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Curvatures of weighted metrics on tangent sphere bundles

Abstract. We determine the curvature equations of natural metrics on tangent bundles and radius r tangent sphere bundles S_rM of a Riemannian manifold M. A family of positive scalar curvature metrics on S_rM is found, for any M with bounded sectional curvature and any chosen constant r.

Keywords. Metric connection, tangent sphere bundle, curvature.

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1 - Introduction

This article continues the study of some structures which identify the tangent sphere bundles $S_rM = \{u \in TM : \ \|u\| = r\}$ of a Riemannian manifold (M, g) with variable radius and weighted Sasaki metric. We use the same notation from [**1**].

Throughout, we assume that M is an m-dimensional manifold with a Riemannian metric g and a compatible metric connection ∇ on M. The manifold TM is endowed with a canonical *vertical* vector field ξ, defined by $\xi_u = u$. Note the map π denotes, where appropriate, any of the bundle projections from SM or TM onto M. And clearly $V = \ker \mathrm{d}\pi \simeq \pi^*TM$ invariantly. Regarding the given connection, $\pi^*\nabla_X\xi = X^v$ and this projection $X \mapsto X^v$ has kernel H. The metric connection thus induces a splitting of $TTM = H \oplus V$ with both H, V parallel and isometric to π^*TM. Indeed $\mathrm{d}\pi$ induces the isomorphism $H \simeq \pi^*TM$ also invariantly (i.e. coordinate-free, see [**1**]). We have a map $\theta \in \mathrm{End}\, TTM$, which identifies H with V, sends V to 0 and is parallel for $\nabla^* = \pi^*\nabla \oplus \pi^*\nabla$.

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We continue our study assuming metrics of the kind $g^{f_1,f_2} = f_1\pi^*g \oplus f_2\pi^*g$ on $H \oplus V$, where f_1, f_2 are given by

$$f_1 = \mathrm{e}^{2\varphi_1}, \qquad f_2 = \mathrm{e}^{2\varphi_2}, \tag{1}$$

for some functions φ_1, φ_2 on M. Obviously we let these functions be composed with π when considered on the manifold TM. Recall the well known Sasaki metric is just $g^S = g^{1,1} = \langle\cdot,\cdot\rangle$ with H induced by the Levi-Civita connection. There is also a horizontal vector field $\theta^t\xi$ known as the spray of the connection. We remark the addition of a third bilinear symmetric form $f_3\mu \otimes \mu$, where $f_3 > 0$ and $\mu = (\theta^t\xi)^\flat = \xi^\flat \circ \theta$, gives a metric with interesting properties on S_rM; the rather much studied Cheeger-Gromov metric, which deviates from g^S by a multiple of $\xi^\flat \otimes \xi^\flat$, is only relevant for TM.

Our treatment of vector fields lets us use canonical projections $X = X^h + X^v$ when necessary, and not permanently recur to lifts of tangent vectors on M to either sections of H or V. The original use of the map θ, for example, expresses the benefit of this feature. We easily turn our attention to tensors defined on TM. Notice the holonomy Lie algebra of any of the metrics above remains unknown in general, even if M is any irreducible Riemannian symmetric space. Our main objective here is to envisage a solution to that problem and so we compute several curvature formulas.

The geometry of tangent bundles has had much study in the past and the Riemannian curvature of the Sasaki metric has been found (cf. the references in [**1, 3, 7**]). Regarding the radius r tangent sphere bundle with the induced metric from g^{f_1,f_2} we achieve in Theorem 1.2 a generalisation of a result from [**6**]: if M has dim ≥ 3 and bounded sectional curvature, and f_1 is sufficiently large or f_2 is sufficiently small, with both constant, then S_rM has positive scalar curvature.

Our purpose with this study is also towards the geometry of the so called *gwistor bundle*, which is the natural G_2-structure existing on S_1M for any oriented Riemannian 4-manifold.

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1.1 - *Computing the curvature of TM*

Let $\nabla = \nabla^g$ denote the Levi-Civita connection of M. As one of the few cases one can cope with, we study the curvature of $G = g^{f_1,f_2}$ where $f_2 = \mathrm{e}^{2\varphi_2}$ is a function on M and f_1 is a constant. We define $\delta = f_2/f_1$.

Recall from [**1**, Theorem 5.2] that the Levi-Civita connection of the tangent bundle is given by determining first

$$\nabla^{*,f_2,\prime}_X Y^v = \nabla^*_X Y^v + X(\varphi_2)Y^v, \tag{2}$$

$$D^* = \nabla^* \oplus \nabla^{*,f_2,\prime} \qquad \text{on} \quad H \oplus V = TTM, \tag{3}$$

$$B(X,Y) = Y(\varphi_2)X^v - \delta\langle X^v, Y^v\rangle \operatorname{grad}\varphi_2, \tag{4}$$

$$\langle A_X Y, Z\rangle = \frac{\delta}{2}\left(\langle \mathcal{R}^\xi(X,Z), Y\rangle + \langle \mathcal{R}^\xi(Y,Z), X\rangle\right). \tag{5}$$

The first connection is metric on the vector bundle V. The tensor $\mathcal{R}^\xi$ is given by $\mathcal{R}^\xi(X,Y) = \pi^* R^\nabla(X,Y)\xi$ and finally $\forall X, Y \in \Gamma(TM, H \oplus V)$, we have

$$\nabla^G_X Y = D^*_X Y - \frac{1}{2}\mathcal{R}^\xi(X,Y) + A(X,Y) + B(X,Y). \tag{6}$$

We recall, for a moment, that if $\nabla' = \nabla + C$ and ∇ are two connections on a vector bundle L, hence with $C \in \Omega^1(\operatorname{End} L)$, then

$$R^{\nabla'} = R^\nabla + \mathrm{d}^\nabla C + C \wedge C \tag{7}$$

where

$$\mathrm{d}^\nabla C(X,Y) = \nabla_X C_Y - \nabla_Y C_X - C_{[X,Y]} \tag{8}$$

and

$$(C \wedge C)(X,Y)Z = C(X, C(Y,Z)) - C(Y, C(X,Z)) \tag{9}$$

with X, Y vector fields and Z a section of L.

