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Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review

Abstract. Kinetic and hyperbolic equations contain small scales (mean free path/time, Debye length, relaxation or reaction time, etc.) that lead to various different asymptotic regimes, in which the classical numerical approximations become prohibitively expensive. Asymptotic-preserving (AP) schemes are schemes that are efficient in these asymptotic regimes. The designing principle of AP schemes is to preserve, *at the discrete level*, the asymptotic limit that drives one (usually the microscopic) equation to its asymptotic (macroscopic) equation. An AP scheme is based on solving the microscopic equation, instead of using a multiphysics approach that couples different physical laws at different scales. When the small scale is not numerically resolved, an AP scheme *automatically* becomes a macroscopic solver for the limiting equation. The AP methodology offers simple, robust and efficient computational methods for a large class of multiscale kinetic, hyperbolic and other physical problems. This paper reviews the basic concept, designing principle and some representative AP schemes.

Keywords. Kinetic equations, hyperbolic equations with relaxation, fluid dynamic limit, asymptotic-preserving schemes, quasi-neutral limit.

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Contents

1 - Introduction

Kinetic and hyperbolic equations often contain small parameters. In the former case, the small parameter is usually the mean free path or time, the average distance or time between two collisions of particles. The latter may contain small relaxation or reaction time. Asymptotic expansions on these parameters usually yield asymptotic equations that describe the averaged, effective, or macroscopic equations. These asymptotic regimes pose tremendous computational challenges since one needs to numerically resolve these small scales which can be prohibitively expensive.

If the asymptotic or macroscopic equations are uniformly valid in the entire domain of interest, it is much more efficient just to solve them. However, there are many problems where the macroscopic models break down in part of the domain, thus the microscopic models are needed, at least locally. We mention a few examples. In the

space shuttle reentry problem, the vehicle will pass from free streaming, rarefied gas (described by the Boltzmann equation), transition to the macroscopic hydrodynamic (described by the Euler or Navier-Stokes equations) regimes [**111**]. For hypersonic flows (Mach number larger than 1.4), it is well-known that the Navier-Stokes equations do not give accurate shock profiles, thus one needs to use the Boltzmann equation or some transition models around shock (and boundary) layers [**1, 42**]. In plasma physics, one has to match the plasma and sheath where the quasineutral (which allows macroscopic modeling) and non-quasineutral (which needs kinetic modeling) models need to be coupled [**43**]. Thus a multiscale and multiphysics approach, that hybridizes the microscopic and the macroscopic models in a domain-decomposition framework, becomes necessary, see for examples [**15, 29, 31, 48, 57, 76, 80, 115, 90, 116**] for multiscale kinetic problems, and [**38, 39, 77**] for broader areas of multiscale modeling and simulation.

A main difficulty of these multiscale and multi-physics domain decomposition type method is that one has to couple models at different scales through an interface or buffer zone where one has to match two different models. This matching requires the transform of data from one-scale to another, which is often difficult and most of the time has no unique solution. The coupling locations may also be difficult to determine in a dynamic problem.

This paper surveys another approach for multiscale problems–the *asymptotic-preserving* (*AP*) *scheme*. This approach has its origin in capturing steady-state solution for neutron transport in the diffusive regime [**86, 87**]. Since the 90s of last century, the AP schemes have been developed for a wide range of time-dependent kinetic and hyperbolic equations. The basic idea is to develop numerical methods that *preserve the asymptotic limits from the microscopic to the macroscopic models, in the discrete setting*. Comparing with multi-physics domain decomposition type methods, the AP schemes solve one set of equations–the microscopic ones–thus avoid the coupling of different models. They become robust macroscopic solvers *automatically* when, in the asymptotic regimes, the small time or spatial scales are *not resolved* numerically.

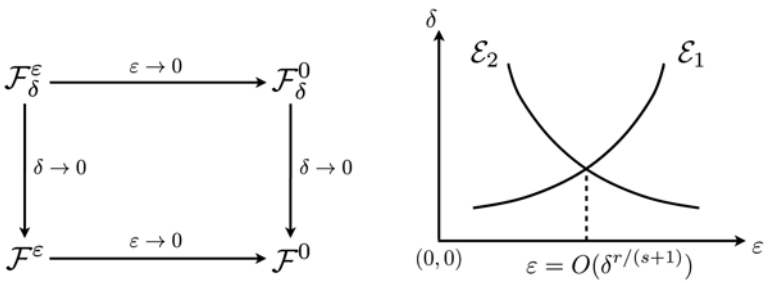

Fig. 1. Left: illustration of AP schemes; right: illustration of uniform convergence of AP schemes.

The idea of AP can be illustrated by the left figure in Fig. 1. Assume one starts with a microscopic model $\mathcal{F}^\varepsilon$ which depends on parameter ε that characterizes the small scale. As $\varepsilon \to 0$ the model is approximated by a macroscopic model $\mathcal{F}^0$ which is independent of ε. Denote the numerical discretization of $\mathcal{F}^\varepsilon$ by $\mathcal{F}^\varepsilon_\delta$, where δ is the numerical parameter (such as mesh size and/or time step). The asymptotic limit of $\mathcal{F}^\varepsilon_\delta$, as $\varepsilon \to 0$ (with δ fixed), if exsits, is denoted by $\mathcal{F}^0_\delta$. *If $\mathcal{F}^0_\delta$ is a good (consistent and stable) approximation of $\mathcal{F}^0$, then the scheme $\mathcal{F}^\varepsilon_\delta$ is called AP.*

Error estimates on an AP scheme can follow the following argument. Typically,

$$\|\mathcal{F}^\varepsilon - \mathcal{F}^0\| = O(\varepsilon)\,. \tag{1.1}$$

Assuming $\mathcal{F}^\varepsilon_\delta$ is an r-th order approximation to $\mathcal{F}^\varepsilon$ for fixed ε. Due to the presence of the small parameter ε, a classical numerical analysis typically gives the following error estimate

$$\mathcal{E}_1 = \|\mathcal{F}^\varepsilon_\delta - \mathcal{F}^\varepsilon\| = O(\delta^r/\varepsilon^s)\,, \quad 1 \le s \le r\,. \tag{1.2}$$

Assume the scheme is AP. This may give the following estimate

$$\|\mathcal{F}^\varepsilon_\delta - \mathcal{F}^0_\delta\| = O(\varepsilon) \qquad \text{uniformly in} \quad \delta\,, \tag{1.3}$$

and

$$\|\mathcal{F}^0_\delta - \mathcal{F}^0\| = O(\delta^r)\,. \tag{1.4}$$

Clearly, if one adds up the errors in (1.1), (1.3) and (1.4), by the triangle inequality, one comes up with the following error estimate:

$$\mathcal{E}_2 = \|\mathcal{F}^\varepsilon_\delta - \mathcal{F}^\varepsilon\| = O(\varepsilon + \delta^r)\,. \tag{1.5}$$

By comparing the two error estimates (1.2) and (1.5), as shown by the right figure in Fig. 1,

$$\|\mathcal{F}^\varepsilon_\delta - \mathcal{F}^\varepsilon\| = \min(\mathcal{E}_1, \mathcal{E}_2)\,,$$

which has an upper bound around $\varepsilon = O(\delta^{r/(s+1)})$. This gives

$$\|\mathcal{F}^\varepsilon_\delta - \mathcal{F}^\varepsilon\| = O(\delta^{r/(s+1)})\,, \qquad \text{uniformly in} \quad \varepsilon\,. \tag{1.6}$$

This argument shows that an AP scheme is *convergent uniformly in* ε. Indeed, if one resolves ε by δ (with $\delta = o(\varepsilon^{s/r})$) one gets a good approximation to the microscopic model $\mathcal{F}^\varepsilon$, as shown by (1.2). If ε is not resolved by δ then one obtains a good approximation to the macroscopic model $\mathcal{F}^0$. This transition is done automatically by the code.

This uniform convergence argument for AP schemes was first used, rigorously, by Golse, Jin and Levermore on numerical methods for linear transport equation in

the diffusive regime [**47**]. The convergence rate in (1.6) may not be sharp. This formal argument applies to any AP schemes, although a rigorous proof will be problem dependent, based on the regularity of the solution and the specific numerical scheme $\mathcal{F}_\delta^\varepsilon$ being used. See [**47, 72**].

The design of an AP scheme needs special care for both time and spatial discretizations. Often the time discretization is more crucial. In the case of kinetic equations passing to the limit of hydrodynamic equations when the Knudsen number goes to 0, as summarized by Jin in [**62**], a scheme is AP if

- it preserves the discrete analogy of the Chapman-Enskog expansion, namely, it is a suitable scheme for the kinetic equation, yet, when holding the mesh size and time step fixed and letting the Knudsen number go to zero, the scheme becomes a suitable scheme for the limiting fluid dynamic Euler equations;
- implicit collision terms can be implemented efficiently.

A typical AP scheme allows time step much larger than ε, thus some implicity is inevitable in order to overcome the stiffness [**45**]. However, an implicit Boltzmann collision term is computational daunting due to the nonlinear, non-local and high dimensional nature of the collision term. Thus one tries to find efficient ways to invert the implicit collision term. Other than the AP property, an efficient implicit solver is also one of the main ingredients in an AP scheme.

In this notes, we will review the basic theory and methods, and give several representative examples of AP schemes. Most examples are kinetic and hyperbolic equations, although the concept also applies to many other equations that admit asymptotic structures.

2 - Hyperbolic systems with stiff relaxations

We begin with some typical physical examples of hyperbolic systems with stiff relaxations.

2.1 - *The shallow-water equations*

The shallow-water equations describe the vertical depth h and mean velocity v of a river with a bottom topography and friction [**118**]

$$\partial_t h + \partial_x(hv) = 0\,, \tag{2.1}$$

$$\partial_t(hv) + \partial_x\left(hv^2 + \frac{1}{2}gh^2\right) = gSh - C_f|v|v\,, \tag{2.2}$$

where $x \in \mathbb{R}$ is the position, $t > 0$ is time, g is the gravitational constant, S the slope of the river bottom, and C_f the friction coefficient. The local equilibrium limit is identified with the long space-time behavior (upon rescaling $x \mapsto \varepsilon x, t \mapsto \varepsilon t$) of solutions of the system that is associated with the description of flooding.

