Riv. Mat. Univ. Parma, Vol. **2** (2011), 199-207

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Graph ideals with linear quotients

Abstract. Monomial ideals corresponding to complete bipartite graphs are considered and the property of having linear quotients is investigated. Standard algebraic invariants are computed.

Keywords. Graph ideals, linear quotients.

Mathematics Subject Classification (2010): 05C99, 05B35, 13C15, 13P10.

Introduction

In several papers on graph theory some algebraic properties of bipartite graphs are studied ([**2**], [**8**]). In particular, classes of monomial ideals in the polynomial ring in two sets of variables $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ over a field K associated to bipartite graphs are introduced in [**6**], [**5**].

In this paper we are interested in some classes of monomial ideals of R with linear quotients that can arise from graph theory.

Let $\{u_1, \ldots, u_t\}$ be the unique minimal set of monomial generators of an ideal $I \subset R$. We say that I has linear quotients if there is an ordering $u_1, \ldots, u_t$ with $\deg u_1 \leq \deg u_2 \leq \ldots \leq \deg u_t$ such that, for each $2 \leq j \leq t$, the colon ideal $(u_1, u_2, \ldots, u_{j-1}) : u_j$ is generated by a subset of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$. It is known that if a monomial ideal I generated in one degree has linear quotients, then I has linear resolution ([**1**]). Important monomial ideals with linear quotients in the polynomial ring in one set of variables are the Veronese ideal and the Veronese-type ideal ([**3**], [**9**]).

In order to find new classes of monomial ideals with this good property we investigate the ideals of Veronese bi-type of R introduced in [**5**]. They arise from the

Received: January 2, 2011; accepted in revised form: March 22, 2011.

walks of a bipartite graph $\mathcal{G}$ with loops and they are called generalized graph ideals. We prove that these ideals have linear quotients and as an application the standard invariants are computed.

The paper is organized as follows. In Section 1 we consider bipartite graphs with loops. A graph on vertex set $V = \{v_1, \ldots, v_n\}$ has loops if it is not requiring $v_i \neq v_j$ for all its edges $\{v_i, v_j\}$. A graph $\mathcal{G}$ is said quasi-bipartite if its vertex set V can be partitioned into disjoint subsets V_1 and V_2, any edge joins a vertex of V_1 with a vertex of V_2 and there exists some vertex of V with a loop. A quasi-bipartite graph G is strong if all the vertices of V_1 are joined to all vertices of V_2 and for each vertex of V there is a loop. The generalized ideals $I_q(\mathcal{G})$ associated to the walks of a strong quasi-bipartite graph $\mathcal{G}$ correspond to the Veronese by-type ideals $L_{q,2} = \sum_{r+s=q} I_{r,2}J_{s,2}$, where $r, s \geq 1$, $I_{r,2}$ is a Veronese type ideal generated in degree r in the variables $X_1, \ldots, X_n$ and $J_{s,2}$ is a Veronese type ideal generated in degree s in the variables $Y_1, \ldots, Y_m$ ([**5**]). We show that $I_q(\mathcal{G})$ associated to a strong quasi-bipartite graph $\mathcal{G}$ has linear quotients for $q \geq 3$ because it is an ideal of Veronese bi-type.

In Section 2 we use the property that $I_q(\mathcal{G})$ has linear quotients to investigate standard algebraic invariants of $R/I_q(\mathcal{G})$. More precisely, formulas to compute the dimension, the projective dimension, the depht, the Castelnuovo-Mumford regularity of $R/I_q(\mathcal{G})$ are stated.

1 - Linear quotients

In this section we investigate monomial ideals arising from quasi-bipartite graphs that have a good property, namely a class of monomial ideals with linear quotients. We recall some preliminary notions given in [**5**].

Definition 1.1. A graph $\mathcal{G}$ with loops is *strong quasi-bipartite* if all the vertices of V_1 are joined to all the vertices of V_2 and for each vertex of V there is a loop.

