 - u_2)\|_2 \|\nabla u_1\|_2 \leq \frac{\mu_f}{2}\|\nabla(u_1 - u_2)\|_2^2 + C\|u_1 - u_2\|_2^2 \cdot \|\nabla u_1\|_2^2,$$

we get

$$\frac{d}{dt}\left(\|(u_1 - u_2)(t)\|_2^2 + \int_0^t \|\nabla(u_1 - u_2)(s)\|_2^2 \mathrm{d}s \right) \leq \tilde{C}\left(\|(u_1 - u_2)(t)\|_2^2 + \int_0^t \|\nabla(u_1 - u_2)(s)\|_2^2 \mathrm{d}s \right).$$

Uniqueness now follows from the Gronwall lemma.

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