Assuming $v > 0$. At local equilibrium, (2.2) gives

$$v = \sqrt{\frac{gSh}{C_f}}$$

which, when submitted to (2.1), yields the "macroscopic" equation

$$\partial_t h + \partial_x \sqrt{\frac{gS}{C_f}} h^{3/2} = 0\,. \tag{2.3}$$

One can even derive the next order correction to (2.3), in the spirit of Chapman-Enskog expansion (see Section 2.3), to get [**60**]:

$$\partial_t h + \partial_x \sqrt{\frac{gs}{C_f}} h^{3/2} = \varepsilon\, \partial_x \left[\frac{1}{2} \sqrt{\frac{gS}{C_f}} \left(\frac{1}{S} - \frac{1}{4C_f} \right) h^{3/2} \partial_x h \right]. \tag{2.4}$$

This equation is dissipative as long as the following (*subcharacteristic*) condition is satisfied [**60**]:

$$s < 4C_f\,.$$

2.2 - *Gas in thermo-non-equilibrium*

The compressible Euler equations in gas dynamics are

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_x \cdot \rho\, u \;=\; 0, \\ \dfrac{\partial \rho\, u}{\partial t} + \nabla_x \cdot (\rho\, u \otimes u + p\, \mathrm{I}) \;=\; 0, \\ \dfrac{\partial E}{\partial t} + \nabla_x \cdot ((E + p)\, u) \;=\; 0, \end{cases} \tag{2.5}$$

where $x \in \mathbb{R}^d$ is the position, $t > 0$ the time, ρ the density, $u \in \mathbb{R}^d$ velocity, p pressure, E total eneregy, while I is the $d \times d$ identity matrix. In the local thermodynamic equilibrium, the system is closed by the constitutive relation

$$p = p(\rho, e)\,, \qquad E = \frac{1}{2}\rho |u|^2 + \rho e\,,$$

where e is the specific internal energy. When the temperature varies over a wide range, the gas may not be in local thermodynamic equilibrium, and the pressure p should then be regarded as a function of only a part ε of the specific internal energy, while another part q is governed by a rate equation [**117, 24**]:

$$\begin{aligned} &\partial_t \rho q + \nabla \cdot (\rho q u) = \frac{Q(\rho,\varepsilon) - q}{\tau(\rho,\varepsilon)}\,, \\ &p = p(\rho,\varepsilon)\,, \qquad E = \frac{1}{2}\,\rho|u|^2 + \rho\varepsilon + \rho q\,, \end{aligned} \tag{2.6}$$

where $\tau(\rho,\varepsilon)$ is the relaxation time.

When $\tau \to 0$, (2.6) implies the local equilibrium

$$q = Q(\rho,\varepsilon)$$

which leads to

$$E = \frac{1}{2}\rho\,(|u|^2 + e) = \frac{1}{2}\,\rho\,(|u|^2 + \varepsilon + Q(\rho,\varepsilon))\,. \tag{2.7}$$

Thus the internal enegry

$$e = \varepsilon + Q(\rho,\varepsilon)$$

which can be inverted to give $\varepsilon = \tilde{\varepsilon}(\rho, e)$. Finally one obtains the equation of state

$$p = \tilde{p}(\rho, e) = p(\rho, \tilde{\varepsilon}(\rho, e))\,. \tag{2.8}$$

Thus the Euler equations (2.5) are closed via (2.8) and (2.7).

2.3 - *An illustrative example: The Jin-Xin model*

We begin the introduction of AP schemes with the following Jin-Xin semilinear hyperbolic relaxation system [**74**]

$$\begin{cases} \partial_t u + \partial_x v = 0\,, \\ \partial_t v + a\,\partial_x u = \dfrac{1}{\varepsilon}(f(u) - v)\,. \end{cases} \tag{2.9}$$

Here ε is the relaxation time.

As $\varepsilon \to 0$, the second equation yields the local equilibrium,

$$v = f(u)\,,$$

which, when submitted to the first equation, gives the "macroscopic" equation

$$u_t + \partial_x f(u) = 0\,. \tag{2.10}$$

This is the so-called zero relaxation limit.

Via the Chapman-Enskog expansion [**21**], one can derive the first order approximation:

$$
\begin{aligned}
v &= f(u) - \varepsilon(\partial_t v + a\partial_x u) \\
&= f(u) - \varepsilon(\partial_t f(u) + a\partial_x u) + O(\varepsilon^2) \\
&= f(u) - \varepsilon(f'(u)\partial_t u + a\partial_x u) + O(\varepsilon^2) \\
&= f(u) - \varepsilon(-f'(u)\partial_x v + a\partial_x u) + O(\varepsilon^2) \\
&= f(u) - \varepsilon(-f'(u)\partial_x f(u) + a\partial_x u) + O(\varepsilon^2) \\
&= f(u) - \varepsilon[(a - f'(u)^2)\partial_x u] + O(\varepsilon^2)\,.
\end{aligned} \tag{2.11}
$$

When applying (2.11) to the first equation of (2.9) and dropping the $O(\varepsilon^2)$ term, one obtains

$$
\partial_t u + \partial_x f(u) = \varepsilon[(a - f'(u)^2)\partial_x u]\,. \tag{2.12}
$$

For (2.12) to be dissipative, we need the following stability condition (the so-called *subcharacteristic condition* [**97**]):

$$
\sqrt{a} \geq |f'(u)| \qquad \text{for all} \quad u\,.
$$

For mathematical justification of this limit see for examples [**101, 102,** 8].

We are concerned with the numerical issue. First, the relaxation term stiff, which should be treated implicitly in order to allow $\Delta t \gg \varepsilon$. Since the convection term is not stiff (the characteristic speeds are $\pm\sqrt{a} = O(1)$), one can use explicit schemes for the convection terms. Here we use the upwind scheme $\left(\text{applied to the Riemann invariants } u \pm \frac{1}{\sqrt{a}}\, v\right)$ for the convection. For time approximations there are two approaches.

• Splitting scheme. One can solve (2.9) in two steps. First one solves the relaxation step implicitly for one time step to get (u^*, v^*) from (u^n, v^n):

$$
\begin{cases}
\dfrac{u^* - u^n}{\Delta t} = 0\,, \\[2ex]
\dfrac{v^* - v^n}{\Delta t} = \dfrac{1}{\varepsilon}(f(u^*) - v^*)\,;
\end{cases} \tag{2.13}
$$

and then the convection step explicitly for one time step to get (u^{n+1}, v^{n+1}) from the initial data (u^*, v^*):

$$
\begin{cases}
\dfrac{u^{n+1} - u^*}{\Delta t} + \dfrac{v^*_{j+1} - v^*_{j-1}}{2\Delta x} = \dfrac{\sqrt{a}\Delta x}{2}\dfrac{u^*_{j+1} - 2u^*_j + u^*_{j-1}}{(\Delta x)^2}, \\[2ex]
\dfrac{v^{n+1} - v^*}{\Delta t} + \dfrac{u^*_{j+1} - u^*_{j-1}}{2\Delta x} = \dfrac{\sqrt{a}\Delta x}{2}\dfrac{v^*_{j+1} - 2v^*_j + v^*_{j-1}}{(\Delta x)^2}.
\end{cases} \tag{2.14}
$$

- Unsplit scheme. One can use an implicit-explicit (IMEX) scheme:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\Delta t}+\dfrac{v^n_{j+1}-v^n_{j-1}}{2\Delta x}-\dfrac{\sqrt{a}\Delta x}{2}\dfrac{u^n_{j+1}-2u^n_j+u^n_{j-1}}{(\Delta x)^2}=0, \\ \dfrac{v^{n+1}-v^n}{\Delta t}+\dfrac{u^n_{j+1}-u^n_{j-1}}{2\Delta x}-\dfrac{\sqrt{a}\Delta x}{2}\dfrac{v^n_{j+1}-2v^n_j+v^n_{j-1}}{(\Delta x)^2} \\ \qquad\qquad =\dfrac{1}{\varepsilon}[v^{n+1}-f(u^{n+1})]\,. \end{cases} \tag{2.15}$$

There are two issues here. First is the time discretization. Notice that both methods treat the relaxation term, which is nonlinear in u, implicitly. Normally one would need to invert a nonlinear operator to find u, v at the unknown time level. However, in the splitting scheme, $u^* = u^n$, thus in the second equation of (2.13) $f(u^*) = f(u^n)$ which is known. Hence one just needs to find v^* in the second equation of (2.13), which is a linear relation that can be solved by hand. For the unsplit scheme, one can also obtain u^{n+1} from the first equation of (2.15) explicitly and then substitute it into $f(u^{n+1})$ and the remaining equation for v^{n+1} is again linear and can be solved analytically. Thus both schemes, although implicit, can be *implemented explicitly*.

The next issue is the AP-property. To study this, let $\varepsilon \to 0$ and keep $\Delta x, \Delta t$ fixed, and assume all quantities are of $O(1)$ except ε. For the splitting scheme the first step of (2.13) implies

$$u^* = u^n, \qquad v^* = f(u^*) = f(u^n)$$

which when substituted to the first equation of (2.14) gives

$$\frac{u^{n+1}-u^n}{\Delta t}+\frac{f(u^n_{j+1})-f(u^n_{j-1})}{2\Delta x}=\frac{\sqrt{a}\Delta x}{2}\,\frac{u^n_{j+1}-2u^n_j+u^n_{j-1}}{(\Delta x)^2}\,. \tag{2.16}$$

This is just the Lax-Friedrichs or Rusanov scheme for the limiting "macroscopic" equation (2.10). Thus the scheme is AP, in both space and time.

For the unsplit scheme, if $\varepsilon \to 0$, the second equation of (2.15) gives

$$v^{n+1} = f(u^{n+1}) \qquad \text{for any} \quad n$$

which when submitted to the first equation of (2.15) also yields (2.16), hence the AP property is satisfied.

