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Almost periodic viscosity solutions of nonlinear parabolic equations in vector fields

Abstract. The aim of this paper is to establish existence and uniqueness for time almost periodic viscosity solutions of parabolic equations in vector fields.

Keywords. Parabolic Hamilton Jacobi equations, almost periodic solutions, vector fields.

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1 - Background and motivation

The viscosity solution notion gives an important answer to the research of a definition of weak solution for strongly nonlinear equations.

It fits in a natural way in the second order equations setting. If H is proper, the maximum principle for the equation

$$H(x, u(x), Du(x), D^2u(x)) = 0$$

where u is a regular function and Du and D^2u are respectively the gradient and the second order derivative matrix (i.e. $H(x_o, u(x_o), D\varphi(x_o), D^2\varphi(x_o)) \leq 0 (\geq 0)$ for every $\varphi \in C^2(\Omega)$ if $x_o \in \Omega$ is a local maximum (minimum) point of $u - \varphi$) doesn't involve any regularity of u and then it can be utilized to define the viscosity subsolution (in case $\leq$) or supersolution (in case $\geq$) notion even if u is only continuous. The definition in the

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parabolic case

$$u_t(t,x) + H(t,x,u(t,x),Du(t,x),D^2u(t,x)) = 0$$

is similar.

However one can refer also to a different but equivalent definition of viscosity sub(super)solution (see Subsection 3.1).

For a better investigation in the Euclidean case we mention the celebrated paper of Crandall et al. [**8**] where both stationary and time dependent problems are considered. The notion of viscosity solution has been introduced in [**10**] about first order equations. These results were extended by the same and other several authors. We just mention the books [**2**], [**1**].

In this paper we prove some results about viscosity solutions for a class of parabolic equations *in vector fields*, that is when the vector fields $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right\}$ in $\mathbf{R}^n$ are replaced by an arbitrary orthonormal (with respect to a Riemannian metric) collection of vector fields or *frame*

$$\Re = \{X_1, X_2, \dots, X_n\}$$

where

$$X_i = \sum_{j=1}^{n} a_{ij}(x)\partial_{x_j}, \quad i = 1, \dots, n$$

for some choice of smooth functions $a_{ij}(x)$. We denote by $\Re u$, $\Re^2 u$ and $(\Re^2 u)^*$ respectively the natural gradient, the natural second order derivative matrix and the symmetrized second order derivative matrix of the function u (a precise definition is given in § 2.2).

Important examples are, besides the canonical frame, the Heisenberg group and the Engel group (see Section 2.2). We consider parabolic equations of the form

$$u_t + H(t,x,u,\Re u,(\Re^2 u)^*) = f(t,x). \tag{1.1}$$

Here we consider functions $H : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times S(n) \to \mathbf{R}$ where $S(n)$ denotes the set of symmetric nxn matrices equipped with its usual order (that is $Y \le X$ when $\langle Yp, p\rangle \le \langle Xp, p\rangle$ for every $p \in \mathbf{R}^n$) and $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$. Moreover we suppose f continuous and H continuous and proper. Recall that H proper means

(H_1) $\quad H(t,x,r,\eta,X) \le H(t,x,s,\eta,Y)$

whenever $r \le s$ and $Y \le X$, for arbitrary t, x and η. Let us note that the derivatives $\Re u$ and $(\Re^2 u)^*$ are taken in the space variable x. Examples of parabolic equations in the frame include the parabolic infinite Laplace equation relative to the frame $\Re$

$$u_t + \triangle_{\Re,\infty} u \equiv u_t - \langle (\Re^2 u)^* \Re u, \Re u \rangle = u_t - \left[\sum_{i,j=1}^{n} (X_i u)(X_j u) X_i X_j u \right] = 0$$

and the parabolic p-Laplace equation for $2 \leq p < \infty$ relative to the frame $\Re$

$$u_t + \triangle_{\Re,p} u \equiv u_t - div_{\Re}(\|\Re u\|^{p-2} \Re u)$$
$$\equiv u_t - \left[\|\Re u\|^{p-2} \triangle_{\Re} u + (p-2) \|\Re u\|^{p-4} \triangle_{\Re,\infty} u \right] = 0.$$

In this framework, following the outline of [**3**], the author of [**5**] proved a maximum principle for viscosity solutions of parabolic equations, and consequently, under additional hypothesis, he established a comparison principle in the same frame. We apply the comparison principle to establish a decay estimate from which we deduce uniqueness and existence of time almost periodic (a.p.) viscosity solutions in vector fields.

We partially follow the method of [**6**], where existence and uniqueness of viscosity solutions of first order evolution Hamilton Jacobi equations in the Euclidean environment are considered.

About the interest to have a.p. solutions we recall in particular the relationship between almost periodicity and stability. Stable electronic circuits exhibit a.p. behavior. In Celestial Mechanics a.p. solutions and stable solutions are strictly related.

