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Near-rings arising from coupling maps

Abstract. By contemporaneous consideration of coupling maps and of "moltiplicative" endomorphisms, a class of near-rings is given. We prove in particular that various of them are local.

Keywords. Near-ring, coupling map, local near-ring.

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1 - Introduction

In this article, all near-rings will be assumed to be left near-rings, that is in which the product is distributive on the left with respect to the sum (see for example [**1**], [**2**], [8]). In order to summarize completely this concept, we recall that a (left) near-ring is a structure $[N; +, \cdot]$, with two operations, addition and multiplication, defined onto N, such that *i*) $[N; +]$ is a group; *ii*) $[N; \cdot]$ is a semigroup; *iii*) $\forall x, y, z \in N \;\; x \cdot (y + z) = x \cdot y + x \cdot z$. A near-ring N is called zero-symmetric ([**1**], Def. 3.2), if $0 \cdot x = 0$ for any $x \in N$, and N is called near-field ([**1**], Def. 2.17) if $[N \setminus \{0\}; \cdot]$ is a group. According to [**5**], Def. 2.1, a zero-symmetric near-ring N with (multiplicative) identity is said to be local if the set $L(N)$ of elements of N without right inverses is a subgroup of $[N; +]$, such that $L(N)N \subseteq N$. A zero-symmetric near-ring N with identity is local if and only if $L(N)$ is a subgroup of $[N; +]$ (by Theorem 2.3 of [**5**], and we notice that in [**5**], [**6**], [**7**] notations for right near-rings, instead of for left, are used).

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We now follow the terminology of [**7**]. Given a ring $[A;+,\cdot]$, a coupling map for A is defined as a function $\varphi : A \to End[A;+,\cdot]$ [1] $(a \to \varphi_a)$ such that $\varphi_0 = 0_A$, and, for any $a, b \in A$, $\varphi_a \circ \varphi_b = \varphi_{a\varphi_a(b)}$. From a coupling map φ for A, a (left) near-ring $[A;+,\circ]$ is introduced, which is said to be the near-ring coupled with A by φ ([**7**] page 155), where $\circ$ is given by $a \circ b = a\varphi_a(b)$ (which implies $\varphi_{a\circ b} = \varphi_a \circ \varphi_b$). In this paper, we generalize and we complicate the description of the local near-rings on elementary abelian p-groups of order p^2 of [**6**], starting from a near-ring $[A;+,\circ]$ coupled with a ring $[A;+,\cdot]$ by a suitable coupling map φ. We introduce a function ρ of $A \setminus \{0\}$ into $A \setminus \{0\}$ satisfying appropriate hypotheses with respect to the previous φ. From φ and from ρ we derive an operation $\circ_1$ onto $A \times A$, finding a class of near-rings; we verify that these near-rings, under some conditions, are local. Furthermore we classify, by a necessary and sufficient condition, all S-near-rings ([**4**]), among the near-rings in such a way introduced. In Section 3, we analyze when, given (arbitrarily) two, of the near-rings here obtained, they can be non-isomorphic.

For further generalities on the theory of near-rings, we refer to the treatises [**1**], [**2**], [8], [**9**],[**10**].

2 - A class of near-rings

Let $[A;+,\cdot]$ be a ring with identity $1 \neq 0$ without divisors of zero. The symbol id_A will be used to designate the identity map on A. Put $A^* = A \setminus \{0\}$. Let φ be a coupling map for A, with φ_a injective for every $a \in A^*$. Since 1 is the only idempotent in $[A^*;\cdot]$, we have then $\varphi_x(1) = 1$ for all $x \in A^*$. Let $R = [A;+,\circ]$ be the (left) near-ring coupled with $[A;+,\cdot]$ by φ. We recall that, here, $x \circ y = x\varphi_x(y)$ and $\varphi_{x\circ y} = \varphi_x \circ \varphi_y$. On the basis of our hypotheses, each element of A^* is left cancelable with respect to $\circ$. By theorem 4.4 of [**7**], R has identity 1. [2] Moreover, 1 is the unique idempotent in $[A^*;\circ]$. [3] It is now clear that 1 is fixed by every endomorphism of $[A^*;\circ]$. For the given $[A;+,\cdot]$ and φ, we denote by A^*_φ the set of all endomorphisms ρ of $[A^*;\circ]$ verifying the condition