Now, we have to compute several d^∇ derivatives of our structure, where $\nabla = \nabla^* \oplus \nabla^*$ respecting the splitting $H \oplus V$. Recall the formula already implicitly used, $R^{\nabla^*} = \pi^* R^\nabla$, easy to see since this is a tensor. Assuming the reader is by now familiar with the notation, we shall let fall the asterisk wherever possible and abbreviate $R^\nabla = R$.

Let $A^{\nabla_X \mathcal{R}^\xi}$ be defined (in the same way as the tensor A is defined):

$$\langle A^{\nabla_X \mathcal{R}^\xi}(Y,Z), W\rangle = \frac{\delta}{2}\left(\langle (\nabla_X \mathcal{R}^\xi)(Y,W), Z\rangle + \langle (\nabla_X \mathcal{R}^\xi)(Z,W), Y\rangle\right). \tag{10}$$

Again we have the properties

$$\nabla_X \mathcal{R}^\xi(Y,Z) = \nabla_X \mathcal{R}^\xi(Y^h, Z^h) \ \in V, \tag{11}$$

$$A^{\nabla_X \mathcal{R}^\xi}(X,Y) = A^{\nabla_X \mathcal{R}^\xi}(X^h, Y^v) + A^{\nabla_X \mathcal{R}^\xi}(X^v, Y^h) \ \in H. \tag{12}$$

Proposition 1.1. *We have:*

1. $R^{\nabla^{*,f_2,\prime}} = R$.
2. $(\nabla_X \mathcal{R}^\xi)(Y,Z) = (\nabla_{X^h} R)(Y,Z)\xi + R(Y,Z)X^v$.
3. $\mathrm{d}^\nabla \mathcal{R}^\xi(X,Y)Z = (\nabla_X R)(Y,Z)\xi - (\nabla_Y R)(X,Z)\xi + R(Y,Z)X^v - R(X,Z)Y^v$.
4. $\mathrm{d}^\nabla A(X,Y)Z = (\mathrm{d}\varphi_2 \wedge A)(X,Y)Z - A^{\nabla_Y \mathcal{R}^\xi}_X Z + A^{\nabla_X \mathcal{R}^\xi}_Y Z + A(\mathcal{R}^\xi(X,Y), Z)$.

Proof. 1. The connection is $\nabla_X Y + \mathrm{d}\varphi_2(X)Y$. Following (7) and letting 1 denote the identity, the C part is $\mathrm{d}\varphi_2.1$. Thence $\mathrm{d}^\nabla(\mathrm{d}\varphi_2.1) = \mathrm{d}\mathrm{d}\varphi_2.1 = 0$. And clearly

$$\mathrm{d}\varphi_2.1 \wedge \mathrm{d}\varphi_2.1 = \mathrm{d}\varphi_2 \wedge \mathrm{d}\varphi_2.1 = 0.$$

2. For any vector fields:

$$\begin{aligned}\nabla_X \mathcal{R}^\xi(Y,Z) &= \nabla^*_X(\pi^* R(Y,Z)\xi) - \pi^* R(\nabla^*_X Y, Z)\xi - \pi^* R(Y, \nabla^*_X Z)\xi \\ &= \pi^*(\nabla_{\mathrm{d}\pi X} R)(Y,Z)\xi + R(Y,Z)\nabla_X\xi \\ &= (\nabla_{X^h} R)(Y,Z)\xi + R(Y,Z)X^v\end{aligned}$$

since we have the identity $\nabla_X \xi = X^v$.

3. Since $\mathcal{R}^\xi_X = \mathcal{R}^\xi_{X^h}$ and $\pi^* T^\nabla = 0$, we have

$$\begin{aligned}&\mathrm{d}^\nabla \mathcal{R}^\xi(X,Y)Z \\ &= (\nabla_X \mathcal{R}^\xi_Y - \nabla_Y \mathcal{R}^\xi_X - \mathcal{R}^\xi_{[X,Y]})Z \\ &= \nabla_X(R(Y,Z)\xi) - R(Y, \nabla_X Z)\xi - \nabla_Y(R(X,Z)\xi) + R(X, \nabla_Y Z)\xi \\ &\qquad - R(\nabla_X Y, Z)\xi + R(\nabla_Y X, Z)\xi \\ &= (\nabla_X R)(Y,Z)\xi - (\nabla_Y R)(X,Z)\xi + R(Y,Z)\nabla_X\xi - R(X,Z)\nabla_Y\xi \\ &= \nabla_X \mathcal{R}^\xi(Y,Z) - \nabla_Y \mathcal{R}^\xi(X,Z).\end{aligned}$$