In Section 2 we describe almost periodic functions and the framework. Section 3 contains all the results of the paper. § 3.1 is devoted to give the definitions of viscosity sub(super)solutions. § 3.2 contains the comparison principle and a decay estimate for viscosity sub(super)solutions. § 3.3 is devoted to establish the existence of viscosity time almost periodic (a.p.) solutions. It largely refers to the results of § 3.2.

Finally in § 3.4 we treat a classical result. We consider a time periodic Hamiltonian of period T and prove that if u is a viscosity sub(super)solution for $t \in (0, T)$, then u is a viscosity sub(super)solution for $t \in \mathbf{R}$.

2 - Basic notions and framework

2.1 - *Almost periodic functions*

In this subsection we recall the definition and some fundamental properties of almost periodic functions. For more details one can refer to [**7**], [**9**].

Definition 2.1. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. We say that f is *almost periodic* if it satisfies the following condition

$$\forall \varepsilon > 0 \ \exists l(\varepsilon) > 0 \text{ such that } \forall a \in \mathbf{R} \ \exists \tau \in [a, a + l(\varepsilon)) \ \text{ satisfying}$$

$$|f(t + \tau) - f(t)| < \varepsilon \ , \ \ \forall t \in \mathbf{R}. \tag{2.1}$$

A number τ verifying (2.1) is called ε *almost period*.

Proposition 2.1. *Assume that* $f : \mathbf{R} \to \mathbf{R}$ *is almost periodic. Then*

i) f *is bounded and uniformly continuous in* $\mathbf{R}$.

ii) $(1/T) \int_a^{a+T} f(t)\, dt$ *converges as* $T \to +\infty$ *uniformly with respect to* $a \in \mathbf{R}$. *The limit is called the average of* f *and denoted by*

$$\langle f \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_a^{a+T} f(t)\, dt \ , \ \ \textit{uniformly w.r.t. } a \in \mathbf{R}.$$

If f *is periodic then* $\langle f \rangle$ *denotes the usual definition of mean of* f *over one period.*

iii) If F *denotes a primitive of* f, *then* F *is almost periodic if and only if* F *is bounded.*

The following definition extends the notion of almost periodicity in order to apply it to differential equations [**11**].

Definition 2.2. Let Ω be a subset of $\mathbf{R}^n$. We say that $u : \mathbf{R} \times \Omega \to \mathbf{R}$ is *almost periodic in t uniformly with respect to x* if u is continuous in t uniformly with respect to x and $\forall \varepsilon > 0 \, \exists \, l(\varepsilon) > 0$ such that all interval of length $l(\varepsilon)$ contains a number τ which is ε almost period for $u(\cdot, x)$, $\forall x \in \Omega$

$$|u(t + \tau, x) - u(t, x)| < \varepsilon \ \forall (t, x) \in \mathbf{R} \times \Omega.$$

2.2 - *Vector fields*

Let

$$\Re = \{X_1, X_2, \ldots, X_n\}$$

be a collection of n linearly independent smooth vector fields or *frame* in $\mathbf{R}^n$ defined as

$$X_i = \sum_{j=1}^{n} a_{ij}(x) \partial_{x_j} \ , \quad i = 1, \ldots, n,$$

for some choice of smooth functions $a_{ij}(x)$. If $\mathbf{A}(x)$ denotes the matrix whose (i, j)-entry is $a_{ij}(x)$, then we suppose $det\mathbf{A}(x) \neq 0$ in $\mathbf{R}^n$. We choose a Riemannian metric so that the frame is orthonormal. The gradient and the second order derivative matrix in the frame are respectively the vector

$$\Re u = \{X_1(u), X_2(u), \ldots, X_n(u)\}$$

and the not necessarily symmetric matrix $\Re^2 u = X_i(X_j(u))$. The symmetrized second order derivative matrix is

$$(\Re^2 u)^* = \frac{1}{2}\{\Re^2 u + (\Re^2 u)^t\}.$$

Fix a point $x \in \mathbf{R}^n$ and let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ denote a vector close to zero. We define the exponential based at x of ξ, denoted by $\Theta_x(\xi)$, as follows : let γ be the unique solution to the system of ordinary differential equations

$$\gamma'(s) = \sum_{i=1}^{n} \xi_i X_i(\gamma(s))$$

satisfying the initial condition $\gamma(0) = x$. We set $\Theta_x(\xi) = \gamma(1)$ and note this is defined in a neighborhood of zero.

As examples of the frame we could cite the canonical frame of the usual first order partial derivatives, the Heisenberg group or the Engel group [**3**].