Definition 1.2. Let $\mathcal{G}$ be strong quasi-bipartite graph on n vertices. A *walk of length* q in $\mathcal{G}$ is an alternating sequence $w = \{v_{i_0}, l_{i_1}, v_{i_1}, l_{i_2}, \ldots, v_{i_{q-1}}, l_{i_q}, v_{i_q}\}$, where v_{i_j} is a vertex of $\mathcal{G}$ and $l_{i_j} = \{v_{i_{j-1}}, v_{i_j}\}$ is the edge joining $v_{i_{j-1}}$ and v_{i_j} or a loop if $v_{i_{j-1}} = v_{i_j}$, $1 \leq i_1 \leq i_2 \leq \ldots \leq i_q \leq n$.

Example 1.1. Let $\mathcal{G}$ be a strong quasi-bipartite graph on disjoint vertex sets $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2\}$. A walk of length 2 is

$$w = \{x_1, l_1, x_1, l_2, y_1\},$$

where $l_1 = \{x_1, x_1\}$ is the loop on x_1 and $l_2 = \{x_1, y_1\}$ is the edge joining x_1 and y_1 (a walk w in $\mathcal{G}$ can not have the edges $\{x_i, x_j\}$, with $i \neq j$ and $\{y_s, y_t\}$ with $s \neq t$, because $\mathcal{G}$ is bipartite).

Let $\mathcal{G}$ be a strong quasi-bipartite graph on disjoint vertex sets $V_1 = \{x_1, \dots, x_n\}$ and $V_2 = \{y_1, \dots, y_m\}$.

The *generalized ideal* $I_q(\mathcal{G})$ associated to $\mathcal{G}$ is the ideal of the polynomial ring $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ generated by the monomials of degree q corresponding to the walks of length $q-1$. Hence the generalized ideal $I_q(\mathcal{G})$ is generated by all the monomials of degree $q \geq 3$ corresponding to the walks of length $q-1$ and the variables in each generator of $I_q(\mathcal{G})$ have at most degree 2.

As described in [**5**] the ideal $I_q(\mathcal{G})$ correspond to the Veronese bi-type ideal $L_{q,2}$ of R generated in degree q. More precisely, one has

$$L_{q,2} = \sum_{r+s=q} I_{r,2} J_{s,2}, \quad r, s \geq 1,$$

where $I_{r,2}$ is the special class of ideals of Veronese-type of degree r in the variables $X_1, \dots, X_n$ generated by the set $\left\{X_1^{a_1} \cdots X_n^{a_n} \mid \sum_{i=1}^{n} a_i = r, \quad 0 \leq a_i \leq 2\right\}$ and $J_{s,2}$ is the special class of ideals of Veronese-type of degree s in the variables $Y_1, \dots, Y_m$ generated by $\left\{Y_1^{b_1} \cdots Y_m^{b_m} \middle| \sum_{j=1}^{m} b_j = s, \quad 0 \leq b_j \leq 2\right\}$.

Hence $L_{q,2} = \left(\left\{X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m} \middle| \sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = q, \quad 0 \leq a_i, b_j \leq 2\right\}\right)$.

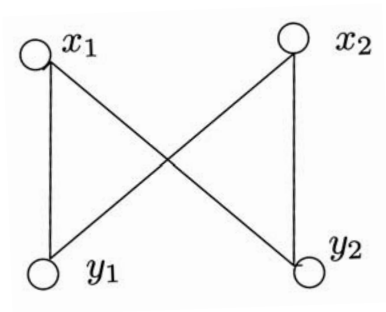
Remark 1.1. By definition the ideal $L_{q,2}$ is not trivial for $2 \leq q \leq 2(n+m) - 1$.

Example 1.2. $R = K[X_1, X_2; Y_1, Y_2]$.
$L_{4,2} = I_{3,2} J_1 + I_1 J_{3,2} + I_2 J_2 = (X_1^2 X_2 Y_1, X_1^2 X_2 Y_2, X_1 X_2^2 Y_1, X_1 X_2^2 Y_2, X_1 Y_1^2 Y_2, X_2 Y_1^2 Y_2, X_1 Y_1 Y_2^2, X_2 Y_1 Y_2^2, X_1^2 Y_1^2, X_1^2 Y_1 Y_2, X_1^2 Y_2^2, X_2^2 Y_1^2, X_2^2 Y_2^2, X_2^2 Y_1 Y_2, X_1 X_2 Y_1^2, X_1 X_2 Y_2^2, X_1 X_2 Y_1 Y_2)$.