4. First we find

$$\begin{aligned}&\langle \nabla_X(A(Y,Z)), W\rangle \\ &= X(\langle A(Y,Z), W\rangle) - \langle A(Y,Z), \nabla_X W\rangle \\ &= \frac{1}{2f_1}(X(f_2))(\langle \mathcal{R}^\xi(Y,W), Z\rangle + \langle \mathcal{R}^\xi(Z,W), Y\rangle) \\ &\quad + \frac{f_2}{2f_1}\big(\langle \nabla_X(\mathcal{R}^\xi(Y,W)), Z\rangle + \langle \mathcal{R}^\xi(Y,W), \nabla_X Z\rangle + \langle \nabla_X(\mathcal{R}^\xi(Z,W)), Y\rangle \\ &\quad + \langle \mathcal{R}^\xi(Z,W), \nabla_X Y\rangle - \langle \mathcal{R}^\xi(Y, \nabla_X W), Z\rangle - \langle \mathcal{R}^\xi(Z, \nabla_X W), Y\rangle\big) \\ &= \langle X(\varphi_2)A(Y,Z), W\rangle + \frac{f_2}{2f_1}\big(\langle (\nabla_X \mathcal{R}^\xi)(Y,W) + \mathcal{R}^\xi(\nabla_X Y, W), Z\rangle \\ &\quad + \langle \mathcal{R}^\xi(Y,W), \nabla_X Z\rangle + \langle (\nabla_X \mathcal{R}^\xi)(Z,W) + \mathcal{R}^\xi(\nabla_X Z, W), Y\rangle + \langle \mathcal{R}^\xi(Z,W), \nabla_X Y\rangle\big) \\ &= \langle X(\varphi_2)A(Y,Z) + A^{\nabla_X \mathcal{R}^\xi}(Y,Z) + A(\nabla_X Y, Z) + A(Y, \nabla_X Z), W\rangle.\end{aligned}$$

Recalling the torsion of ∇^* is $\mathcal{R}^\xi$, cf. [**1**, Proposition 5.1], we then have

$$\begin{aligned}
&\mathrm{d}^\nabla A(X,Y)Z\\
&= (\nabla_X A_Y)Z - (\nabla_Y A_X)Z - A_{[X,Y]}Z\\
&= \nabla_X(A(Y,Z)) - A(Y,\nabla_X Z) - \cdots\\
&= X(\varphi_2)A(Y,Z) + A^{\nabla_X\mathcal{R}^\xi}(Y,Z) + A(\nabla_X Y,Z) - Y(\varphi_2)A(X,Z)\\
&\qquad - A^{\nabla_Y\mathcal{R}^\xi}(X,Z) - A(\nabla_Y X,Z) - A(\nabla_X Y - \nabla_Y X - \mathcal{R}^\xi(X,Y),Z)\\
&= \mathrm{d}\varphi_2\wedge A(X,Y)Z + A^{\nabla_X\mathcal{R}^\xi}(Y,Z) - A^{\nabla_Y\mathcal{R}^\xi}(X,Z) + A(\mathcal{R}^\xi(X,Y),Z)
\end{aligned}$$

as we wished. □

In a very similar computation as the above we find:

Proposition 1.2. *The B tensor satisfies*

$$\begin{aligned}
\mathrm{d}^\nabla B(X,Y)Z =&\langle\nabla_X\,\mathrm{grad}\,\varphi_2,Z\rangle Y^v - \langle\nabla_Y\mathrm{grad}\,\varphi_2,Z\rangle X^v + Z(\varphi_2)\mathcal{R}^\xi(X,Y)\\
&- \delta(2X(\varphi_2)\langle Y^v,Z^v\rangle - 2Y(\varphi_2)\langle X^v,Z^v\rangle - \langle\mathcal{R}^\xi(X,Y),Z\rangle\mathrm{grad}\,\varphi_2\\
&- \langle Y^v,Z^v\rangle\nabla_X\,\mathrm{grad}\,\varphi_2 + \langle X^v,Z^v\rangle\nabla_Y\,\mathrm{grad}\,\varphi_2).
\end{aligned}\tag{13}$$

Now, we want to compute the curvature of ∇^G. As the reader might see, the development of $\mathrm{d}^\nabla C + C\wedge C$ is quite long when $C = \mathrm{d}\varphi_2.1^v - 1/2\mathcal{R}^\xi + A + B$. So we shall proceed with two particular cases. The first is below, while the second is in the next section.

Theorem 1.1. *Suppose $f_1 > 0$ is a constant, $f_2 = \mathrm{e}^{2\varphi_2}$ and the connection ∇ is flat, so that*

$$\nabla^G_X Y = \nabla_X Y + X(\varphi_2)Y^v + Y(\varphi_2)X^v - \delta\langle X^v,Y^v\rangle\mathrm{grad}\,\varphi_2.\tag{14}$$

Then the Riemannian curvature tensor of TM with metric $G = g^{f_1,f_2}$ is given by

$$\begin{aligned}
R^G(X,Y)Z \;=\; &\big(X(\varphi_2)Z(\varphi_2) + \delta\varepsilon^2\langle X^v,Z^v\rangle + \langle\nabla_X\,\mathrm{grad}\,\varphi_2,Z\rangle\big)Y^v\\
&- \big(Y(\varphi_2)Z(\varphi_2) + \delta\varepsilon^2\langle Y^v,Z^v\rangle + \langle\nabla_Y\mathrm{grad}\,\varphi_2,Z\rangle\big)X^v\\
&- \delta\big(X(\varphi_2)\langle Y^v,Z^v\rangle - Y(\varphi_2)\langle X^v,Z^v\rangle\big)\mathrm{grad}\,\varphi_2\\
&- \delta\langle Y^v,Z^v\rangle\nabla_X\,\mathrm{grad}\,\varphi_2 + \delta\langle X^v,Z^v\rangle\nabla_Y\mathrm{grad}\,\varphi_2
\end{aligned}\tag{15}$$

where $\varepsilon = \|\mathrm{grad}\,\varphi_2\|$.