The Heisenberg group. We consider the Riemannian frame of the left invariant vector fields $\{X_1, X_2, X_3\}$ in $\mathbf{R}^3$ given, for $p = (x, y, z) \in \mathbf{R}^3$, by

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \ , \ X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \ , \ X_3 = \frac{\partial}{\partial z} \, .$$

The Engel group. We consider the Riemannian frame of the vector fields $\{X_1, X_2, X_3, X_4\}$ in $\mathbf{R}^4$ given, for $p = (x, y, z, w) \in \mathbf{R}^4$, by

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} + \left(\frac{-xy}{12} - \frac{z}{2}\right)\frac{\partial}{\partial w} \ , \ X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2}{12}\frac{\partial}{\partial w}$$

$$X_3 = \frac{\partial}{\partial z} + \frac{x}{2}\frac{\partial}{\partial w} \ , \ X_4 = \frac{\partial}{\partial w} \, .$$

3 - The results

3.1 - *Parabolic jets and viscosity solutions to parabolic equations in vector fields*

Let $\mathcal{O} \subseteq \mathbf{R}^n$ be an open set and let $T > 0$. We define the set $\mathcal{O}_T := (0, T) \times \mathcal{O}$. If u is a function defined in $\mathcal{O}_T$ and $(t_o, x_o) \in \mathcal{O}_T$ we denote by $P^{2,+}u(t_o, x_o)$ the *parabolic* *superjet of u at* (t_o, x_o) defined as $P^{2,+}u(t_o,x_o) = \Big\{(a,\eta,\mathcal{X}) \in \mathbf{R}\times\mathbf{R}^n\times S(n)$ such that

$$u(t,\Theta_{x_o}(\xi)) \le u(t_o,x_o) + a(t-t_o) + \langle\eta,\xi\rangle + \frac{1}{2}\langle\mathcal{X}\xi,\xi\rangle + o(|t-t_o|+|\xi|^2)$$

as $t\to t_o$ and $\xi\to 0\Big\}$. The *parabolic subjet of u at* (t_o,x_o) is defined as

$$P^{2,-}u(t_o,x_o) = -P^{2,+}(-u)(t_o,x_o).$$

We also define the set theoretic closure of the superjet, denoted $\bar{P}^{2,+}u(t_o,x_o)$ by requiring that $(a,\eta,\mathcal{X})\in\bar{P}^{2,+}u(t_o,x_o)$ exactly when there is a sequence $(a_n,t_n,x_n,u(t_n,x_n),\eta_n,\mathcal{X}_n)\to(a,t_o,x_o,u(t_o,x_o),\eta,\mathcal{X})$ with the triple $(a_n,\ \eta_n,\ \mathcal{X}_n) \in P^{2,+}u(t_o,x_o)$. A similar definition holds for the closure of the subjet.

We use these jets to define viscosity subsolutions and supersolutions to equation (1.1).

Definition 3.1. Let $(t_o,x_o)\in\mathcal{O}_T$ be as above. An *Upper SemiContinuous* (*USC*) function u is a viscosity subsolution in $\mathcal{O}_T$ of (1.1) if for all $(t_o,x_o)\in\mathcal{O}_T$ whenever $(a,\eta,\mathcal{X})\in P^{2,+}u(t_o,x_o)$ we have

$$a + H(t_o,x_o,u(t_o,x_o),\eta,\mathcal{X}) \le f(t_o,x_o). \tag{3.1}$$

A *Lower SemiContinuous* (*LSC*) function u is a viscosity supersolution in $\mathcal{O}_T$ if for all $(t_o,x_o)\in\mathcal{O}_T$ whenever $(b,\nu,\mathcal{Y})\in P^{2,-}u(t_o,x_o)$ we have

$$b + H(t_o,x_o,u(t_o,x_o),\nu,\mathcal{Y}) \ge f(t_o,x_o). \tag{3.2}$$

A continuous function u is a viscosity solution in $\mathcal{O}_T$ of (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

We observe that the continuity of the function H allows (3.1) and (3.2) to hold when $(a,\eta,\mathcal{X})\in\bar{P}^{2,+}u(t_o,x_o)$ and $(b,\nu,\mathcal{Y})\in\bar{P}^{2,-}u(t_o,x_o)$.

An equivalent definition of sub(super)solution can be obtained from the following result. Let $\mathcal{O}$ and T as above. If u is a real function defined in $\mathcal{O}_T$ and $(t_o,x_o)\in\mathcal{O}_T$ we set

$$\mathcal{A}u(t_o,x_o) := \{\phi\in C^2(\mathcal{O}_T)\ :\ u(t,x)-\phi(t,x)\le u(t_o,x_o)-\phi(t_o,x_o) \text{ for } (t,x)\in\mathcal{O}_T\}$$

consisting of all test functions that “touch from above”. We define the set of all test functions that “touch from below”, denoted $\mathcal{B}u(t_o,x_o)$, by

$$\mathcal{B}u(t_o,x_o) := \{\phi\in C^2(\mathcal{O}_T)\ :\ u(t,x)-\phi(t,x)\ge u(t_o,x_o)-\phi(t_o,x_o) \text{ for } (t,x)\in\mathcal{O}_T\}.$$

We have [**5**, Lemma 2.1]

$$P^{2,+}u(t_o,x_o) = \{(\phi_t(t_o,x_o),\Re\phi(t_o,x_o),(\Re\phi)^*(t_o,x_o))\ :\ \phi\in\mathcal{A}u(t_o,x_o)\},$$
$$P^{2,-}u(t_o,x_o) = \{(\phi_t(t_o,x_o),\Re\phi(t_o,x_o),(\Re\phi)^*(t_o,x_o))\ :\ \phi\in\mathcal{B}u(t_o,x_o)\}.$$