Therefore we can write:

$$I_q(\mathcal{G}) = L_{q,2} = \sum_{r+s=q} I_{r,2} J_{s,2}, \text{ for } q \geq 3.$$

Example 1.3. Let $R = K[X_1, X_2; Y_1, Y_2]$ and $\mathcal{G}$ be the strong quasi-bipartite graph on disjoint vertex sets $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2\}$:



A walk of length 6 in $\mathcal{G}$ is: $w = \{x_1, l_1, x_1, l_2, y_1, l_3, y_1, l_4, x_2, l_5, x_2, l_6, y_2\}$, where $l_1 = \{x_1, x_1\}$ is the loop on x_1, $l_2 = \{x_1, y_1\}$ is the edge joining x_1 and y_1, $l_3 = \{y_1, y_1\}$ is the the loop on y_1, $l_4 = \{x_2, y_1\}$ is the edge joining x_2 and y_1, $l_5 = \{x_2, x_2\}$ is the loop on x_2, $l_6 = \{x_2, y_2\}$ is the edge joining x_2 and y_2. The walk w corresponds to the monomial $X_1^2X_2^2Y_1^2Y_2$ of R.

All the walks of length 6 correspond to the generators of the generalized ideal of degree $q = 7$:

$$I_7(\mathcal{G}) = L_{7,2} = (X_1^2X_2^2Y_1^2Y_2, X_1^2X_2^2Y_1Y_2^2, X_1^2X_2Y_1^2Y_2^2, X_1X_2^2Y_1^2Y_2^2).$$

Remark 1.2. For $q = 2$ the ideal $L_{q,2}$ doesn't describe the edge ideal $I(\mathcal{G}) = I_2(\mathcal{G})$ of a strong quasi-bipartite graph. In fact, if we consider the strong quasi-bipartite graph on vertices x_1, x_2, y_1, y_2 then

$$I(\mathcal{G}) = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2, X_1^2, X_2^2, Y_1^2, Y_2^2),$$

but $L_{2,2} = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2)$. Hence $I(\mathcal{G}) \neq L_{2,2}$.

For this reason in the sequel we consider $I_q(\mathcal{G})$ for $q \geq 3$.

The following definitions give some algebraic properties of monomial ideals, our aim is to investigate them for $I_q(\mathcal{G})$, where $q \geq 3$ and $\mathcal{G}$ is a strong quasi-bipartite graph.

Definition 1.3. Let $L \subset R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be a monomial ideal and $G(L)$ be its unique set of minimal generators. L satisfies the *bi-exchange condition* if for all pairs of monomials $u = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}$ and $v = X_1^{c_1} \cdots X_n^{c_n} Y_1^{d_1} \cdots Y_m^{d_m}$ in $G(L)$ and for each i with $a_i > c_i$ or k with $b_k > d_k$ there exist $j \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m\}$ such that $a_j < c_j$ or $b_l < d_l$ and $X_j u / X_i \in G(L)$ or $Y_l u / Y_k \in G(L)$.

Definition 1.4. Let $L \subset R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be a monomial ideal and $G(L)$ be its unique set of minimal generators. L has *linear quotients* if there is an ordering $u_1, \ldots, u_t$ of the monomials belonging to $G(L)$ with

$\deg(u_1) \leq \cdots \leq \deg(u_t)$ such that for each $2 \leq j \leq t$ the colon ideal $(u_1, \ldots, u_{j-1}) : u_j$ is generated by a subset of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$.

Proposition 1.1. *Let $\mathcal{G}$ be a strong quasi-bipartite graph, then $I_q(\mathcal{G})$ satisfies the bi-exchange condition, for $q \geq 3$.*