Proof. After some computations we find

$$B \wedge B(X,Y)Z = \delta\varepsilon^2(\langle X^v, Z^v\rangle Y^v - \langle Y^v, Z^v\rangle X^v)$$

and

$$\begin{aligned} C \wedge C(X,Y)Z &= (\mathrm{d}\varphi_2.1^v \wedge B + B \wedge \mathrm{d}\varphi_2.1^v + B \wedge B)(X,Y)Z \\ &= X(\varphi_2)Z(\varphi_2)Y^v - Y(\varphi_2)Z(\varphi_2)X^v + Y(\varphi_2)B(X,Z^v) \\ &\quad - X(\varphi_2)B(Y,Z^v) + B \wedge B(X,Y)Z \\ &= X(\varphi_2)(Z(\varphi_2)Y^v + \delta\langle Y^v, Z^v\rangle \operatorname{grad}\varphi_2) \\ &\quad - Y(\varphi_2)(Z(\varphi_2)X^v + \delta\langle X^v, Z^v\rangle \operatorname{grad}\varphi_2) + B \wedge B(X,Y)Z. \end{aligned}$$

Adding to $\mathrm{d}^\nabla C = \mathrm{d}^\nabla B$ above, we deduce $R^G = \mathrm{d}^\nabla C + C \wedge C$. □

The case when $\operatorname{grad}\varphi_2$ is parallel may be further developed. Straightforward computations yield the following result.

Corollary 1.1. *Suppose (M,g) is a flat Riemannian manifold and the function f_2 verifies $\nabla \mathrm{d}\varphi_2 = 0$. Then the sectional curvature of the metric $G = g^{f_1,f_2}$ on a plane Π spanned by the orthonormal basis X, Y is*

$$\begin{aligned} k(\Pi) &= G(R^G(X,Y)Y,X) \\ &= -f_2\varepsilon^4\|bX^v - aY^v\|^2 - f_2\varepsilon^2\delta(\|X^v\|^2\|Y^v\|^2 - \langle X^v, Y^v\rangle^2), \end{aligned} \tag{16}$$

where $X = a\operatorname{grad}\varphi_2 + X' + X^v$, $Y = b\operatorname{grad}\varphi_2 + Y' + Y^v$ and $X', Y' \in H \cap (\operatorname{grad}\varphi_2)^\perp$, $a, b \in \mathbb{R}$. In particular, $k(\Pi) \leq 0$.

Hence on points x where $\operatorname{grad}\varphi_2 \neq 0$ the fibres T_xM are hyperbolic totally geodesic submanifolds. The other fibres are flat.

In the previous conditions, we observe that the equations of a geodesic curve Θ in TM appear as:

$$\begin{cases} \nabla_{\dot\Theta}\dot\Theta^h - f_2\langle\dot\Theta^v, \dot\Theta^v\rangle \operatorname{grad}\varphi_2 = 0 \\ \nabla_{\dot\Theta}\dot\Theta^v + 2\dot\Theta(\varphi_2)\dot\Theta^v = 0. \end{cases} \tag{17}$$

So it would be interesting at least in this case to solve the problem of knowing when is ∇^G complete (the completeness of a pull-back connection seems to be an open problem).

If M is a simply connected flat Riemannian manifold and ∇^G is a complete connection, then we suppose (TM, G, I^G, ω^G) is very close to being a Stein manifold (cf.

[**1**] for such Hermitian structure). To deduce this applying a famous result of Wu, cf. [**10**], we would need TM to be Kähler with $k \leq 0$. But then we would be asking the function f_2 to be a constant, by [**1**, Corollary 6.3], and so we would be referring only to the complex plane with standard metric.

1.2 - *Curvature of* g^{f_1,f_2} *with* f_1, f_2 *constants*

The second particular situation we must try to investigate is when f_2 is a constant. So we continue with $\nabla = \nabla^g$ the Levi-Civita connection of M. We may write simply

$$\nabla^G = \nabla + C \qquad \text{with} \qquad C = -\frac{1}{2}\mathcal{R}^\xi + A. \tag{18}$$

The connection $D^* = \nabla^* \oplus \nabla^*$, so we write it as ∇. Since $\mathcal{R}^\xi \wedge \mathcal{R}^\xi = 0$, the curvature of G is

$$R^G = R^\nabla - \frac{1}{2}\mathrm{d}^\nabla\mathcal{R}^\xi + \mathrm{d}^\nabla A - \frac{1}{2}\mathcal{R}^\xi \wedge A - \frac{1}{2}A \wedge \mathcal{R}^\xi + A \wedge A. \tag{19}$$

Notice R^∇ stands for $R^{\nabla^*} \oplus R^{\nabla^*}$. Some parts of the tensor R^G were computed in Proposition 1.1, namely those involving d^∇. Now

$$\begin{aligned} \mathrm{d}^\nabla\mathcal{R}^\xi(X,Y)Z &= (\nabla_X R)(Y,Z)\xi - (\nabla_Y R)(X,Z)\xi + R(Y,Z)X^v - R(X,Z)Y^v \\ &= (\nabla_X\mathcal{R}^\xi)(Y,Z) - (\nabla_Y\mathcal{R}^\xi)(X,Z), \end{aligned} \tag{20}$$

$$\mathrm{d}^\nabla A(X,Y)Z = -A^{\nabla_Y\mathcal{R}^\xi}(X,Z) + A^{\nabla_X\mathcal{R}^\xi}(Y,Z) + A(\mathcal{R}^\xi(X,Y),Z). \tag{21}$$

The others parts do not simplify nor cancel each other, as the reader may notice reading their nature in $H \oplus V$.