Numerical approximations of hyperbolic systems with stiff relaxations were first studied in [**66, 61**]). It was also shown in [**74**] that if one uses a second order upwind scheme in (2.14) (with a slope limiter to suppress numerical oscillations [**91**]) then the limiting scheme, when $\varepsilon \to 0$, becomes a total-variation-diminishing (TVD) scheme for (2.10). Thus the second order spatial discretization combined with any of the aforementioned two time-discretizations will also be AP.

A necessary condition for a scheme for (2.9) to be AP is that the solution must be driven to the local equilibrium when $\varepsilon \to 0$, namely,

$$v^n - f(u^n) = O(\varepsilon)\,, \quad \text{for} \quad n \geq 1 \tag{2.17}$$

for *any* initial data v^0. Namely, the numerical solution projects *any* data into the local equilibrium, with an accuracy of $O(\varepsilon)$, in *one step*. This can usually be achieved by a backward Euler or any imlicit L-stable ODE solvers for the collision term [**61**]. Such AP schemes will be referred to as *strongly AP*.

2.4 - *Nonlinear hyperbolic systems with relaxation*

We consider the more general nonlinear hyperbolic system with relaxation (without loss of generality we use the following 2×2 system):

$$\begin{cases} \dfrac{\partial u}{\partial t} + f_1(u,v)_x = 0\,, \\ \dfrac{\partial v}{\partial t} + f_2(u,v)_x = \dfrac{1}{\varepsilon} R(u,v)\,. \end{cases} \tag{2.18}$$

The relaxation term $R : \mathbb{R}^2 \mapsto \mathbb{R}$ is dissipative in the sense of [**22**]:

$$\partial_v R \leq 0\,. \tag{2.19}$$

It possesses a unique local equilibrium, namely, $R(u,v) = 0$ implies $v = g(u)$. At the local equilibrium, one has the macroscopic system

$$u_t + f_1(u, g(u))_x = 0\,.$$

Here we consider the case in which inverting an implicit R is not as simple as in the case of the Jin-Xin relaxation model. A penalty method for the Botzmann collision operator using the BGK operator was introduced by Filbet and Jin (see [**41**]). Here we apply the method, by adding and subtracting the right hand side of (2.18) by a simple relaxation term $\dfrac{\beta}{\varepsilon}(v - g(u))$ and then use the following temporal approximation to (2.18):

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + f_1(u^n, v^n)_x = 0\,, \\ \dfrac{v^{n+1} - v^n}{\Delta t} + f_2(u^n, v^n)_x \\ \qquad\qquad = \dfrac{1}{\varepsilon}\left[R(u^n, v^n) + \beta(v^n - g(u^n))\right] - \dfrac{\beta}{\varepsilon}[v^{n+1} - g(u^{n+1})]\,. \end{cases} \tag{2.20}$$

Here β is a parameter to be determined. Assume all functions are smooth. Some simple mathematical manipulations on (2.20) give

$$v^{n+1} - g(u^{n+1}) = -\left[f_2(u^n, v^n)_x + (g(u^{n+1}) - g(u^n))/\Delta t\right] \frac{\varepsilon \Delta t}{\varepsilon + \beta \Delta t} + \frac{1 + \frac{\Delta t}{\varepsilon}\left[\beta + \frac{R(u^n, v^n)}{v^n - g(u^n)}\right]}{1 + \beta \frac{\Delta t}{\varepsilon}} (v^n - g(u^n)) .$$

Note that

$$\frac{R(u^n, v^n)}{v^n - g(u^n)} = \frac{R(u^n, v^n) - R(u^n, g(u^n))}{v^n - g(u^n)} = \partial_v R(u^n, w^n) < 0 \quad \text{for some} \quad w^n,$$

thus if

$$\beta > \frac{1}{2} \sup |\partial_v R| , \tag{2.21}$$

there exists a constant C, and $0 < r < 1$ independent of ε and Δt, such that

$$|v^{n+1} - g(u^{n+1})| \le C \frac{\varepsilon \Delta t}{\varepsilon + \beta \Delta t} + r |v^n - g(u^n)| . \tag{2.22}$$

From here it is easy to see that

$$|v^n - g(u^n)| \le \frac{C}{1 - r} \frac{\varepsilon \Delta t}{\varepsilon + \beta \Delta t} + r^n |v^0 - g(u^0)| . \tag{2.23}$$

This clearly gives

$$|v^n - g(u^n)| \le \frac{C}{(1 - r)\beta} \varepsilon + r^n |v^0 - g(u^0)| \tag{2.24}$$

in which the first term on the right hand side is $O(\varepsilon)$ independent of Δt. For any $\varepsilon \ll 1$, there exists an $N_\varepsilon \ge 1$ such that

$$r^n |v^0 - g(u^0)| \le \varepsilon ,$$

provided that the stability condition (2.21) is satisfied. Thus,

$$v^n = g(u^n) + O(\varepsilon) , \quad \text{for} \quad n \ge N_\varepsilon . \tag{2.25}$$

Thus the solution will be projected to the local equilibrium beyond an initial layer. This is a weaker condition compared to (2.17), and will be referred to as *relaxed AP*.

Since β in (2.20) contributes to the truncation error of the scheme, it should be chosen to be the smallest value satisfying (2.21). It can be made time, space or even u-dependent in order to minimize the numerical error.

We point out that the penalty by the relaxation operator is not merely for stability. For example, consider the linear penalty method

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\Delta t}+f_1(u^n,v^n)_x=0\,, \\ \dfrac{v^{n+1}-v^n}{\Delta t}+f_2(u^n,v^n)_x=\dfrac{1}{\varepsilon}[R(u^n,v^n)+\beta v^n]-\dfrac{\beta}{\varepsilon}v^{n+1}\,. \end{cases} \tag{2.26}$$

A simple mathematical manipulation on (2.26) gives

$$\begin{aligned} v^{n+1}-g(u^{n+1}) =& -f_2(u^n,v^n)_x\frac{\varepsilon\Delta t}{\varepsilon+\beta\Delta t}-[g(u^{n+1})-g(u^n)] \\ &+\frac{1+\dfrac{\Delta t}{\varepsilon}\left[\mu+\dfrac{R(u^n,v^n)}{v^n-g(u^n)}\right]}{1+\mu\dfrac{\Delta t}{\varepsilon}}\,(v^n-g(u^n))\,. \end{aligned} \tag{2.27}$$

The first two terms on the right hand side of (2.27) can only be bounded by $C(\varepsilon+\Delta t)$, while the third term, under the condition (2.21), is similar to the second term on the right hand side of (2.22). In conclusion, corresponding to (2.24), here one can only obtain

$$|v^n-g(u^n)|\le C(\varepsilon+\Delta t)+r^n|v^0-g(u^0)| \tag{2.28}$$

which, if $\Delta t\gg\varepsilon$, gives

$$v^n-g(u^n)=O(\Delta t)\,,\quad n\ge N_\varepsilon\,. \tag{2.29}$$

This clearly is as not good as the relaxed-AP property (2.25).

Another observation is the following. For prepared initial data

$$v^0=g(u^0)+O(\varepsilon)\,, \tag{2.30}$$

from (2.23), one sees that

$$v^n=g(u^n)+O(\varepsilon)\,,\quad \text{for any}\quad n\ge 1\,.$$

Namely, if the data are within $O(\varepsilon)$ of the local equilibrium, they remain so for all future times (later, we will refer this property as "weakly AP"). However, for the linear penalty method, even if the initial data are well prepared as in (2.30), from (2.28) one sees that

$$v^1=g(u^1)+O(\varepsilon+\Delta t)\,, \tag{2.31}$$

so the deviation from the local equilibrium at later times is always of $O(\Delta t)$ rather than $O(\varepsilon)$, for $\Delta t\gg\varepsilon$.

Another classical method uses Taylor expansion [**120**]

$$R^{n+1} \approx R^n + \partial_u R(u^n, v^n)(u^{n+1} - u^n) + \partial_v R(u^n, v^n)(v^{n+1} - v^n) \tag{2.32}$$

with implicit term R^{n+1} on (2.18) replaced by the right hand side of (2.32). This also introduces an implicit scheme that can be implemented explicitly, with stability condition independent of ε. However, one can show the same properties (2.28) and (2.29) [**41**] thus is not very desirable.

We only discussed first order schemes here. Higher order AP schemes in time can be developed by Runge-Kutta splitting schemes [**16**] or the more general implicit-explicit (IMEX) methods, see [**106, 110**].

2.5 - *Well-balanced schemes*

Since mid 90s there have also been extensive works on developing the so-called *well-balanced* schemes for hyperbolic systems with source terms. A prototype example is the shallow-water equations (2.1) and (2.2) by ignoring the friction term ($C_f = 0$):

$$\partial_t h + \partial_x(hv) = 0\,, \tag{2.33}$$

$$\partial_t(hv) + \partial_x\left(hv^2 + \frac{1}{2}gh^2\right) = gSh\,, \tag{2.34}$$

where the bottom slope is given by

$$S(x) = -B'(x)$$

with $B(x)$ the function of the bottom topography. A typical shock capturing method encounters difficulties when $B(x)$ is *discontinuous*. *Well-balanced* schemes aim at capturing the steady state of the system with good performance even for a discontinuous B. The stready state solution can be understood as the long time limit, with the rescaling $t \mapsto \varepsilon t$ and sending $\varepsilon \to 0$. Then the limit equations become

$$\partial_x(hv) = 0\,, \tag{2.35}$$

$$\partial_x\left(hv^2 + \frac{1}{2}gh^2\right) = -gB'h\,. \tag{2.36}$$

After a simple mathematical manipulation one gets

$$hv = C_1\,, \qquad \frac{1}{2}u^2 + g(h + B) = C_2 \tag{2.37}$$

where C_1 and C_2 are constants determined by boundary conditions. The Well-balanced schemes are schemes that preserve the steady state solutions (2.37) either exactly, or

formally with at least second order accuracy. The concept of well-balanced scheme was first introduced by Greenberg and Le Roux [**56**], although this principle was explored earlier [**112**]. We refer to [**5, 11, 12, 50, 51, 56, 53, 54, 63, 73, 92, 83, 107, 113, 119**] for the development and analysis of well-balanced schemes. The connections between the well-balanced schemes and asymptotic-preserving schemes were studied in [**53, 52**].