Proof. Let $I_q(\mathcal{G})$ and $G(I_q(\mathcal{G}))$ be its unique set of minimal generators. Set $u = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}, v = X_1^{c_1} \cdots X_n^{c_n} Y_1^{d_1} \cdots Y_m^{d_m} \in G(I_q(\mathcal{G}))$, then $X_1^{a_1} \cdots X_n^{a_n} \in I_{r,2}$ and $Y_1^{b_1} \cdots Y_m^{b_m} \in J_{s,2}$ such that $\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = q$, and $0 \leq a_i, b_j \leq 2$. Let $B_q = \{(a; b) \in \mathbb{Z}_+^{n+m} : |a| = r, |b| = s, r + s = q, 0 \leq a_i, b_j \leq 2\}$ be the set of the vector exponents of the elements of $G(I_q(\mathcal{G}))$. Let $(a, b), (c, d) \in B_q$ with $a_i > c_i$ or $b_k > d_k$, for some $j \in \{1, \ldots, n\}$ with $a_j < c_j$ and $l \in \{1, \ldots, m\}$ with $b_l < d_l$, one has $(a; b) - (e_i; 0) + (e_j; 0) = (a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_n; b_1, \ldots, b_m)$ and $(a; b) - (0; e'_k) + (0; e'_l) = (a_1, \ldots, a_n; b_1, \ldots, b_k - 1, \ldots, b_l + 1, \ldots, b_m)$, where e_i and e'_k denote the standard basis vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Hence $(a; b) - (e_i; 0) + (e_j; 0) \in B_q$ or $(a; b) - (0; e'_k) + (0; e'_l) \in B_q$ by construction. It follows that $X_j u / X_i \in G(I_q(\mathcal{G}))$ or $Y_l u / Y_k \in G(I_q(\mathcal{G}))$. □

Theorem 1.1. *Let $\mathcal{G}$ be a strong quasi-bipartite graph, then $I_q(\mathcal{G})$ has linear quotients, for $q \geq 3$.*

Proof. Let $G(I_q(\mathcal{G}))$ be the unique set of minimal generators of $I_q(\mathcal{G})$. Let $u \in G(I_q(\mathcal{G}))$, we set $I = (v \in G(I_q(\mathcal{G})) | \ v \prec u)$ with $\prec$ the lexicographical order. We prove that $I : u = (v/GCD(u, v) | v \in I)$ is generated by a subset of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$. Therefore we must prove that for all $v \prec u$ there exists a variable of R in $I : u$ such that it divides $v/GCD(u, v)$. Let $u = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}$ and $v = X_1^{c_1} \cdots X_n^{c_n} Y_1^{d_1} \cdots Y_m^{d_m}$ in $G(I_q(\mathcal{G}))$. Since $v \prec u$ there exists an integer $i \in \{1, \ldots, n\}$ with $a_i > c_i$ and $a_k = c_k$ for $k = 1, \ldots, i - 1$. Hence by Proposition 1.1 there exists an integer $j \in \{1, \ldots, n\}$ with $c_j > a_j$ such that $w = X_j(u/X_i) \in G(I_q(\mathcal{G}))$. Since $j > i$, it follows that $w \in I$ and $w = X_j(u/X_i) \in G(I_q(\mathcal{G}))$ implies $wX_i = X_j u$, that is $X_j \in I : u$. Since the j-th component of the vector exponent of $v/GCD(u, v)$ is given by $c_j - \min\{c_j, a_j\} = c_j - a_j > 0$, then X_j divides $v/GCD(u, v)$ as required. If we suppose that $a_k = c_k$ for all $k = 1, \ldots, n$, $b_i > d_i$ and $b_l = d_l$ for all $l = 1, \ldots, i - 1, i \in \{1, \ldots, m\}$, then we obtain $Y_j \in I : u$ and Y_j divides $v/GCD(u, v)$. The thesis follows. □

Corollary 1.1. *Let $\mathcal{G}$ be a strong quasi-bipartite graph, then $I_q(\mathcal{G})$ has linear resolution, for $q \geq 3$.*

Proof. It is known that if a monomial ideal generated in the same degree has linear quotients, then it has a linear resolution ([**1**]). □

2 - Applications

Let $\mathcal{G}$ be a strong quasi-bipartite graph. Let $I_q(\mathcal{G})$ be the ideal of R with linear quotients with respect to the ordering $u_1, \ldots, u_t$ of the monomials of $G(I_q(\mathcal{G}))$. We denote by $q_j(I_q(\mathcal{G}))$ the number of the variables which is required to generate the ideal $(u_1, \ldots, u_{j-1}) : u_j$.