3.2 - *Comparison principle and decay estimate*

Let $\Omega \subseteq \mathbf{R}^n$ be an open bounded set. Let us start by listing the usual hypothesis, besides H proper, for the existence and uniqueness results. Sometimes in the following we will consider H independent of the time t or dependent on t variable in a finite interval. We will formulate all the hypothesis for time dependent Hamiltonians, with $t \in \mathbf{R}$, whereas when dealing with Hamiltonians not depending on time, stationary variants have to be considered. We need obvious variants also when the time is restricted to a finite interval.

(H_2) $\quad H(t,y,r,\xi,X) - H(t,x,r,\eta,Y) \leq \omega_1(|x-y|, \|\xi-\eta\|, \|X-Y\|),$

whenever $(t,r) \in \mathbf{R}\times\mathbf{R}$, $x, y \in \Omega$, $\xi,\eta \in \mathbf{R}^n$, X, $Y \in S(n)$, where the function $\omega_1 : [0,+\infty] \to [0,+\infty]$ satisfies $\lim\limits_{z\to 0^+} \omega(z) = 0$.

(H_3) $\quad H(t,x,r,\eta,X) - H(t,x,s,\eta,X) \geq \gamma(r-s)$

where γ is a suitable positive constant, for any $r \geq s$, $(t,x,\eta,X) \in \mathbf{R}\times\Omega\times\mathbf{R}^n\times S(n)$.

Example. $H = -\nu(\Re^2 u)^* + f(x,u,\Re u)$, where $\nu > 0$, $f(x,u,\eta) - f(x,v,\eta) \geq \gamma(u-v)$ if $u \geq v$, for all x,η and a constant $\gamma > 0$, $f(x,u,\xi) - f(y,u,\eta) \leq \omega(|x-y|, \|\xi-\eta\|)$ for all u.

This case may be regarded as a first order Hamilton-Jacobi equation perturbed by an additional "viscosity term" $-\nu(\Re^2 u)^*$.

We now formulate the comparison principle for the problem

$$
(3.3)\quad
\begin{aligned}
&(E)\ u_t(t,x) + H(t,x,u(t,x),\Re u(t,x),(\Re^2 u)^*(t,x)) = f(t,x) \text{ in } (0,T)\times\Omega\\
&(BC)\ u(t,x) = h(t,x) \text{ in } [0,T)\times\partial\Omega\\
&(IC)\ u(0,x) = \varphi(x) \text{ in } \bar\Omega\,.
\end{aligned}
$$

Here $\varphi \in C(\bar\Omega)$ and $h \in C([0,T]\times\partial\Omega)$. We also adopt the convention in [8] that a viscosity subsolution $u(t,x)$ to problem (3.3) is a viscosity subsolution to (E) such that $u(t,x) \leq h(t,x)$ on $[0,T)\times\partial\Omega$ and $u(0,x) \leq \varphi(x)$ on $\bar\Omega$. Viscosity supersolutions and solutions are defined in an analogous way.

Lemma 3.1 (Comparison principle). *Let Ω be an open bounded subset of $\mathbf{R}^n$ and let $T > 0$. Assume that $H \in C\big([0,T]\times\bar\Omega\times\mathbf{R}\times\mathbf{R}^n\times S(n)\big)$ satisfies (H_1) and (H_2). If u is a viscosity subsolution and v is a viscosity supersolution to problem (3.3) then $u \leq v$ on $[0,T)\times\Omega$.*

Proof. See [**5**, Theorem 3.3] $\square$

Theorem 3.1. *Let Ω be an open bounded subset of $\mathbf{R}^n$ and let $T > 0$. Assume that $H \in C\big([0,T] \times \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S(n)\big)$ satisfies (H_1), (H_2) and (H_3). Let u, v be bounded viscosity subsolution of*

$$u_t + H(t,x,u,\Re u,(\Re^2 u)^*) = f^1(t,x)$$

in $(0,T) \times \Omega$, $u(t,x) = g^1(t,x)$ for $x \in \partial\Omega$ and $0 \le t < T$, respectively supersolution of

$$v_t + H(t,x,v,\Re v,(\Re^2 v)^*) = f^2(t,x)$$

in $(0,T) \times \Omega$, $v(t,x) = g^2(t,x)$ for $x \in \partial\Omega$ and $0 \le t < T$, where $f^1, f^2 \in C\big([0,T] \times \bar{\Omega}\big)$, g^1, $g^2 \in C([0,T] \times \partial\Omega)$. Then for all $t \in [0,T]$ we have

$$\begin{aligned} (3.4) \quad e^{\gamma t} \sup_{x \in \bar{\Omega}} (u(t,x) - v(t,x))_+ \le\ & \|(u(0,\cdot) - v(0,\cdot))_+\|_{L^\infty(\bar{\Omega})} \\ & + \int_0^t e^{\gamma s} \|\big(f^1(s,\cdot) - f^2(s,\cdot)\big)_+\|_{L^\infty(\bar{\Omega})}\, ds + \sup_{(s,x) \in [0,t] \times \partial\Omega} e^{\gamma s}(g^1 - g^2)_+(s,x). \end{aligned}$$