3 - Kinetic equations: the Euler regime

3.1 - *The Boltzmann equation*

The Boltzmann equation describes the time evolution of the density distribution of a dilute gas of particles when the only interactions taken into account are binary elastic collisions. For space variable $x \in \Omega \subset \mathbb{R}^{d_x}$, particle velocity $v \in \mathbb{R}^{d_v}$ ($d_v \geq 2$), the Boltzmann equation reads:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon} \mathcal{Q}(f), \tag{3.1}$$

where $f := f(t, x, v)$ is the particle distribution function in the phase space. The parameter $\varepsilon > 0$ is the dimensionless Knudsen number defined as the ratio of the mean free path over a typical length scale such as the size of the spatial domain, which measures the rarefiedness of the gas. The Boltzmann collision operator $\mathcal{Q}$ is a quadratic operator,

$$\mathcal{Q}(f)(v) = \int_{\mathbb{R}^{d_v}} \int_{\mathbb{S}^{d_v-1}} B(|v - v_\star|, \cos\theta) \left(f'_\star f' - f_\star f\right) d\sigma \, dv_\star, \tag{3.2}$$

where we used the shorthanded notation $f = f(v)$, $f_\star = f(v_\star)$, $f' = f(v')$, $f'_\star = f(v'_\star)$. The pre-collision velocity pairs $(v, v_\star)$ and the post-collision velocities $(v', v'_\star)$ are related by

$$\begin{cases} v' = v - \dfrac{1}{2}\big((v - v_\star) - |v - v_\star|\,\sigma\big), \\ v'_\star = v - \dfrac{1}{2}\big((v - v_\star) + |v - v_\star|\,\sigma\big), \end{cases}$$

with $\sigma \in \mathbb{S}^{d_v-1}$. The collision kernel B is a non-negative function which by physical arguments of invariance only depends on $|v - v_\star|$ and $\cos\theta = \hat{v} \cdot \sigma$ (where $\hat{v} = (v - v_\star)/|v - v_\star|$ is the normalized relative velocity).

Boltzmann's collision operator has the fundamental properties of conserving mass, momentum and energy:

$$\int_{\mathbb{R}^{d_v}} \mathcal{Q}(f)\,\phi(v)\,dv = 0, \quad \text{for} \quad \phi(v) = \left(1, v, \frac{|v|^2}{2}\right)^T, \tag{3.3}$$

and it satisfies the well-known Boltzmann's H theorem

$$\frac{d}{dt}\int_{\mathbb{R}^{d_v}} f\,\log f\,dv = \int_{\mathbb{R}^{d_v}} \mathcal{Q}(f)\log(f)\,dv \le 0.$$

The functional $-\int f\log f$ is the *entropy* of the solution. Boltzmann's H theorem implies that any equilibrium distribution function, *i.e.*, any function which is a maximum of the entropy, has the form of a local Maxwellian distribution

$$\mathcal{M}(v) = \frac{\rho}{(2\pi T)^{d_v/2}}\exp\left(-\frac{|u-v|^2}{2T}\right), \tag{3.4}$$

where ρ, u, T are the *density*, *macroscopic velocity* and *temperature* of the gas, defined by

$$\begin{aligned} &\rho = \int_{\mathbb{R}^{d_v}} f(v)\,dv = \int_{\mathbb{R}^{d_v}} \mathcal{M}(v)\,dv, \quad u = \frac{1}{\rho}\int_{\mathbb{R}^{d_v}} v f(v)\,dv = \frac{1}{\rho}\int_{\mathbb{R}^{d_v}} v\,\mathcal{M}(v)\,dv, \\ &T = \frac{1}{d_v\rho}\int_{\mathbb{R}^{d_v}} |u-v|^2 f(v)\,dv = \frac{1}{d_v\rho}\int_{\mathbb{R}^{d_v}} |u-v|^2\,\mathcal{M}(v)\,dv. \end{aligned} \tag{3.5}$$

When the Knudsen number $\varepsilon > 0$ becomes very small, the macroscopic model, which describes the evolution of ρ, u and T by fluid dynamics equations, namely, the compressible Euler or Navier-Stokes equations, become adequate [**2, 3, 13, 49**]. More specifically, as $\varepsilon \to 0$, f will converge to the local Maxwellian $\mathcal{M}$, and system (2.5) becomes a closed system for the $2 + d_v$ moments ρ, u and E, with the total energy E given by

$$E = \frac{1}{2}\rho u^2 + \frac{d_v}{2}\rho T. \tag{3.6}$$

The pressure is related to the internal energy by the constitutive relation for a polytropic gas

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho|u|^2\right), \tag{3.7}$$

where the polytropic constant $\gamma = (d_v + 2)/d_v$ represents the ratio between specific heat at constant pressure and at constant volume, thus yielding $p = \rho T$. For small but non zero values of the Knudsen number ε, the evolution equation for the moments can be derived by the so-called Chapman-Enskog expansion [**21**], applied to the Boltzmann equation. This approach gives the Navier-Stokes equations as a second order approximation with respect to ε to the solution of the Boltzmann equation:

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \nabla_x \cdot \rho_\varepsilon u_\varepsilon = 0, \\ \dfrac{\partial \rho_\varepsilon u_\varepsilon}{\partial t} + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon + p_\varepsilon \mathrm{I}) = \varepsilon \nabla_x \cdot [\mu_\varepsilon \sigma(u_\varepsilon)], \\ \dfrac{\partial E_\varepsilon}{\partial t} + \nabla_x \cdot (E_\varepsilon + p_\varepsilon) u_\varepsilon) = \varepsilon \nabla_x \cdot (\mu_\varepsilon \sigma(u_\varepsilon) u + \kappa_\varepsilon \nabla_x T_\varepsilon). \end{cases} \tag{3.8}$$

In these equations $\sigma(u)$ denotes the strain-rate tensor given by

$$\sigma(u) = \nabla_x u + (\nabla_x u)^T - \frac{2}{d_v} \nabla_x \cdot u \, \mathrm{I},$$

while the viscosity $\mu_\varepsilon = \mu(T_\varepsilon)$ and the thermal conductivity $\kappa_\varepsilon = \kappa(T_\varepsilon)$ are defined according to the linearized Boltzmann operator with respect to the local Maxwellian [**2**].

3.2 - *The BGK model*

We begin our study of AP schemes for the kinetic equation with the BGK model [**7**]:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} [\mathcal{M} - f]. \tag{3.9}$$

Again we consider both the splitting and unsplitting schemes.

Splitting scheme

First one uses an implicit collision:

$$\frac{f^* - f^n}{\Delta t} = \frac{1}{\varepsilon} [\mathcal{M}^* - f^*] \tag{3.10}$$

and then solves the explicit convection:

$$\frac{f^{n+1} - f^*}{\Delta t} + v \cdot \nabla_x f^* = 0. \tag{3.11}$$

Let

$$U := (\rho, \rho u, E)^T = \int \phi(v) f(v)\, dv$$

be the hydrodynamic variables (moments). Due to the important property (3.5) one sees that (3.10) gives $\mathcal{M}^* = \mathcal{M}^n$ upon taking the moments (multiplying (3.10) by $\phi(v)$ and integrating over v gives $U^* = U^n$), hence this step becomes

$$f^* = \frac{1}{1 + \Delta t/\varepsilon} f^n + \frac{\Delta t/\varepsilon}{1 + \Delta t/\varepsilon} \mathcal{M}^n . \tag{3.12}$$

Therefore, although (3.10) is an implicit scheme, it can be *implemented explicitly*. The second step is already explicit. So the entire scheme can be implemented explicitly without inverting any (system of) nonlinear equations. See [**25**].

This time discretization is strongly AP. In (3.12), let $\varepsilon \to 0$, with Δt fixed, then for any f^n, one gets $f^* = \mathcal{M}^n$. Applying this to (3.11) and taking the moments give the forward Euler time discretization of the compressible Euler equations (2.5) with (3.6) and (3.7).

Remark 3.1. *In the first step of the time-splitting, one could also integrate the space homogeneous BGK equation*

$$\partial_t f = \frac{1}{\varepsilon} [\mathcal{M} - f]$$

exactly to get

$$f^* = e^{-\Delta t/\varepsilon} f^n + (1 - e^{-\Delta t/\varepsilon}) \mathcal{M}^n \tag{3.13}$$

rather than using (3.12). *This method is also strongly AP since, as* $\varepsilon \to 0$, (3.13) *gives* $f^* = \mathcal{M}^n + o(\varepsilon)$.

An unsplit scheme

One can also solve the BGK equation using an unsplit IMEX scheme

$$\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x f^n = \frac{1}{\varepsilon} [\mathcal{M}^{n+1} - f^{n+1}] . \tag{3.14}$$

Now, due to the convection term, $\mathcal{M}^{n+1} \neq \mathcal{M}^n$. However, one can take the moments of (3.14) by multiplying $\phi(v)$ defined in (3.3), and using the property of $\mathcal{M}$ in (3.5). Then from (3.14),

$$U^{n+1} = \int \phi(v) (f^n - \Delta t\, v \cdot \nabla_x f^n)\, dv = U^n - \Delta t \int \phi(v) v \cdot \nabla_x f^n\, dv .$$

Thus U^{n+1} can be obtained explicitly, which defines $\mathcal{M}^{n+1}$ via (3.4). Now f^{n+1} can be obtained from (3.14) explicitly. This time discretization is also strongly AP since, for $\varepsilon \ll 1$, (3.14) implies $f^n = \mathcal{M}^n + O(\varepsilon)$ for any $n \geq 1$ regardless of the initial data f^0.