Set $q(I_q(\mathcal{G})) = \max_{2 \leq j \leq t} q_j(I_q(\mathcal{G}))$. The integer $q(I_q(\mathcal{G}))$ is independent of the choice of the ordering of the generators that gives linear quotients ([**3**]).

A *vertex cover* of $I_q(\mathcal{G})$ is a subset W of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$ such that each $u \in G(I_q(\mathcal{G}))$ is divided by some variables of W.

Denote by $h(I_q(\mathcal{G}))$ the minimal cardinality of the vertex covers of $I_q(\mathcal{G})$.

Now we investigate some algebraic invariants of $R/I_q(\mathcal{G})$.

Theorem 2.1. *Let* $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ *with* $n, m > 1$ *and* $I_q(\mathcal{G})$ *with* $3 \leq q \leq 2(n + m - 1)$. *Then:*

1) $\dim_R(R/I_q(\mathcal{G})) = n + m - \min\{n, m\}$
2) $\mathrm{pd}_R(R/I_q(\mathcal{G})) = n + m$
3) $\mathrm{depth}_R(R/I_q(\mathcal{G})) = 0$
4) $\mathrm{reg}_R(R/I_q(\mathcal{G})) = q - 1$.

Proof. Let $I_q(\mathcal{G}) = \left(\left\{X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m} \,\middle|\, \sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = q,\ 0 \leq a_i, b_j \leq 2\right\}\right)$ with $3 \leq q \leq 2(n + m - 1)$. We order the generators of $I_q(\mathcal{G})$ with respect to the lexicographical order with $X_1 \succ X_2 \succ \cdots \succ X_n \succ Y_1 \succ Y_2 \succ \cdots \succ Y_m$. Let $j \in \{1, \ldots, n\}$ such that $2(j-1) + a_j = q - 1$, $a_j = 1, 2$, we have: $X_1^2 X_2^2 \cdots X_{j-1}^2 X_j^{a_j} Y_1 \succ X_1^2 X_2^2 \cdots X_{j-1}^2 X_j^{a_j} Y_2 \succ \cdots \succ X_1^2 X_2^2 \cdots X_{j-1}^2 X_j^{a_j} Y_m \succ \cdots$ and so on up to $X_n Y_l^{b_l} Y_{l+1}^2 \cdots Y_m^2$, with $2(m - l) + b_l = q - 1$. By the computation of the linear quotients it follows that the maximum system of their generators is $\{X_1, \ldots, X_n; Y_1, \ldots, Y_{m-1}\}$. Hence $q(I_q(\mathcal{G})) = n + m - 1$. The minimal cardinality of the vertex covers of $I_q(\mathcal{G})$ is $h(I_q(\mathcal{G})) = \min\{n, m\}$ being $\mathcal{G}$ a quasi-bipartite graph. Hence:

1) $\dim_R (R/I_q(\mathcal{G})) = \dim_R R - h(I_q(\mathcal{G})) = n + m - \min\{n, m\}$ ([**3**]).

2) The length of the minimal free resolution of $R/I_q(\mathcal{G})$ over R is equal to $q(I_q(\mathcal{G})) + 1$ ([**4**], Corollary 1.6). Then $\mathrm{pd}_R(R/I_q(\mathcal{G})) = n + m$.

3) is a consequence of 1) and 2), by using Auslander-Buchsbaum formula

$$\mathrm{depth}_R(R/I_q(\mathcal{G})) = n + m - \mathrm{pd}_R(R/I_q(\mathcal{G})) = 0.$$

4) $I_q(\mathcal{G})$ has linear resolution, then $\mathrm{reg}_R(R/I_q(\mathcal{G})) = q - 1$. □

Theorem 2.2. *Let* $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ *with* $n, m > 1$ *and* $I_q(\mathcal{G})$ *with* $q = 2(n + m) - 1$. *Then:*

1) $\dim_R(R/I_q(\mathcal{G})) = n + m - 1$
2) $\mathrm{pd}_R(R/I_q(\mathcal{G})) = 2$
3) $\mathrm{depth}_R(R/I_q(\mathcal{G})) = n + m - 2$
4) $\mathrm{reg}_R(R/I_q(\mathcal{G})) = 2(n + m - 1)$.