Let $e_1, \ldots, e_m$ be a real g-orthonormal basis of TM at a given point. This is immediately lifted to H and then to V by θ, giving a g^S-orthonormal basis. Writing

$$A(X,Y) = \sum\langle A(X,Y), e_i\rangle e_i = \frac{\delta}{2}\sum(\langle\mathcal{R}^\xi(X,e_i),Y\rangle + \langle\mathcal{R}^\xi(Y,e_i),X\rangle)e_i, \tag{22}$$

we have the Gauss-Codazzi type equations

$$\begin{aligned} -\frac{1}{2}\mathcal{R}^\xi \wedge A(X,Y)Z &= -\frac{1}{2}\mathcal{R}^\xi(X,A(Y,Z)) + \frac{1}{2}\mathcal{R}^\xi(Y,A(X,Z)) \\ &= -\frac{\delta}{4}\sum_j\big((\langle\mathcal{R}^\xi(Y,e_j),Z\rangle + \langle\mathcal{R}^\xi(Z,e_j),Y\rangle)\mathcal{R}^\xi(X,e_j) \\ &\qquad - (\langle\mathcal{R}^\xi(X,e_j),Z\rangle + \langle\mathcal{R}^\xi(Z,e_j),X\rangle)\mathcal{R}^\xi(Y,e_j)\big), \end{aligned} \tag{23}$$

$$
\begin{aligned}
-\frac{1}{2}A\wedge\mathcal{R}^\xi(X,Y)Z &= -\frac{1}{2}\,A(X,\mathcal{R}^\xi(Y,Z))+\frac{1}{2}\,A(Y,\mathcal{R}^\xi(X,Z))\\
&= -\frac{\delta}{4}\sum_i(\langle\mathcal{R}^\xi(X,e_i),\mathcal{R}^\xi(Y,Z)\rangle-\langle\mathcal{R}^\xi(Y,e_i),\mathcal{R}^\xi(X,Z)\rangle)e_i
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
A\wedge A(X,Y)Z &= A(X,A(Y,Z))-A(Y,A(X,Z))\\
&= \frac{\delta}{2}\sum_i(\langle\mathcal{R}^\xi(A(Y,Z),e_i),X\rangle-\langle\mathcal{R}^\xi(A(X,Z),e_i),Y\rangle)e_i\\
&= \frac{\delta^2}{4}\sum_{i,j}^{m}\big((\langle\mathcal{R}^\xi(Y,e_j),Z\rangle+\langle\mathcal{R}^\xi(Z,e_j),Y\rangle)\langle\mathcal{R}^\xi(e_j,e_i),X\rangle\\
&\quad-(\langle\mathcal{R}^\xi(X,e_j),Z\rangle+\langle\mathcal{R}^\xi(Z,e_j),X\rangle)\langle\mathcal{R}^\xi(e_j,e_i),Y\rangle\big)e_i.
\end{aligned}
\tag{25}
$$

Also $A(X,\mathcal{R}^\xi(Y,Z))=\dfrac{\delta}{2}\sum\langle\mathcal{R}^\xi(X,e_i),\mathcal{R}^\xi(Y,Z)\rangle e_i$. Now we have

$$
\begin{aligned}
R^G(X^h,Y^h)Z^h &= R(X^h,Y^h)Z^h-\frac{1}{2}(\nabla_{X^h}\mathcal{R}^\xi)(Y^h,Z^h)\\
&\quad+\frac{1}{2}(\nabla_{Y^h}\mathcal{R}^\xi)(X^h,Z^h)+A(\mathcal{R}^\xi(X^h,Y^h),Z^h)\\
&\quad-\frac{1}{2}\,A(X^h,\mathcal{R}^\xi(Y^h,Z^h))+\frac{1}{2}\,A(Y^h,\mathcal{R}^\xi(X^h,Z^h)),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&R^G(X^v,Y^h)Z^h\\
&= -\frac{1}{2}(\nabla_{X^v}\mathcal{R}^\xi)(Y^h,Z^h)-A^{\nabla_{Y^h}\mathcal{R}^\xi}(X^v,Z^h)+\frac{\delta}{4}\sum\langle\mathcal{R}^\xi(Z^h,e_j),X^v\rangle\mathcal{R}^\xi(Y^h,e_j)\\
&= -\frac{1}{2}R(Y^h,Z^h)X^v-A^{\nabla_{Y^h}\mathcal{R}^\xi}(X^v,Z^h)+\frac{\delta}{4}\sum\langle\mathcal{R}^\xi(Z^h,e_j),X^v\rangle\mathcal{R}^\xi(Y^h,e_j),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&R^G(X^v,Y^h)Z^v\\
&= A^{\nabla_{X^v}\mathcal{R}^\xi}(Y^h,Z^v)+\frac{\delta^2}{4}\sum\langle\mathcal{R}^\xi(Y^h,e_j),Z^v\rangle\langle\mathcal{R}^\xi(e_j,e_i),X^v\rangle e_i,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
R^G(X^h,Y^h)Z^v &= R(X^h,Y^h)Z^v-A^{\nabla_{Y^h}\mathcal{R}^\xi}(X^h,Z^v)+A^{\nabla_{X^h}\mathcal{R}^\xi}(Y^h,Z^v)\\
&+\frac{\delta}{4}\sum\big(\langle\mathcal{R}^\xi(X^h,e_j),Z^v\rangle\mathcal{R}^\xi(Y^h,e_j)-\langle\mathcal{R}^\xi(Y^h,e_j),Z^v\rangle\mathcal{R}^\xi(X^h,e_j)\big),
\end{aligned}
\tag{29}
$$

$$R^G(X^v,Y^v)Z^h = -A^{\nabla_{Y^v}\mathcal{R}^\xi}(X^v,Z^h)+A^{\nabla_{X^v}\mathcal{R}^\xi}(Y^v,Z^h) \tag{30}$$
$$+\frac{\delta^2}{4}\sum\big(\langle\mathcal{R}^\xi(Z^h,e_j),Y^v\rangle\langle\mathcal{R}^\xi(e_j,e_i),X^v\rangle-\langle\mathcal{R}^\xi(Z^h,e_j),X^v\rangle\langle\mathcal{R}^\xi(e_j,e_i),Y^v\rangle\big)e_i$$

and, clearly, $R^G(X^v,Y^v)Z^v=0$.