Remark 3.2. *Due to the high dimensionality, the Direct Simulation Monte-Carlo (DSMC) method, first introduced by Bird (see* [**9**]*), has been one of the main computational methods for the Boltzmann equation. Notice that* (3.12) *and* (3.13) *can both be written as*

$$f^* = \alpha f^n + (1-\alpha)\mathcal{M}^n \tag{3.15}$$

where $0<\alpha<1$. *If one considers* f *as a probability density function, then* (3.15) *can have a Monte-Carlo interpretation:*

With probability α*, do not resample; with probability* $1-\alpha$*, sample from* $\mathcal{M}^n$.

This is the basis of the AP time-relaxed [**44**] *Monte-Carlo method for the Boltzmann equation. See* [**17, 104, 105**].

Spatial discretization

Consider the 1d case. Divide the spatial domain into a number of cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i \in \mathbb{Z}$. Each cell is centered at x_i with a uniform length Δx. Then a first order upwind scheme for equation (3.11) can be written as

$$\frac{f_i^{n+1}-f_i^*}{\Delta t} + \frac{\frac{v+|v|}{2}(f_i^*-f_{i-1}^*) + \frac{v-|v|}{2}(f_{i+1}^*-f_i^*)}{\Delta x} = 0. \tag{3.16}$$

Multiplying (3.16) by $\phi(v)$ and integrating with respect to v, one can get

$$\frac{U_i^{n+1}-U_i^*}{\Delta t} + \frac{F_{i+\frac{1}{2}}^* - F_{i-\frac{1}{2}}^*}{\Delta x} = 0 \tag{3.17}$$

with

$$F_{i+\frac{1}{2}} = \int_{-\infty}^{\infty} \phi(v)\left(\frac{v+|v|}{2} f_i + \frac{v-|v|}{2} f_{i+1}\right) dv.$$

Define

$$F_i^{\pm} = \int_{-\infty}^{\infty} \phi(v)\, \frac{v \pm |v|}{2} f_i \, dv, \tag{3.18}$$

then $F_{i+\frac{1}{2}} = F_i^+ + F_{i+1}^-$. Consider the splitting scheme. If $\varepsilon \to 0$, then (3.12) implies $f^* = \mathcal{M}^* = \mathcal{M}^n$. Plugging this into (3.18), then $F_i^{\pm}$ can be evaluated in closed form

[35, 109]:

$$F_i^{\pm} = \int \phi(v) \frac{v \pm |v|}{2} \mathcal{M}_i \, dv = \begin{pmatrix} \rho_i u_i A_i^{\pm} \pm \rho_i B_i \\ \left(\rho_i T_i + \rho_i u_i^2\right) A_i^{\pm} \pm \rho_i u_i B_i \\ \left(\rho_i T_i u_i + \frac{1}{2}\rho_i u_i^3\right) A_i^{\pm} \pm \left(\frac{1}{2}\rho_i u_i^2 + \rho_i T_i\right) B_i \end{pmatrix}, \tag{3.19}$$

where

$$A_i^{\pm} = \frac{1}{2}(1 \pm \operatorname{erf}(s_i)), \quad B_i = e^{-s_i^2} \sqrt{\frac{T_i}{2\pi}} \quad \text{and} \quad s_i = u_i \sqrt{\frac{1}{2T_i}}, \tag{3.20}$$

and $\operatorname{erf}(s)$ is the error function

$$\operatorname{erf}(s) = \frac{2}{\pi} \int_0^s e^{-z} \, dx \, .$$

This is the classical kinetic scheme **[35, 109]** for the compressible Euler equations (2.5). Thus the spatial discretization (3.16) is also AP.

3.3 - *AP schemes for the Boltzmann equation*

We now consider the Boltzmann equation (3.1). For the Boltzmann collision operator $\mathcal{Q}$, an implcit approximation cannot be implemented explicitly. Due to the complexity of $\mathcal{Q}$ (nonlocal, nonlinear and high-dimensional), this is computationally daunting. An idea introduced by Filbet and Jin **[41]** is to penalize $\mathcal{Q}$ by the BGK operator $\mathcal{P}$:

$$\mathcal{P}(f) = \beta \left[\mathcal{M}(v) - f(v)\right], \tag{3.21}$$

where β is a parameter to be determined. Namely, we rewrite the Boltzmann equation (3.1) as

$$\partial_t f \, + \, v \nabla_x f \, = \, \frac{\mathcal{Q}(f) \, - \, \mathcal{P}(f)}{\varepsilon} + \frac{\mathcal{P}(f)}{\varepsilon} \, . \tag{3.22}$$

An IMEX Discretization to the Boltzmann equation

Now, similar to what was done in Section 2.4, we take the following IMEX time discretization:

$$\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x f^n \, = \, \frac{\mathcal{Q}(f^n) \, - \, \mathcal{P}(f^n)}{\varepsilon} + \frac{P(f^{n+1})}{\varepsilon} \, . \tag{3.23}$$

This scheme can be written as

$$f^{n+1} = \frac{\varepsilon}{\varepsilon + \beta^{n+1}\Delta t}\left[f^n - \Delta t\, v\, \nabla_x f^n\right] + \Delta t\, \frac{\mathcal{Q}(f^n) - \mathcal{P}(f^n)}{\varepsilon + \beta^{n+1}\Delta t} + \frac{\beta^{n+1}\Delta t}{\varepsilon + \beta^{n+1}\Delta t}\, \mathcal{M}^{n+1}, \tag{3.24}$$

where $\beta^{n+1} = \beta(t^{n+1}, x)$.

Although (3.24) appears nonlinearly implicit, it can be computed explicitly. Specifically, upon multiplying (3.24) by $\phi(v)$ defined in (3.3), and using the conservation property of $\mathcal{Q}$ and $\mathcal{P}$ and the definition of $\mathcal{M}$ in (3.5), the macroscopic quantity $U := (\rho, \rho u, E)$ can be computed from f as

$$U^{n+1} = \frac{\varepsilon}{\varepsilon + \beta^{n+1}\Delta t}\int \phi(v)\,(f^n - \Delta t\, v \cdot \nabla_x f^n)\, dv + \frac{\beta^{n+1}\Delta t}{\varepsilon + \beta^{n+1}\Delta t}\, U^{n+1},$$

or simply

$$U^{n+1} = \int \phi(v)\,(f^n - \Delta t v \cdot \nabla_x f^n)\, dv.$$

Thus U^{n+1} can be obtained explicitly, which defines $\mathcal{M}^{n+1}$. Now f^{n+1} can be obtained from (3.24) explicitly. In summary, although (3.23) is nonlinearly implicit, it can be solved *explicitly*.

Scheme (3.23) satisfies the following properties [**41**]:

Proposition 3.3. *Consider the numerical solution given by* (3.23). *Then,*

(*i*) *If* $\varepsilon \to 0$ *and* $f^n = \mathcal{M}^n + O(\varepsilon)$, *then* $f^{n+1} = \mathcal{M}^{n+1} + O(\varepsilon)$. *Thus, the scheme is weakly AP.*

(*ii*) *Assume* $\varepsilon \ll 1$ *and* $f^n = \mathcal{M}^n + O(\varepsilon)$. *If there exists a constant* $C > 0$ *such that*

$$\left\| \frac{f^{n+1} - f^n}{\Delta t} \right\| + \left\| \frac{U^{n+1} - U^n}{\Delta t} \right\| \le C, \tag{3.25}$$

then the scheme (3.23) *asymptotically becomes a first order in time approximation of the compressible Navier-Stokes equations* (3.8).

Remark 3.4. *A relaxed AP result analogous to* (2.24) *has not been established for the scheme* (3.23), *although it was numerically confirmed in* [**41**].

Note that $\mathcal{P}(f)$ can be computed from an expansion of the Boltzmann operator with respect to $\mathcal{M}$:

$$\mathcal{Q}(f) \simeq \mathcal{Q}(\mathcal{M}) + \nabla \mathcal{Q}(\mathcal{M})\,[\mathcal{M} - f\,].$$

Thus, we choose $\beta > 0$ as the least upper bound of the operator $\nabla \mathcal{Q}(\mathcal{M})$. Other choices of β are also possible, for example

$$\beta = \sup \left| \frac{\mathcal{Q}(f) - \mathcal{Q}(\mathcal{M})}{f - \mathcal{M}} \right| = \sup \left| \frac{\mathcal{Q}(f)}{f - \mathcal{M}} \right|,$$

or, at time t^n,

$$\beta^n = \sup \left| \frac{\mathcal{Q}(f^n) - \mathcal{Q}(f^{n-1})}{f^n - f^{n-1}} \right|.$$

This approach was also extended to quantum Boltzmann equation [**40**]. A distinguished feature there is that, in stead of using the quantum Maxwellian–which is the local equilibrium of the quantum collision operator– for penalty, it was suggested to use the *classical* Maxwellian, with temperature replaced by internal enegry. This avoids the inversion of complicated quantum (Bose-Einstein or Fermi-Dirac) distribution functions during the time-evolution steps.

The exponential integration method

One can write (3.22) as

$$\partial_t f + v \cdot \nabla_x f = \frac{\beta}{\varepsilon} \left(\frac{\mathcal{Q}^+(f)}{\beta} - \mathcal{M} \right) + \frac{\beta}{\varepsilon} (\mathcal{M} - f), \tag{3.26}$$

with

$$\mathcal{Q}^+(f) = \mathcal{Q}(f) + \beta f .$$

Now we use an operator splitting to separate the convection and collision term in (3.26), as was done by Dimarco and Pareschi in [**37**] (see also a related approach by Lemou [88]). Since during the collision, $\mathcal{M}$ is independent of t, the space homogeneous part of (3.26) is equivalent to

$$\partial_t [(f - \mathcal{M}) e^{\beta t/\varepsilon}] = \frac{1}{\varepsilon} [\mathcal{Q}^+(f) - \beta \mathcal{M}] e^{\beta t/\varepsilon} . \tag{3.27}$$

If one discretizes the above equation by the forward Euler method one arrives at

$$f^* = e^{-\beta \Delta t/\varepsilon} f^n + \left[1 - e^{-\beta \Delta t/\varepsilon} - \frac{\beta}{\varepsilon} e^{-\beta \Delta t/\varepsilon} \right] \mathcal{M}^n + \frac{\beta}{\varepsilon} e^{-\beta \Delta t/\varepsilon} \frac{\mathcal{Q}^+(f^n)}{\beta} . \tag{3.28}$$

Clearly, in the above equation, $f^n, \mathcal{M}^n$ and $\mathcal{Q}^+(f^n)$ are all non-negative (for β sufficiently large), and the right hand side is a convex combination of these non-negative quantities, *for any* ε and Δt, since all coefficients are non-negative and add to 1, thus the scheme is *positivity preserving*. Moreover, for $\varepsilon \ll 1$, (3.28) gives $f^* = M^n + o(\varepsilon)$,

which when substituted into the explicit convection step yields the correct Euler limit for the moments of f. Thus it is strongly AP in time.