Proof. For $(t,x) \in [0,T] \times \bar{\Omega}$, set $w^1(t,x) := e^{\gamma t} u(t,x)$ and $w^2(t,x) := e^{\gamma t} v(t,x) + A(t)$, where

$$\begin{aligned} A(t) :=\ & \|(u(0,\cdot) - v(0,\cdot))_+\|_{L^\infty(\bar{\Omega})} + \sup_{(s,x) \in [0,t] \times \partial\Omega} e^{\gamma s}(g^1 - g^2)_+(s,x) \\ & + \int_0^t e^{\gamma s} \|\big(f^1(s,\cdot) - f^2(s,\cdot)\big)_+\|_{L^\infty(\bar{\Omega})}\, ds. \end{aligned}$$

It is not hard to see that w^1 and w^2 are a subsolution and resp. a supersolution of the problem

$$\begin{cases} \varphi_t(t,x) + \widetilde{H}(t,x,\varphi,\Re\varphi,(\Re^2\varphi)^*) = e^{\gamma t} f^1(t,x), & \text{in } (0,T) \times \Omega \\ \varphi(t,x) = e^{\gamma t} g^1(t,x), & \text{on } [0,T) \times \partial\Omega \end{cases}$$

where $\widetilde{H}(t,x,\varphi,\Re\varphi,(\Re^2\varphi)^*) = e^{\gamma t} H(t,x,e^{-\gamma t}\varphi, e^{-\gamma t}\Re\varphi, e^{-\gamma t}(\Re^2\varphi)^*) - \gamma\varphi$ and where, in virtue of (H_3), $\widetilde{H}$ satisfies (H_1), (H_2).

In fact, about w^1, let (t_o, x_o) be a maximum point for $w^1 - \psi$, where $\psi \in C^2((0,T) \times \Omega)$. We can suppose $\psi(t,x) = e^{\gamma t}\alpha(t,x)$ where $\alpha \in C^2((0,T) \times \Omega)$. So $w^1 - \psi = e^{\gamma t}(u - \alpha)$. Let $\psi_o(t,x) = \psi(t,x) + L$ where $L = w^1(t_o,x_o) - \psi(t_o,x_o)$. So (t_o,x_o) is a maximum point for $w^1 - \psi_o$ and $(w^1 - \psi_o)(t,x) \le (w^1 - \psi_o)(t_o,x_o) = 0$. Let $\alpha_o = \alpha + Le^{-\gamma t}$. Then $w^1 - \psi_o = (u - \alpha_o)e^{\gamma t}$ and $(u - \alpha_o)(t,x) = e^{-\gamma t}(w^1 - \psi_o)(t,x) \le 0 \le e^{-\gamma t_o}(w^1 - \psi_o)(t_o,x_o) = (u - \alpha_o)(t_o,x_o)$, that is (t_o,x_o) is a maximum point also for

$(u - \alpha_o)$. The definition of subsolution implies that

$$(\alpha_o)_t(t_o, x_o) + H(t_o, x_o, u(t_o, x_o), \Re\alpha_o(t_o, x_o), (\Re^2\alpha_o)^*(t_o, x_o)) \leq f^1(t_o, x_o)$$

that is, as $\alpha_o = e^{-\gamma t}\psi_o$,

$$\psi_t(t_o, x_o) + e^{\gamma t_o} H(t_o, x_o, e^{-\gamma t_o} w^1(t_o, x_o), e^{-\gamma t_o}\Re\psi(t_o, x_o), e^{-\gamma t_o}(\Re^2\psi)^*(t_o, x_o)) \\ -\gamma w^1(t_o, x_o) \leq e^{\gamma t_o} f^1(t_o, x_o).$$

With regard to w^2 take into account that, by virtue of (H_3), we have

$$e^{\gamma t_o} H(t_o, x_o, e^{-\gamma t_o}(w^2(t_o, x_o) - A(t_o)), e^{-\gamma t_o}\Re\psi(t_o, x_o), e^{-\gamma t_o}(\Re^2\psi)^*(t_o, x_o)) \\ \leq e^{\gamma t_o} H(t_o, x_o, e^{-\gamma t_o} w^2(t_o, x_o), e^{-\gamma t_o}\Re\psi(t_o, x_o), e^{-\gamma t_o}(\Re^2\psi)^*(t_o, x_o)) - \gamma A(t_o)$$