The simplification in formula (27) is due to property 2 in Proposition 1.1. In order to find the Ricci curvature of G we let $R^G(X,Y,Z,W)$ denote the 4-tensor $G(R^G(X,Y)Z,W)$. The same we agree in denoting R with the metric g. We only need

$$\begin{aligned} R^G(X^h,Y^h,Y^h,W^h) &= f_1R(X^h,Y^h,Y^h,W^h)+f_1\langle A(\mathcal{R}^\xi(X^h,Y^h),Y^h),W^h\rangle \\ &\quad +\frac{f_1}{2}\langle A(Y^h,\mathcal{R}^\xi(X^h,Y^h)),W^h\rangle \\ &= f_1R(X^h,Y^h,Y^h,W^h)+\frac{f_2}{2}\langle\mathcal{R}^\xi(Y^h,W^h),\mathcal{R}^\xi(X^h,Y^h)\rangle \\ &\quad +\frac{f_2}{4}\langle\mathcal{R}^\xi(Y^h,W^h),\mathcal{R}^\xi(X^h,Y^h)\rangle \\ &= f_1R(X^h,Y^h,Y^h,X^h)+\frac{3}{4}f_2\langle\mathcal{R}^\xi(Y^h,W^h),\mathcal{R}^\xi(X^h,Y^h)\rangle, \end{aligned} \tag{31}$$

$$\begin{aligned} R^G(X^h,Y^v,Y^v,W^h) &= -f_1\langle A^{\nabla_{Y^v}\mathcal{R}^\xi}(X^h,Y^v),W^h\rangle \\ &\quad -\frac{f_1\delta^2}{4}\sum_{i,j=1}^m\langle\mathcal{R}^\xi(X^h,e_j),Y^v\rangle\langle\mathcal{R}^\xi(e_j,e_i),Y^v\rangle\langle e_i,W^h\rangle \\ &= -\frac{f_2}{2}\langle(\nabla_{Y^v}\mathcal{R}^\xi)(X^h,W^h),Y^v\rangle \\ &\quad +\frac{f_1\delta^2}{4}\sum\langle\mathcal{R}^\xi(X^h,e_j),Y^v\rangle\langle\mathcal{R}^\xi(W^h,e_j),Y^v\rangle \\ &= \frac{f_1\delta^2}{4}\sum\langle\mathcal{R}^\xi(X^h,e_j),Y^v\rangle\langle\mathcal{R}^\xi(W^h,e_j),Y^v\rangle, \end{aligned} \tag{32}$$

$$R^G(X^v,Y^h,Y^h,W^h)=R^G(W^h,Y^h,Y^h,X^v)=\frac{f_2}{2}\langle(\nabla_{Y^h}\mathcal{R}^\xi)(W^h,Y^h),X^v\rangle, \tag{33}$$

$$R^G(X^v,Y^h,Y^h,W^v)=\frac{f_2\delta}{4}\sum_j\langle\mathcal{R}^\xi(Y^h,e_j),W^v\rangle\langle\mathcal{R}^\xi(Y^h,e_j),X^v\rangle, \tag{34}$$

$$R^G(X^v,Y^v,Y^v,W^h)=0, \tag{35}$$

$$R^G(X^h,Y^h,Y^h,W^v)=\frac{f_2}{2}\langle(\nabla_{Y^h}\mathcal{R}^\xi)(X^h,Y^h),W^v\rangle \tag{36}$$

and of course $R^G(X^h, Y^v, Y^v, W^v) = 0$. The simplification in formula (32) is due to property 2 in Proposition 1.1 and the skew-symmetries of R. Henceforth the Ricci curvature of G, the trace of the Ricci endomorphism, is given by

$$\begin{aligned}
&\mathrm{ric}^G(X^h, Y^h)\\
&\quad= \sum_{i=1}^m R^G\Big(X^h, \frac{e_i}{\sqrt{f_1}}, \frac{e_i}{\sqrt{f_1}}, Y^h\Big) + R^G\Big(X^h, \frac{\theta e_i}{\sqrt{f_2}}, \frac{\theta e_i}{\sqrt{f_2}}, Y^h\Big)\\
&\quad= \mathrm{ric}(X^h, Y^h) - \frac{3}{4}\delta \sum_{j=1}^m \langle \mathcal{R}^\xi(X^h, e_j), \mathcal{R}^\xi(Y^h, e_j)\rangle\\
&\qquad\qquad + \frac{\delta}{4} \sum_{i,j=1}^m \langle \mathcal{R}^\xi(X^h, e_j), \theta e_i\rangle \langle \mathcal{R}^\xi(Y^h, e_j), \theta e_i\rangle\\
&\quad= \mathrm{ric}(X^h, Y^h) - \frac{\delta}{2} \sum_{j=1}^m \langle \mathcal{R}^\xi(X^h, e_j), \mathcal{R}^\xi(Y^h, e_j)\rangle,
\end{aligned} \tag{37}$$

$$\mathrm{ric}^G(X^v, Y^v) = \frac{\delta^2}{4} \sum_{i,j=1}^m \langle \mathcal{R}^\xi(e_i, e_j), X^v\rangle \langle \mathcal{R}^\xi(e_i, e_j), Y^v\rangle, \tag{38}$$

$$\mathrm{ric}^G(X^h, Y^v) = -\frac{\delta}{2} \sum_{i=1}^m \langle (\nabla_i \mathcal{R}^\xi)(e_i, X^h), Y^v\rangle. \tag{39}$$

And the scalar curvature is

$$\begin{aligned}
S^G &= \sum_{k=1}^m \frac{1}{f_1}\mathrm{ric}^G(e_k, e_k) + \frac{1}{f_2}\mathrm{ric}^G(\theta e_k, \theta e_k)\\
&= \frac{S}{f_1} - \frac{f_2}{4f_1^2} \sum_{i,j,k=1}^m ({\mathcal{R}^\xi}_{ijk})^2
\end{aligned} \tag{40}$$

where ${\mathcal{R}^\xi}_{ijk} = \langle \mathcal{R}^\xi(e_i, e_j), \theta e_k\rangle = \langle R(e_i, e_j)u, e_k\rangle$ on each point $u \in TM$. Of course, ric and S above denote respectively the Ricci and scalar curvatures of M.