One can also integrate (3.27) by higher order Runge-Kutta method, then higher order AP schemes will result, see [**37**].

This method can also be implemented in the Monte-Carlo framework [**37**]. For (3.28) one can do the following:

With probability $e^{-\beta\Delta t/\varepsilon}$*, do not resample; with probability* $1-e^{-\beta\Delta t/\varepsilon}-\dfrac{\beta}{\varepsilon}e^{-\beta\Delta t/\varepsilon}$ *sample from* $\mathcal{M}^n$*; with probability* $\dfrac{\beta}{\varepsilon}e^{-\beta\Delta t/\varepsilon}$ *sample from* $\dfrac{\mathcal{Q}^+(f^n)}{\beta}$.

3.4 - *The Fokker-Planck-Landau equation*

The nonlinear Fokker-Planck-Landau (nFPL) equation is widely used in plasma physics. It is a Boltzmann type kinetic equation that describes the dynamics of the phase space density distribution function $f=f(t,x,v)$ of charged particles at position x, time t with velocity v. The rescaled nFPL equation reads

$$\partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}Q(f), \qquad x\in\mathbb{R}^{d_x}, \quad v\in\mathbb{R}^{d_v} \tag{3.29}$$

with the nFPL operator

$$Q(f)=\nabla_v\cdot\int_{\mathbb{R}^{N_v}} A(v-v_*)(f(v_*)\nabla_v f(v)-f(v)\nabla_{v_*}f(v_*))dv_*, \tag{3.30}$$

where the semi-positive definite matrix $A(z)$ is given by

$$A(z)=|z|^{\gamma+2}\left(I-\frac{z\otimes z}{|z|^2}\right).$$

Here ε is the Knudsen number. The parameter γ is determined by the type of interaction between particles. In the case of inverse power law relationship, that is, when two particles at distance r interact with a force proportional to $1/r^s$, $\gamma=\dfrac{s-5}{s-1}$. For example, in the cases of the Maxwell molecules $\gamma=0$ (corresponding to $s=5$) and for the Coulomb potential $\gamma=-3$ (corresponding to $s=2$).

The nFPL equation is derived as a limit of the Boltzmann equation when all the collisions become grazing. It is more relevant in physics for charged particles, where the Coulomb potential is presented. The first derivation was due to Landau ([**84**], [**85**]). In this article we will always take $\gamma=-3$, while the scheme itself can be applied to any γ.

Similar to the classical Boltzmann operator, the nFPL operator (3.30) also preserves mass, momentum and energy. In addition, due to the semi-positivity of $A(z)$, one also obtains the entropy dissipation inequality,

$$\int_{\mathbb{R}^{N_v}} Q(f)\log f \le 0. \tag{3.31}$$

Here the equality holds only if f is the local Maxwellian $f=\mathcal{M}$.

The diffusive nature of the nFPL operator introduces new stiffness. An explicit collision method requires $\Delta t = O(\varepsilon(\Delta v)^2)$, which is more restrictive than solving the Boltzmann collision operator $\mathcal{Q}$ explicitly. The BGK operator is not suitable as a penalization any more. In [**75**], the following Fokker-Planck (FP) operator was used as the penalty operator:

$$P_{FP}(f) = P_{FP}^{M} f = \nabla_v \cdot \left(\mathcal{M} \nabla_v \left(\frac{f}{\mathcal{M}} \right) \right). \tag{3.32}$$

The FP operator is a linear operator when the Maxwellian $\mathcal{M}$ is time independent, in the case of the space homogeneous Fokker-Planck equation

$$\partial_t f = P_{FP}^{M} f.$$

Since $P_{FP}^{M} f$ preserves the macroscopic variables (density, momentum and energy), the Maxwellian $\mathcal{M}$ does not change in time.

A first order scheme for the nFPL equation (3.29) (3.30) reads

$$\frac{f^{n+1}-f^n}{\Delta t} + v\cdot\nabla_x f^n = \frac{1}{\varepsilon}\left(Q(f^n) - \beta P^n f^n + \beta P^{n+1} f^{n+1}\right), \tag{3.33}$$

where $P^n f^n = P_{FP}^{M^n} f^n$ is the FP operator (3.32) and β is given by

$$\beta = \max_v \lambda(D_A(f)). \tag{3.34}$$

Here $\lambda(D_A)$ is the spectral radius of the positive symmetric matrix D_A, with $D_A(f)$ defined by

$$D_A(f) = \int A(v-v_*) f_* dv_*. \tag{3.35}$$

Discretization of $\mathcal{P}(f)$

First we define a symmetric operator

$$\tilde{P}^M h = \frac{1}{\sqrt{\mathcal{M}}} \nabla_v \cdot \left(M \nabla_v \left(\frac{h}{\sqrt{\mathcal{M}}} \right) \right). \tag{3.36}$$

Note

$$P^M f = \sqrt{\mathcal{M}}\,\tilde{P}^M \frac{f}{\sqrt{\mathcal{M}}} \tag{3.37}$$

and (3.33) can be rewritten as

$$\begin{aligned}\left(\frac{f}{\sqrt{\mathcal{M}}}\right)^{n+1} = &\left(1-\frac{\Delta t\beta}{\varepsilon}\tilde{P}^{n+1}\right)^{-1}\left\{\frac{1}{\sqrt{\mathcal{M}^{n+1}}}\left(f^n - \Delta t v\cdot\nabla_x f^n\right.\right.\\ &\left.\left.+\frac{\Delta t}{\varepsilon}\left(Q(f^n)-\beta\sqrt{\mathcal{M}^n}\tilde{P}^n\frac{f^n}{\sqrt{\mathcal{M}^n}}\right)\right)\right\}.\end{aligned} \tag{3.38}$$

Now we give the discretization of $\tilde{P}$ in one dimension. The extension to higher dimension is similar.

$$\begin{aligned}(\tilde{P}^M h)_j = &\frac{1}{(\Delta v)^2}\frac{1}{\sqrt{\mathcal{M}_j}}\cdot\left[\sqrt{\mathcal{M}_j\mathcal{M}_{j+1}}\left(\left(\frac{h}{\sqrt{\mathcal{M}}}\right)_{j+1}-\left(\frac{h}{\sqrt{\mathcal{M}}}\right)_j\right)\right.\\ &\left.-\sqrt{\mathcal{M}_j\mathcal{M}_{j-1}}\left(\left(\frac{h}{\sqrt{\mathcal{M}}}\right)_j-\left(\frac{h}{\sqrt{\mathcal{M}}}\right)_{j-1}\right)\right]\\ = &\frac{1}{(\Delta v)^2}\left(h_{j+1}-\frac{\sqrt{\mathcal{M}_{j+1}}+\sqrt{\mathcal{M}_{j-1}}}{\sqrt{\mathcal{M}_j}}h_j+h_{j-1}\right).\end{aligned}$$

Then $\tilde{P}$ is symmetric (under the normal inner product). It is also positive definite, as shown in [**75**]. Thus one can invert the non-singular matrix $\left(1-\frac{\Delta t\beta}{\varepsilon}\tilde{P}^{n+1}\right)^{-1}$ by the Conjugate-Gradient method. Besides, after this discretization, we have the well balanced property

$$P^M\mathcal{M} = \sqrt{\mathcal{M}}\,\tilde{P}^M\sqrt{\mathcal{M}} = 0.$$

4 - Linear transport equations: the diffusion regime

4.1 - *The linear transport equation and its diffusion limit*

Consider the multidimensional linear transport equation under the diffusive scaling. Let $f(t,x,v)$ be the probability density distribution for particles at space point $x\in\mathbb{R}^d$, time t, and travel with velocity $v\in\Omega\subset\mathbb{R}^d$. Here Ω is symmetric in v, thus $\int_\Omega g(v)dv = 0$ for any function g odd in v. f solves the linear transport

equation

$$\varepsilon\,\partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}\mathcal{L}(f) + \varepsilon G\,. \tag{4.1}$$

In (4.1), $G = G(t,x)$ is the source term, ε is the mean free path, $\mathcal{L}(f)$ is the anisotropic collision term defined by

$$\mathcal{L}(f) = \int \sigma(v,w)\{M(v)f(w) - \mathcal{M}(w)f(v)\}dw\,,$$

with the normalized Maxwellian M defined by

$$M(v) = \frac{1}{(\pi)^{d/2}}\exp\left(-|v|^2\right).$$

The anisotropic scattering kernel σ is rotationally invariant and satisfies

$$\sigma(v,w) = \sigma(w,v) > 0\,.$$

This positivity assumption means that the collision operator is a regularized version of the physically relevant models.