where (t_o, x_o) is a maximum point for $(w^2 - A) - \psi$, i.e. for $w^2 - \phi$, where $\varphi = \psi + A$ (which is C^1 in the variable t after regularization A^ε of A, that is after substitution of $\sup_{(s,x)\in[0,t]\times\partial\Omega} e^{\gamma s}(g^1 - g^2)_+(s, x)$ with a regularization $\sigma^\varepsilon(t)$, and C^2 in the variable x). Let us observe that $\Re\psi = \Re\varphi$, $(\Re^2\psi)^* = (\Re^2\varphi)^*$, $\psi_t = \varphi_t - A^\varepsilon_t$ and, as $A^\varepsilon_t \geq e^{\gamma t}\|\big(f^1(t,\cdot) - f^2(t,\cdot)\big)_+\|_{L^\infty(\bar\Omega)}$ (A is increasing , so we can suppose $\sigma^\varepsilon_t \geq 0$)

$$e^{\gamma t_o} f^2(t_o, x_o) + A^\varepsilon_t(t_o) \geq e^{\gamma t_o} f^1(t_o, x_o).$$

Moreover, for $(t, x) \in [0, T) \times \partial\Omega$

$$w^2(t, x) = e^{\gamma t} v(t, x) + A(t) \geq e^{\gamma t} g^2(t, x) + e^{\gamma t}(g^1 - g^2)(t, x) = e^{\gamma t} g^1(t, x).$$

It is also clear that $w^1(0, x) \leq w^2(0, x)$ on $\bar\Omega$. By the comparison principle, Lemma 3.1, we get

$$w^1(t, x) \leq w^2(t, x) \text{ on } [0, T] \times \Omega$$

and the conclusion follows. □

Corollary 3.1. *Let the hypothesis of Theorem* 3.1 *be in force. Then for all* $t \in [0, T]$

$$\sup_{x\in\bar\Omega}(u(t, x) - v(t, x)) \leq e^{-\gamma t}\|(u(0,\cdot) - v(0,\cdot))_+\|_{L^\infty(\bar\Omega)} \\ + \sup_{0\leq s\leq t}\int_s^t \sup_{x\in\bar\Omega}\big(f^1(\sigma, x) - f^2(\sigma, x)\big)\, d\sigma + \sup_{(s,x)\in[0,t]\times\partial\Omega}(g^1 - g^2)_+(s, x). \tag{3.5}$$

Proof. Let us fix $t \in [0,T]$. We denote by $h:[0,T] \to \mathbf{R}$ the function $h(\sigma) := \sup_{x\in\bar{\Omega}} \big(f^1(\sigma,x) - f^2(\sigma,x)\big)$. Consider the function $w:[0,T]\times\Omega \to \mathbf{R}$ given by

$$w(s,x) := v(s,x) + \int_0^s h(\sigma)\,d\sigma + \sup_{0\le\tau\le t}\left(-\int_0^\tau h(\sigma)\,d\sigma\right)$$
$$+ \sup_{(\alpha,x)\in[0,t]\times\partial\Omega} (g^1 - g^2)_+(\alpha,x) \ , \ \ (s,x)\in[0,t)\times\Omega.$$

It is easily seen that w is a bounded viscosity supersolution of $\partial_s + H = f^1$, $(s,x)\in(0,t)\times\Omega$, since $w \ge v$ on $(s,x)\in(0,t)\times\Omega$ and H is nondecreasing with respect to the third variable. Moreover we have on $[0,t)\times\partial\Omega$,

$$w(s,x) \ge g^2(s,x) + \left[\int_0^s h(\sigma)\,d\sigma + \sup_{0\le\tau\le t}\left(-\int_0^\tau h(\sigma)\,d\sigma\right)\right]$$
$$+ \sup_{(\alpha,x)\in[0,t]\times\partial\Omega} (g^1-g^2)_+(\alpha,x) \ge g^2(s,x)$$
$$+ \sup_{(\alpha,x)\in[0,t]\times\partial\Omega} (g^1-g^2)_+(\alpha,x) \ge g^1(s,x).$$

We deduce from Theorem 3.1 that for any $(s,x)\in(0,t)\times\Omega$

$$e^{\gamma s}(u(s,x) - w(s,x)) \le \sup_{x\in\bar{\Omega}}(u(0,x) - w(0,x))_+ \le \sup_{x\in\bar{\Omega}}(u(0,x) - v(0,x))_+$$

implying that

$$u(s,x) - v(s,x) \le e^{-\gamma s}\sup_{x\in\bar{\Omega}}(u(0,x)-v(0,x))_+ + \int_0^s h(\sigma)\,d\sigma$$
$$+ \sup_{0\le\tau\le t}\left(-\int_0^\tau h(\sigma)\,d\sigma\right) + \sup_{(\alpha,x)\in[0,t]\times\partial\Omega}(g^1-g^2)_+(\alpha,x).$$

In particular for $s = t$ one gets for any $x\in\Omega$

$$u(t,x) - v(t,x) \le e^{-\gamma t}\sup_{x\in\bar{\Omega}}(u(0,x)-v(0,x))_+$$
$$+ \sup_{0\le\tau\le t}\left(\int_\tau^t \sup_{x\in\bar{\Omega}}\big(f^1(\sigma,x)-f^2(\sigma,x)\big)\,d\sigma\right) + \sup_{(s,x)\in[0,t]\times\partial\Omega}(g^1-g^2)_+(s,x)\,.$$

□

Note that in the right hand side term of (3.5) we have now $\sup_{x\in\bar{\Omega}}\big(f^1(\sigma,x)-f^2(\sigma,x)\big)$ and not $\sup_{x\in\bar{\Omega}}\big(f^1(\sigma,x)-f^2(\sigma,x)\big)_+$.