The following result generalises another from [**9**] strictly for the Sasaki metric.

Proposition 1.3. *The Riemannian manifold* (TM, G) *is Einstein* $\Leftrightarrow$ *TM is flat* $\Leftrightarrow$ *M is flat.*

Proof. If TM is Einstein then S^G is constant. In the present case it has a quadratic part varying in $\|u\|$, unless all ${\mathcal{R}^\xi}_{ijk} = 0,\ \forall u$. □

It is worth recalling the following results. The Sasaki metric of TM is locally symmetric if and only if M is flat ([**5**]). And, regarding what we continue studying next, the tangent unit sphere bundle is locally symmetric if and only if (M, g) is flat or locally $(S^2(1), g_{\mathrm{std}})$. Conformally flat is stronger: reserved for the locally standard 2-sphere (cf. [**3**]). More recently it was proved semi-symmetric is the same as locally symmetric ([**4**]).

1.3 - *The second fundamental form of S_rM and the Ricci and scalar curvature*

Let us start by recalling the theory of the second fundamental form of a Riemannian embedding. Suppose Q^q is a submanifold of a Riemannian manifold (N^{q+p}, G) and Q inherits the induced metric from N. Let ∇' denote the Levi-Civita connection of N and let X, Y be two vectors tangent to Q. Then we have the Gauss formula

$$\nabla'_X Y = \nabla_X Y + \alpha(X, Y) \tag{41}$$

where the sum respects the orthogonal decomposition $TQ \oplus TQ^\perp$. Passed the formality, $\nabla_X Y$ is the Levi-Civita connection of Q. The clearly symmetric tensor

$$\alpha : \Omega^0(TQ \otimes TQ) \longrightarrow \Omega^0(TQ^\perp) \tag{42}$$

is called the second fundamental form. Its trace H^α is the mean curvature vector. Let $\eta \in \Omega^0(TQ^\perp)$. Then we have the Weingarten formula $\nabla'_X\eta = -A_\eta X + D_X\eta$ where A_η is a self-adjoint tensor on TQ since $\langle A_\eta X, Y\rangle = -G(\nabla'_X\eta, Y) = G(\eta, \nabla'_X Y) = G(\eta, \alpha(X, Y))$ and D is a metric connection on $TQ^\perp$. Finally we have the Gauss equation

$$R(X, Y, Z, W) = R'(X, Y, Z, W) - G(\alpha(X, Z), \alpha(Y, W)) + G(\alpha(Y, Z), \alpha(X, W)). \tag{43}$$

We now resume with the study of the induced metric $G = g^{f_1, f_2}$ on the tangent sphere bundle S_rM with radius function $r \in \mathrm{C}^\infty_M$, with $\nabla = \nabla^g$ and f_1, f_2 constant. Recall $m = n + 1$ is the dimension of M.

Proposition 1.4. $TS_rM = \{X \in TM : \langle X, \xi\rangle = rX(r)\}$.

Proof. Indeed we have $\langle \xi, \xi\rangle - r^2 = 0$ defining the submanifold. Differentiating,

$$X(\langle \xi, \xi\rangle - r^2) = 2\langle \nabla^*_X\xi, \xi\rangle - 2rX(r) = 2(\langle X^v, \xi\rangle - rX(r))$$

we find the tangent space. □

In order to write the second fundamental form, we may write α as a scalar tensor:

$$\alpha(X,Y)=G(\nabla^G_XY,U^G) \tag{44}$$

with U^G a unit vector field defined on S_rM and such that $U^G\perp^G TS_rM$. Writing

$$U^G=a\operatorname{grad}r+b\xi \tag{45}$$

for some functions a,b, we find the solution

$$a=-\delta br \qquad\text{and}\qquad b=\frac{1}{r\sqrt{f_2+\delta f_2\tau^2}} \tag{46}$$

where $\delta=f_2/f_1$ and $\tau=\|\operatorname{grad}r\|$.

Proposition 1.5. *The second fundamental form of $S_rM\subset TM$ with the induced metric g^{f_1,f_2} and where f_1,f_2 are constants, is given by*

$$\alpha(X,Y)=af_1(A(X,Y)(r)-\langle Y,\nabla_X\operatorname{grad}r\rangle)+bf_2(X(r)Y(r)-\langle Y^v,X^v\rangle). \tag{47}$$

If $\nabla\mathrm{d}r=0$, then the mean curvature is $H^\alpha=-\dfrac{n}{r\sqrt{f_2+\delta f_2\tau^2}}$.