The collision operator $\mathcal{L}$ has the following properties:

$$\mathcal{L}(f) = 0 \Longrightarrow f(t,x,v) = \rho(x,t)\,M(v)\,,$$

with

$$\rho(t,x) = \int f(v)\,dv := \langle\, f\,\rangle\,.$$

We also assume that the collision frequency λ satisfies the following bound for some positive constant μ,

$$\lambda(v) = \int \sigma(v,w)\,\mathcal{M}(w)\,dw \le \mu\,.$$

As $\varepsilon \to 0$, one can show that $f(x,v,t)$ is approximated by

$$f \approx \rho(x,t)\,M(v)$$

where ρ satisfies the diffusion equation [**108, 99**]

$$\partial_t\rho = \nabla_x\cdot(D\nabla_x\rho) + G \tag{4.2}$$

with the diffusion coefficient matrix

$$D = \int \frac{M(v)}{\lambda(v)}\,v\otimes v\,dv\,.$$

4.2 - *Parity equations-based AP schemes*

Split (4.1) into two equations, one for v and one for $-v$:

$$\varepsilon\,\partial_t f(v) + v\cdot\nabla_x f(v) = \frac{1}{\varepsilon}\mathcal{L}(f)(v) + \varepsilon G, \tag{4.3}$$

$$\varepsilon\,\partial_t f(-v) - v\cdot\nabla_x f(-v) = \frac{1}{\varepsilon}\mathcal{L}(f)(-v) + \varepsilon G, \tag{4.4}$$

Define the even- and odd-parities [**94**] as

$$r(t,x,v) = \frac{1}{2}\,[f(t,x,v) + f(t,x,-v)], \tag{4.5}$$

$$j(t,x,v) = \frac{1}{2\varepsilon}\,[f(t,x,v) - f(t,x,-v)]. \tag{4.6}$$

Adding and subtracting the two equations in (4.4) lead to

$$\partial_t r + v\cdot\nabla_x j = \frac{1}{\varepsilon^2}\mathcal{L}(r) + G, \tag{4.7}$$

$$\partial_t j + \frac{1}{\varepsilon^2} v\cdot\nabla_x r = -\frac{1}{\varepsilon^2}\,\lambda j\,; \tag{4.8}$$

where we used the property that

$$\int \sigma(v,w) j(w)\,dw = 0$$

since $j(w)$ is an odd-function in w.

Remark 4.1. *If $\sigma(v,w) = \sigma(|v|,|w|)$, then it is possible to use the even and odd parities only for the positive components of v and w, as is the case for neutron-transport equation* [**94**].

The idea of [**70**] was to rewrite (4.7) and (4.8) into the following form

$$\partial_t r + v\cdot\nabla_x j = \frac{1}{\varepsilon^2}\mathcal{L}(r) + G, \tag{4.9}$$

$$\partial_t j + v\cdot\nabla_x r = -\frac{1}{\varepsilon^2}\left[\,\lambda j + (1-\varepsilon^2\psi) v\cdot\nabla_x r\right], \tag{4.10}$$

where $\psi = \psi(\varepsilon)$ is a free parameter satisfying $0 \le \psi \le 1/\varepsilon^2$. The simplest choice of ψ is

$$\psi(\varepsilon) = \min\left\{\,1, \frac{1}{\varepsilon}\,\right\}.$$

One can easily derive the diffusion equation from (4.9) and (4.10). As $\varepsilon \to 0$, they give

$$\mathcal{L}(r) = 0\,, \tag{4.11}$$

$$\lambda j = -\, v \cdot \nabla_x r\,. \tag{4.12}$$

Solving (4.11) gives

$$r = \rho(x,t)\, M(v)\,, \tag{4.13}$$

where

$$\rho(x,t) = \langle f(x,\cdot,t)\rangle = \langle r(x,\cdot,x)\rangle\,.$$

Equation (4.12) gives

$$j = \frac{M(v)}{\lambda(v)}\,[\,-\,v \cdot \nabla_x \rho]\,. \tag{4.14}$$

Applying (4.13) and (4.14) in (4.9), and integrating over v, one gets the drift diffusion equation (4.2). Thus (4.9) and (4.10) set the foundation for AP schemes that compute the diffusion limit correctly when $\varepsilon \to 0$.

(4.9) and (4.10) allow one to use the conventional splitting employed for the relaxation system like those studied in Section 2.3. Specifically, a natural splitting on (4.9) and (4.10) consists of combining the relaxation step

$$\partial_t r = \frac{1}{\varepsilon^2}\,\mathcal{L}(r) + G, \tag{4.15}$$

$$\partial_t j = \frac{1}{\varepsilon^2}\,\left[-\lambda j - (1-\varepsilon^2\phi)(v \cdot \nabla_x r)\right]\,, \tag{4.16}$$

with the transport step

$$\partial_t r + v \cdot \nabla_x j = 0, \tag{4.17}$$

$$\partial_t j + v \cdot \nabla_x r = 0. \tag{4.18}$$

Equations (4.17) and (4.18) can be solved using an explicit scheme, whereas for step (4.15)-(4.16) one uses an implicit scheme.

The key is how to solve the collision step (4.15) implicitly in an efficient way. In the case of neutron transport, where $L(r) = \rho - r$, the implicit term can be integrated explicitly [**70**]. Otherwise, one can use the penalty method of Filbet-Jin [**41**], as was done in [**34**]. The same can be said about the scheme in Section 4.3.

As far as spatial discretization is concerned, one can use any upwind type (high order shock capturing) scheme for convection terms in (4.17) and (4.18), while in the right hand side of (4.16), it was suggested in [**70**] to use center difference for the

gradient of r. When $\varepsilon \to 0$, these spatial discretizations become consistent and stable discretization of (4.2), thus is AP spatially. See [**58, 82**] for more studies on this type of schemes. However, the limiting discrete diffusion equation is not compact. In 1d it is a five-point rather then a three-point discretization of the drift-diffusion equation. This problem can be fixed using staggered grid for r and j, as pointed out in [**68**]. See also Section 4.3. However, it is not clear how such staggered grids can be used in higher space dimensions.

Remark 4.2. *The scheme with $\psi = 0$ was introduced by Klar* [**78**]. *The difference with $\psi > 0$ is that* (4.17) (4.18) *give a strongly hyperbolic system, thus one can use well-developed hyperbolic solvers* (*such as the upwind scheme and its high resolution shock-capturing extension* [**91**], *which have good stability property and can capture discontinuous solutions without numerical oscillations.*

4.3 - *Micro-macro decomposition based AP schemes*

A different approach, based on the so-called micro-macro decomposition, was proposed by Lemou and Mieussens [**89**]. The idea begins with the decomposition of f into the equilibrium part ρM and the non-equilibrium part g:

$$f = \rho M + \varepsilon g\,. \tag{4.19}$$

The nonequilibrium part g clearly satisfies $\langle g\rangle = 0$. Applying (4.19) in (4.1) gives

$$\varepsilon M\partial_t \rho + \varepsilon^2 \partial_t g + v\cdot M\nabla_x \rho + \varepsilon v\cdot \nabla_x g = Lg + \varepsilon G\,. \tag{4.20}$$

Integrating this equation on Ω with respect to v gives the following continuity equation with a source term:

$$\partial_t \rho + \nabla_x \cdot \langle vg\rangle = G\,. \tag{4.21}$$

Define operator $\Pi : \Pi(\,\cdot\,)(v) := M\langle\,\cdot\,\rangle$, and I the identity operator. Then an evolution equation on g is found by applying the orthogonal projection $I - \Pi$ to (4.20):

$$\varepsilon^2\partial_t g + \varepsilon(I-\Pi)(v\cdot\nabla_x g) + v\cdot M\nabla_x\rho = \mathcal{L}g + (I-\Pi)\varepsilon G\,. \tag{4.22}$$

(4.21) and (4.22) constitute the micro-macro formulation of (4.1).

Time discretization

It was suggested in [**89**] to use the following time discretization:

$$\frac{g^{n+1}-g^n}{\Delta t} + \frac{1}{\varepsilon}(I-\Pi)(v\cdot\nabla_x g^n) = \frac{1}{\varepsilon^2}\mathcal{L}g^{n+1} - \frac{1}{\varepsilon^2}v\cdot M\nabla_x\rho^n\,. \tag{4.23}$$

In the macroscopic equation (4.21) there is no stiff term, but to recover the correct diffusion limit, the flux of g is taken at t_{n+1}, which gives

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla_x \cdot \langle v g^{n+1} \rangle = G\,. \tag{4.24}$$

Note that as $\varepsilon \to 0$, (4.23) gives

$$\mathcal{L} g^{n+1} = v \cdot M \nabla_x \rho^n$$

which gives

$$g^{n+1}(v) = \mathcal{L}^{-1}(vM)\cdot \nabla_x \rho^n = \frac{M(v)}{\lambda(v)}\left[\int \sigma(v,w) g^{n+1}(w)\,dw - v\cdot\nabla_x \rho^n\right].$$

Applying this to (4.24), and using the rotational invariance of σ, yield the following time explicit discretization of the diffusion equation

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla_x \cdot \langle D \nabla \rho^n \rangle = G\,.$$

Thus this time discretization is strongly AP.

One-dimensional Spatial discretization

In one space dimension, a staggered grid can be used by also defining $x_{i+1/2} = (i+1/2)\Delta x$. Now the macroscopic density ρ will be defined at gridpoint x_i, while g is defined at $x_{i+1/2}$. Using upwind discretization for the space derivative, one arrives at

$$\frac{\rho_i^{n+1}-\rho_i^n}{\delta t} + \left\langle v \frac{g_{i+1/2}^{n+1} - g_{i-1/2}^{n+1}}{\Delta x}\right\rangle = G\,, \tag{4.25}$$

$$\begin{aligned}\frac{g_{i+1/2}^{n+1}-g_{i+1/2}^n}{\Delta x} + \frac{1}{\varepsilon \Delta x}(I-\Pi)\Big(v^+(g_{i+1/2}^n - g_{i-1/2}^n) + v^-(g_{i+3/2}^n - g_{i+1/2}^n)\Big) \\ = \frac{1}{\varepsilon^2}\mathcal{L} g_{i+1/2}^{n+1} - \frac{1}{\varepsilon^2} vM\,\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\,,\end{aligned} \tag{4.26}$$

where $v^\pm = (v \pm |v|)/2$.

As $\varepsilon \to 0$, (4.26) gives

$$g_{i+1/2}^{n+1} = \mathcal{L}^{-1}(vM)\frac{\rho_{i+1}^n - \rho_i^n}{\delta x}$$

which when applied to (4.25) gives the following scheme

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + D\,\frac{\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n}{(\Delta x)^2} = G\,.$$

This is the classical three point explicit discretization of the diffusion equation (4.2).