The following Theorem 3.2 and Corollary 3.2 have an interest independently of the aim of the paper.

Theorem 3.2. *Let Ω be an open bounded subset of $\mathbf{R}^n$. Assume that $H \in C\big(\mathbf{R} \times \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S(n)\big)$ satisfies (H_1), (H_2) and (H_3). Let u, v be bounded subsolution of*

$$u_t + H(t, x, u, \Re u, (\Re^2 u)^*) = f^1(t, x)$$

in $\mathbf{R} \times \Omega$, $u(t, x) = g^1(t, x)$ for $(t, x) \in \mathbf{R} \times \partial\Omega$, respectively supersolution of

$$v_t + H(t, x, v, \Re v, (\Re^2 v)^*) = f^2(t, x)$$

in $\mathbf{R} \times \Omega$, $v(t, x) = g^2(t, x)$ for $(t, x) \in \mathbf{R} \times \partial\Omega$, where f^1, $f^2 \in BUC\big(\mathbf{R} \times \bar{\Omega}\big)$ and g^1, $g^2 \in BUC(\mathbf{R} \times \partial\Omega)$. Then one has for all $t \in \mathbf{R}$

$$\sup_{x \in \bar{\Omega}} (u(t, x) - v(t, x)) \tag{3.6}$$
$$\le e^{-\gamma t} \int_{-\infty}^{t} e^{\gamma s} \big\| \big(f^1(s, \cdot) - f^2(s, \cdot)\big)_+ \big\|_{L^\infty(\bar{\Omega})} \, ds + e^{-\gamma t} \sup_{(s,x) \in [-\infty, t] \times \partial\Omega} e^{\gamma s} (g^1 - g^2)_+(s, x)$$

Proof. Take t_o, $t \in \mathbf{R}$, $t_o \le t$ and using the proof of Theorem 3.1 write for all $x \in \bar{\Omega}$

$$u(t, x) - v(t, x) \le e^{-\gamma(t - t_o)} \cdot \big(\|u\|_\infty + \|v\|_\infty\big)$$
$$+ e^{-\gamma t} \int_{t_o}^{t} e^{\gamma s} \big\| \big(f^1(s, \cdot) - f^2(s, \cdot)\big)_+ \big\|_{L^\infty(\bar{\Omega})} \, ds + e^{-\gamma t} \sup_{(s,x) \in [t_o, t] \times \partial\Omega} e^{\gamma s} (g^1 - g^2)_+(s, x).$$

The conclusion follows by passing $t_o \to -\infty$. □

Corollary 3.2. *Let the hypothesis of Theorem 3.2 be in force. Then for every $t \in \mathbf{R}$*

$$\sup_{x \in \bar{\Omega}} (u(t, x) - v(t, x)) \tag{3.7}$$
$$\le \sup_{s \le t} \int_{s}^{t} \sup_{x \in \bar{\Omega}} \big(f^1(\sigma, x) - f^2(\sigma, x)\big) \, d\sigma + \sup_{(s,x) \in [-\infty, t] \times \partial\Omega} (g^1 - g^2)_+(s, x).$$

Proof. Repeating the proof of Corollary 3.1 in $[t_o, t] \times \Omega$ we obtain

$$u(t, x) - v(t, x) \le e^{-\gamma(t - t_o)} \sup_{x \in \bar{\Omega}} (u(t_o, x) - v(t_o, x))_+$$
$$+ \sup_{t_o \le \tau \le t} \left(\int_{\tau}^{t} \sup_{x \in \bar{\Omega}} \big(f^1(\sigma, x) - f^2(\sigma, x)\big) \, d\sigma \right) + \sup_{(s,x) \in [t_o, t] \times \partial\Omega} (g^1 - g^2)_+(s, x).$$

The conclusion follows passing to the limit $t_o \to -\infty$. □

3.3 - *Existence*

We are now in a position to state the existence of almost periodic (periodic) viscosity solutions of the Dirichlet problem

$$
\begin{gathered}
u_t + H(x, u, \Re(u), (\Re^2 u)^*) = f(t) \text{ in } \mathbf{R} \times \Omega \\
u(t, x) = 0 \text{ in } \mathbf{R} \times \partial\Omega
\end{gathered}
\tag{3.8}
$$

where H is independent of t.