Proof. Let $q = 2(n + m) - 1$. The generators of $I_q(\mathcal{G})$ are the following:

$$\begin{aligned}
f_1 &= X_1^2X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}^2Y_m,\\
f_2 &= X_1^2X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}Y_m^2,\\
f_3 &= X_1^2X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-2}Y_{m-1}^2Y_m^2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
f_{n+m-1} &= X_1^2X_2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}^2Y_m^2,\\
f_{n+m} &= X_1X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}^2Y_m^2.
\end{aligned}$$

The linear quotients are:

$$(f_1) : (f_2) = (Y_{m-1}),\ (f_1,f_2) : (f_3) = (Y_{m-2}),\ \dots,\ (f_1, \dots, f_{m-1}) : (f_m) = (Y_1),$$
$$(f_1, \dots, f_m) : (f_{m+1}) = (X_n),\ \dots,\ (f_1, \dots, f_{m+n-1}) : (f_{m+n}) = (X_1).$$

It follows that $q(I_q(\mathcal{G})) = 1$. The minimal cardinality of the vertex covers of $I_q(\mathcal{G})$ is $h(I_q(\mathcal{G})) = 1$ being $W = \{X_1\}$ a minimal vertex cover of $I_q(\mathcal{G})$. Hence:

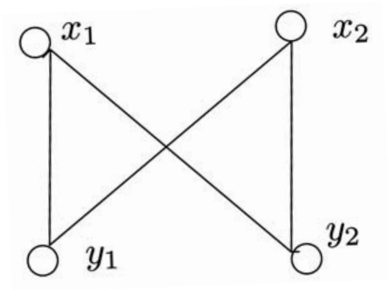
1) $\dim_R(R/I_q(\mathcal{G})) = \dim_R R - h(I_q(\mathcal{G})) = n + m - 1$ ([**3**]).

2) The length of the minimal free resolution of $R/I_q(\mathcal{G})$ over R is equal to $q(I_q(\mathcal{G})) + 1$ ([**4**], Corollary 1.6). Hence $\mathrm{pd}_R(R/I_q(\mathcal{G})) = 2$.

3) As a consequence of 1) and 2) we compute $\mathrm{depth}_R(R/I_q(\mathcal{G})) = n + m - \mathrm{pd}_R(R/I_q(\mathcal{G})) = n + m - 2$.

4) $\mathrm{reg}_R(R/I_q(\mathcal{G})) = q - 1 = 2(n + m) - 1 - 1 = 2(n + m - 1)$ because $I_q(\mathcal{G})$ has linear resolution. □

Example 2.1. Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring over a field K and $\mathcal{G}$ be the strong quasi-bipartite graph:



$I_7(\mathcal{G}) = (X_1^2X_2^2Y_1^2Y_2, X_1^2X_2^2Y_1Y_2^2, X_1^2X_2Y_1^2Y_2^2, X_1X_2^2Y_1^2Y_2^2)$.

Set $f_1 = X_1^2X_2^2Y_1^2Y_2, f_2 = X_1^2X_2^2Y_1Y_2^2, f_3 = X_1^2X_2Y_1^2Y_2^2, f_4 = X_1X_2^2Y_1^2Y_2^2$.

The linear quotients are:

$\mathcal{I}_2 = (f_1) : (f_2) = (Y_1)$
$\mathcal{I}_3 = (f_1, f_2) : (f_3) = (X_2)$
$\mathcal{I}_4 = (f_1, f_2, f_3) : (f_4) = (X_1)$.

Then $q(I_7(\mathcal{G})) = \max_{2\leq i\leq 4}\{q_i(I_7(\mathcal{G}))\} = 1$.

The minimal cardinality of a vertex cover of $I_7(\mathcal{G})$ is $h(I_7(\mathcal{G})) = 1$ and $W = \{X_1\}$ is such a vertex cover. Then:

1) $\dim_R(R/I_7(\mathcal{G})) = 3$
2) $\mathrm{pd}_R(R/I_7(\mathcal{G})) = 2$
3) $\mathrm{depth}_R(R/I_7(\mathcal{G})) = 2$
4) $\mathrm{reg}_R(R/I_7(\mathcal{G})) = 6$.

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