The uniform stability condition ($\Delta t \le C(\Delta x)^2$), uniformly in ε) of this method was proved in [**95**].

Remark 4.3. *The micro-macro decomposition* (4.19) *was first used by Liu and Yu for theoretical study of the fluid limit of the Boltzmann equation* [**98**]. *It was then used in* [**4**] *to develop an AP scheme for the Boltzmann equation. The method was extended to multispecies Boltzmann equation in* [**71**]. *For the transport equation in the diffusive the micro-macro decomposition was also used in* [**20, 81**].

5 - The Euler-Poisson equations: the quasineutral regime

5.1 - *The Euler-Poisson system and its quasineutral limit*

Consider the one-species recaled Euler-Poisson (EP) equations for charged particles:

$$\partial_t n + \nabla \cdot q = 0\,, \tag{5.1}$$

$$\partial_t q + \nabla \cdot \Big(\frac{q \otimes q}{n}\Big) + \nabla p(n) = n \nabla \phi\,, \tag{5.2}$$

$$\varepsilon^2 \Delta \phi = n - 1\,, \tag{5.3}$$

where $n = n(x,t)$ is the particle number density, $q = q(x,t) = nu$ is the momentum (u is the average velocity), $p(n) = n^\gamma$ is the pressure law with $\gamma \ge 1$, and $\phi = \phi(x,t)$ is the electric potential. Here we consider negatively charged electrons with scaled charge equal to -1, and the right hand side of the Poisson equation invovles uniform ion background density equal to 1. The dimensionless parameter $\varepsilon = \lambda_D/L$ is the scaled Debye length, i.e., the ratio of the actual Debye length λ_D to the macroscopic length scale L.

In many applications, the Debye length is very small compared to the macroscopic length, thus $\varepsilon \ll 1$. This is the so-called *quasineutral regime*. A typical stable time discretization of the Euler-Poisson system requires

$$\Delta t \le \varepsilon$$

which is very inefficient in the quasi-neutral regime $\varepsilon \ll 1$. On the other hand, by sending $\varepsilon \to 0$ in (5.3), one has $n = 1$, i.e., the electron density equals to the density of the background ion, thus one obtains the following quasineutral limiting equations

from (5.1) and (5.2):

$$\nabla \cdot q = 0\,, \tag{5.4}$$

$$\partial_t q + \nabla \cdot (q \otimes q) = \nabla \phi\,. \tag{5.5}$$

This is the incompressible Euler equations.

The design of an AP scheme for the EP system is based on the following framework, first introduced by Degond etc. in [**26, 27**]. In the passage to the quasi-neutral limit, the electric potential ϕ changes dramatically, from the Poisson equation (5.3) into

$$\Delta \phi = \nabla^2 : (q \otimes q) \tag{5.6}$$

where ∇^2 denotes the Hessian and ":" the contracted product of rank two tensors. (5.6) is obtained by taking the divergence of (5.5). Any attempt to design a uniformly stable scheme for $\varepsilon \to 0$ requires finding a different (but equivalent) formulation of the Poisson equation in which the transition from (5.3) toward (5.6) appears explicitly.

To this end, we take ∂_t on (5.1), $\nabla\cdot$ on (5.2) and ∂_{tt} on (5.3) to get

$$\partial_{tt} n + \nabla \cdot \partial_t q = 0\,, \tag{5.7}$$

$$\nabla \cdot \partial_t q + \nabla^2 : \Big(\frac{q \otimes q}{n} + p(n)\mathrm{I}\Big) = \nabla \cdot (n \nabla \phi)\,, \tag{5.8}$$

$$\varepsilon^2 \Delta \partial_{tt} \phi = \partial_{tt} n\,. \tag{5.9}$$

Now eliminating $\nabla \cdot \partial_t q$ by combining (5.7) and (5.8) and using (5.9), one gets

$$-\nabla \cdot [(n + \varepsilon^2 \partial_{tt}) \nabla \phi] + \nabla^2 : f(n, q) = 0 \tag{5.10}$$

where the momentum flux tensor is given by

$$f(n, q) = \frac{q \otimes q}{n} + p(n)\mathrm{I}\,.$$

Equation (5.10) is formally a second order wave-like formulation of the Euler equations, equivalent to the Poisson equation (5.3) provided (n, q) satisfies the mass and momentum equations (5.1) and (5.2) and that the Poisson equation and its time derivative are satisfied at $t = 0$:

$$(\varepsilon^2 \Delta \phi - n + 1)|_{t=0} = 0\,, \quad (\partial_t(\varepsilon^2 \Delta \phi - n + 1))|_{t=0} = 0\,.$$

The important advantage of (5.10) over (5.3) is that, as long as $n > 0$, it does not degenerate when $\varepsilon \to 0$ and, moreover, reduces to the quasi-neutral elliptic equation (5.6) for ϕ when $\varepsilon = 0$. Thus it is *uniformly elliptic*. Therefore, one can expect the asymptotic stability with respect to ε if a suitable time discretization is used.

5.2 - *Time discretization*

First, consider a standard time discretization for solving the EP system (5.1)-(5.3):

$$\frac{n^{m+1}-n^m}{\Delta t}+\nabla\cdot q^m=0\,, \tag{5.11}$$

$$\frac{q^{m+1}-q^m}{\Delta t}+\nabla\cdot f(n^m,q^m)=n^{m+1}\nabla\phi^{m+1}\,, \tag{5.12}$$

$$\varepsilon^2\Delta\phi^{m+1}=n^{m+1}-1\,. \tag{5.13}$$

Note that although the scheme involves an implicit ϕ in the momentum equation, the cost stays the same as that of a fully explicit scheme for the momentum equation, since ϕ^{m+1} is obtained from the Poisson equation (5.13). This time discretization is constrained by the stability condition (5.1) which is prohibitive.

The scheme of [**26**] consists in computing the mass flux in (5.11) implicitly:

$$\frac{n^{m+1}-n^m}{\Delta t}+\nabla\cdot q^{m+1}=0\,, \tag{5.14}$$

while (5.12) and (5.13) remain the same. With (5.12)-(5.14), one can use the discrete analogy of the same procedure which leads from the Poisson equation (5.3) to (5.10) to obtain the following discrete equation

$$\begin{aligned}&-\nabla\cdot((\varepsilon^2/\Delta t^2+n^m)\nabla\phi^{m+1})\\ &=-\nabla^2:f(n^m,q^m)-(2n^m-n^{m-1}-1)/(\Delta t)^2:=G(n^m,q^m,n^{m-1})\,.\end{aligned} \tag{5.15}$$

This is a new time discrete elliptic equation for ϕ^{m+1} which is consistent to the original one (5.10), thus does not degenerate when $\varepsilon\to 0$.

To verify the AP property, we assume that at step m, the data n^m and q^m satisfy $n=1$ and $\nabla\cdot q^m=0$. Thus the initial data are "well-prepared", namely, they are already in the quasi-neutral regime. Letting $\varepsilon\to 0$ in (5.13) gives that $n^{m+1}=1$. Then, given that $n^m=1$, (5.14) yields

$$\nabla\cdot q^{m+1}=0\,. \tag{5.16}$$

Finally, (5.12) leads to

$$\frac{q^{m+1}-q^m}{\Delta t}+\nabla\cdot f(1,q^m)=\nabla\phi^{m+1}\,. \tag{5.17}$$

Clearly, (5.16) and (5.17) are consistent discretization of the quasi-neutral Euler system (5.4), (5.5). Thus the solution remain quasi-neutral if the initial data do, which is a weak AP property.

The numerical implementation of this scheme consists of solving (5.15) for ϕ^{m+1}, then obtaining q^{m+1} from (5.12), and finally (5.11) for n^{m+1}. The reformulated Poisson equation (5.15) allows one to compute ϕ^{m+1} in terms of known quantities $G(n^m, q^m, n^{m-1})$ from the previous time steps. Since it is a variable coefficient elliptic equation, solving it is a bit more costly (by, for example a preconditioned Conjugate-Gradient method) than solving the (constant coefficient) Poisson equation (5.3) for ϕ directly which can use fast Poisson solvers. Nevertheless, the gain in numerical stability merits the extra computational effort.

A stability analysis for the linearized EP system [**32**] shows that the stability condition of the time discretization (5.14), (5.11)-(5.12) is

$$\delta = O(1)\,, \quad \text{as} \quad \varepsilon \to 0\,. \tag{5.18}$$

which is much better than (5.1). Nevertheless, the scheme is still constrained by the CFL condition of the hydrodynamic system (which is an $O(1)$ quantity).

We also note that similar ideas can be applied to the low Mach number limit of compressible Euler equations [**30, 33, 59**].

6 - Conclusions

In this article we surveyed the basic concept, designing principle, and the development of asymptotic-preserving (AP) schemes for multiscale kinetic and hyperbolic problems in the last two decades. Unlike a typical multiscale and multiphysics approach, the AP schemes solve only the microscopic equations, yet automatically capture the macroscopic behavior when the mesh size and time step do not resolve the microscopic scales, thus they realize the numerical transition from microscopic to macroscopic scales in a seamless and extremely simple way. Although the design of an AP scheme is strongly guided by the macroscopic equations, the code does not need to couple a micro solver with a macro solver, thus it bypasses some of the bottleneck difficulties in multiscale problems.

There are still many challenging open problems. It is well-known that the asymptotic limit from the micro to the macro scales breaks down at boundaries, sharp interfaces between different media or materials, etc. where one needs to find suitable boundary or interface conditions to be used for the AP schemes [**64, 65, 89, 6**]. Other approximate models, such as transition models [**93, 114**], could also be combined to further reduce the computational cost.

In recent years, kinetic equations have also found many applications in emerging areas, such as multiphase flows [**14, 19, 55, 36, 103**], complex fluids [**10, 96**], granular materials [**46**], social and economic behavior [**100**], neuronal networks [**18**] etc., where AP schemes can find interesting applications.

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