Proof. Continuing from (44),

$$\begin{aligned}
&\alpha(X,Y)\\
&\quad=f_1\langle\nabla_XY^h+A(X,Y),a\operatorname{grad}r\rangle+f_2\langle\nabla_XY^v-\tfrac12\mathcal{R}^\xi(X,Y),b\xi\rangle\\
&\quad=af_1\langle\nabla_XY^h+A(X,Y),\operatorname{grad}r\rangle+bf_2\langle\nabla_XY^v,\xi\rangle\\
&\quad=af_1(X(Y(r))-\langle Y,\nabla_X\operatorname{grad}r\rangle+af_1A_{X,Y}(r)+bf_2(X(rY(r))-\langle Y^v,\nabla_X\xi\rangle)\\
&\quad=(af_1+bf_2r)X(Y(r))+af_1(A_{X,Y}(r)-\langle Y,\nabla_X\operatorname{grad}r\rangle)\\
&\qquad\qquad+bf_2(X(r)Y(r)-\langle Y,X^v\rangle)
\end{aligned}$$

and the result follows. For the mean curvature we take a horizontal g-orthonormal frame $e_1,\dots,e_m$ with $e_m=u/r$. Then the $Y_i=\frac{1}{\sqrt{f_2}}\theta e_i$ for $i=1,\dots,n$ constitute a vertical frame tangent to S_rM. There must also exist an extension of these vectors to an o.n. frame of T_uS_rM, and therefore a $m\times m$-matrix $a_{ip}\in\mathbb{R}$ inducing m vectors $X_i=\sum_p a_{ip}e_p+x_i\xi/r$, tangent and o.n. to each other and to the Y_j; in particular with $x_i=X_i(r)\in\mathbb{R}$. Now the condition $\nabla\operatorname{grad}r=0$ implies $A(X,Y)(r)=0$ for all X,Y because in the definition we find the symmetrization of

$$\langle\mathcal{R}^\xi(X,\operatorname{grad}r),Y\rangle=-\langle R(u,\theta^tY)\operatorname{grad}r,X^h\rangle=0.$$

Finally,

$$H^{\alpha}=\sum_{i=1}^{m}\alpha(X_i,X_i)+\sum_{j=1}^{n}\alpha(Y_j,Y_j)$$
$$=\sum bf_2(X_i(r))^2-bf_2x_i^2-\sum_j b\ =-nb\,.$$
□

So one has the formulas to compute the Riemannian curvature $\tilde{R}$ of S_rM.

From now on we assume r is a constant. Then

$$b=\frac{1}{r\sqrt{f_2}},\qquad a=-\frac{\sqrt{f_2}}{f_1}\qquad\text{and}\qquad \alpha(X,Y)=-\frac{\sqrt{f_2}}{r}\langle X^v,Y^v\rangle. \tag{48}$$

Henceforth, by Gauss formula (43), the curvature $\tilde{R}^G(X,Y,Z,W)$ does not differ from that one, given previously for the ambient manifold, except if all four vectors are vertical. Minor adaptations must follow in the Ricci and scalar curvatures, respectively $\tilde{\mathrm{ric}}^G$ and $\tilde{S}^G$, of the tangent sphere bundle.

Proposition 1.6. *With* ric^G *and* S^G *restricted to* S_rM*, we have*

1. $\tilde{\mathrm{ric}}^G=\mathrm{ric}^G+\dfrac{n-1}{r^2}g_{|_{V\otimes V}}$.
2. $\tilde{S}^G=S^G+\dfrac{(n-1)n}{f_2r^2}$.

Proof. The fibres are n-dimensional spheres. The differences $\tilde{\mathrm{ric}}^G-\mathrm{ric}^G$ and $\tilde{S}^G-S^G$ are easy to check from (48) and the Gauss equation. More closely

$$\tilde{\mathrm{ric}}^G(X,Y)=\mathrm{ric}^G(X,Y)+\frac{1}{f_2}\sum_{i=1}^{n}\tilde{R}^G(X^v,\theta e_i,\theta e_i,Y^v)$$
$$=\mathrm{ric}^G(X,Y)+\frac{1}{f_2}\sum\big(-\alpha(X,\theta e_i)\alpha(\theta e_i,Y)+\alpha(\theta e_i,\theta e_i)\alpha(X,Y)\big)$$
$$=\mathrm{ric}^G(X,Y)+\frac{n}{r^2}\langle X^v,Y^v\rangle-\frac{1}{r^2}\langle X^v,Y^v\rangle.$$

Looking at formula (37), we see the sum in i of the $R^G(X,\theta e_i,\theta e_i,Y)$ up to $m=n+1$ gives the same as the sum up to n. This is because we may take an orthonormal basis of V at each point u such that u/r is the last vector and then we notice $\langle\mathcal{R}^{\xi}(X^h,e_j),\xi\rangle=0$. Recall $u\perp T_uS_rM$ and $\xi_u=u$. The same question is not put in formulas (38, 39). The same observations are made for $\tilde{S}^G$. □

Theorem 1.2. *Let the radius* r *be a fixed constant. We have the following:*

1. *For a surface* M *the bundles* TM *and* S_rM *have the same Ricci and scalar curvatures.*

2. *Let* $m \geq 3$ *and suppose* M *has bounded sectional curvatures (e.g. if it is compact). Then:*
   (a) *for any* f_2 *there exists a sufficiently large* f_1 *such that the tangent sphere bundle* (S_rM, g^{f_1,f_2}) *has positive scalar curvature.*
   (b) *for any* f_1 *there exists a sufficiently small* f_2 *such that the tangent sphere bundle* (S_rM, g^{f_1,f_2}) *has positive scalar curvature.*

Proof. It is clear by a polarisation process that all values $\mathcal{R}^{\xi}{}_{ijk}$ in formula (40) remain bounded on S_rM. The result follows combining with Proposition 1.6. □

In the present setting, we immediately generalise Theorems 1 and 2 in [**6**].

Theorem 1.3 [**6** Theorem 1]. *Let* $\dim M \geq 3$ *and suppose* M *has bounded sectional curvatures (e.g. if it is compact). Then the tangent sphere bundle* (S_rM, g^{f_1,f_2}) *has positive scalar curvature for all sufficiently small constant radius* $r > 0$.

We just remark that [**6**, Theorem 2] essentially gives conditions for achieving *negative* scalar curvature. We may state analogous result for the weighted metric.

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