Theorem 3.3. *Let Ω be a bounded open subset of $\mathbf{R}^n$. Assume that $H \in C(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S(n))$ satisfies (H_1), (H_2), (H_3). Assume that $f : \mathbf{R} \to \mathbf{R}$ is a continuous almost periodic (periodic) function. If u is a bounded viscosity solution of* (3.8) *in $BUC([a, b] \times \bar{\Omega})$ for any $a \le b$, then it is a.p. (periodic) in t uniformly with respect to x and it is in $BUC(\mathbf{R} \times \bar{\Omega})$.*

Proof. Let u be a bounded viscosity solution of (3.8) in $BUC([a, b] \times \bar{\Omega})$ for any $a \le b$. As f is a.p., then for all $\varepsilon > 0$ there exists $l(\varepsilon\gamma)$ such that any interval of length $l(\varepsilon\gamma)$ contains a number τ which is an $\varepsilon\gamma$ almost period for f. We will show that any interval of length $l(\varepsilon\gamma)$ contains a number τ which is an ε almost period for $u(\cdot, x)$, for all $x \in \Omega$.

Consider an interval of length $l(\varepsilon\gamma)$, fix τ an $\varepsilon\gamma$ almost period of f and fix $\bar{t} \in \mathbf{R}$. Observe that for any integer $n \ge 0$ the function u solves in the viscosity sense

$$
\begin{gathered}
u_t + H(x, u, \Re(u), (\Re^2 u)^*) = f(t) \text{ in } [-n, +\infty) \times \Omega \\
u(t, x) = 0 \text{ in } [-n, +\infty) \times \partial\Omega
\end{gathered}
$$

and the function $u_\tau(t, x) = u(t + \tau, x)$ solves in the viscosity sense

$$
\begin{gathered}
u_{\tau,t} + H(x, u_\tau, \Re(u_\tau), (\Re^2 u_\tau)^*) = f(t + \tau) \text{ in } [-n - \tau, +\infty) \times \Omega \\
u_\tau(t, x) = 0 \text{ in } [-n - \tau, +\infty) \times \partial\Omega.
\end{gathered}
$$

If M is a positive constant such that $|u| \le M$ in $\mathbf{R} \times \Omega$, then, by Corollary 3.1 we have for all $t \ge t_n = max\{-n, -n - \tau\}$, $x \in \Omega$

$$
e^{\gamma t} |u(t, x) - u_\tau(t, x)| \le e^{\gamma t_n} \left(\|u\|_\infty + \|u_\tau\|_\infty \right) + \int_{t_n}^{t} e^{\gamma s} |f(\sigma + \tau) - f(\sigma)| \, d\sigma.
$$

In particular, for $t = \bar{t}$ and n large enough we deduce

$$
|u(\bar{t}, x) - u(\bar{t} + \tau, x)| \le 2M e^{-\gamma(\bar{t} - t_n)} + e^{-\gamma\bar{t}} \int_{t_n}^{\bar{t}} e^{\gamma\sigma} \gamma\varepsilon \, d\sigma \le 2M e^{-\gamma(\bar{t} - t_n)} + \varepsilon.
$$

Letting $n \to +\infty$ we have $t_n \to -\infty$ and therefore $|u(\bar{t},x) - u(\bar{t}+\tau,x)| \leq \varepsilon$ for all $(\bar{t},x) \in \mathbf{R} \times \Omega$. Since we already know that $u \in BUC([a,b] \times \bar{\Omega})$ for every $a \leq b$, by time almost periodicity we deduce also that $u \in BUC(\mathbf{R} \times \bar{\Omega})$.

In case of f periodic of period $T > 0$ the same calculations give for any fixed $\bar{t} \in \mathbf{R}$ if the integer n is large enough

$$|u(\bar{t},x) - u(\bar{t}+T,x)| \leq 2Me^{-\gamma(\bar{t}+nT)}$$

for any $x \in \Omega$, as $u(t,x)$ and $u_T(t,x) := u(t+T,x)$ solve the same problem (3.8) in $(-nT, +\infty) \times \Omega$, and the result follows again by passing $n \to +\infty$. □

3.4 - *About periodic solutions*

Lemma 3.2. *Assume $H \in C\big([0,T] \times \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S(n)\big)$ and let $u \in C((0,T] \times \bar{\Omega})$ be a viscosity subsolution (resp. supersolution) of*

$$u_t + H(t,x,u,\Re u,(\Re^2 u)^*) = 0 \tag{3.9}$$

in $(0,T) \times \Omega$. Then u is a viscosity subsolution (resp. supersolution) of (3.9) in $(0,T] \times \Omega$.

Proof. The proof is the same as [**6**, Lemma 1]. □

Proposition 3.1. *Assume that $H \in C\big(\mathbf{R} \times \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S(n)\big)$ and $u \in C(\mathbf{R} \times \bar{\Omega})$ are T periodic. Let u be a viscosity subsolution (resp. supersolution) of (3.9) in $(0,T) \times \Omega$. Then u is a viscosity subsolution (resp. supersolution) of (3.9) in $\mathbf{R} \times \Omega$.*

Proof. The proof is the same as [**6**, Proposition 2]. □

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