$$\forall x \in A^* \quad \varphi_x = \varphi_{\rho(x)}.$$

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[1] We will note 0_A the function $A \to A$ which sends every element of A to 0.

[2] For a direct verification, we remember that $\varphi_1(1) = 1$; thus $\varphi_1 = \varphi_{1\varphi_1(1)} = \varphi_1 \circ \varphi_1$. Since φ_1 is injective, $\varphi_1 = id_A$; hence 1 is identity for R.

[3] If $e \neq 0$ is such that $e \circ e = e$, then $e \circ e = e \circ 1$, which implies $e = 1$, by left-cancellability of e.

Examples. First of all, if $[M;\cdot]$ is any monoid and if $z \in M$ has (two-sided) inverse z^{-1}, we will call inner automorphism of M induced by z the function $M \to M$, $y \to z^{-1}yz$. Now we consider the above mentioned structure $[A^*;\circ]$ and we observe that, if β is an inner automorphism of $[A^*;\circ]$ induced by an element z (endowed with inverse in $[A^*;\circ]$, which we denote $z^{\underline{-1}}$ by [**10**] page 69) with φ_z belonging to the center of $[\varphi(A);\circ]$, then $\beta \in A^*_\varphi$. Indeed, by our assumption, $\forall x \in A^*$

$$\varphi_{\beta(x)} = \varphi_{z^{\underline{-1}}\circ x\circ z} = \varphi_z^{-1} \circ \varphi_x \circ \varphi_z = \varphi_x$$

(see also [**10**], page 69).

Moreover, let $[C;+,\cdot]$ be an unitary integral domain, $T \in Aut\,[C;+,\cdot]$, $B = C[X]$. We denote by $\overline{T}$ the natural extension of T to B (defined by $\overline{T}(c_0 + \ldots + c_nX^n) = T(c_0) + \ldots + T(c_n)X^n$). Assume the order of T in $Aut\,[C;+,\cdot]$ to be a natural number h. Let m be a natural number congruent to 1 mod h. Consider the coupling map φ' for B defined by (see also [**7**], paragraph 6):

$$\varphi'_0 = 0_B, \quad \forall f \in B^* = B \setminus \{0\} \qquad \varphi'_f = \overline{T}^{\deg f}.$$

Then the function $\rho' : B^* \to B^*$, $\rho' : f \to f^m$ (f^m calculated in $[B;\cdot]$) is an element of $B^*_{\varphi'}$ (by our general notations). In fact, $\overline{T}^m = \overline{T}$, hence for $f \in B^*$ we have

$$\varphi'_{\rho'(f)} = \varphi'_{f^m} = \overline{T}^{\deg f^m} = (\overline{T}^m)^{\deg f} = \overline{T}^{\deg f} = \varphi'_f,$$

and, given the near-ring $[B;+,\circ']$, coupled with $[B;+,\cdot]$ by φ', for $f,g \in B^*$, the following steps holds

$$\rho'(f) \circ' \rho'(g) = f^m \circ' g^m = f^m\varphi'_{f^m}(g^m) = f^m \cdot (\overline{T}^{\deg f^m}(g^m)) = f^m \cdot ((\overline{T}^m)^{\deg f}(g^m))$$
$$= f^m \cdot (\overline{T}^{\deg f}(g^m)) = f^m \cdot (\overline{T}^{\deg f}(g))^m = (f \cdot (\overline{T}^{\deg f}(g)))^m = (f\varphi'_f(g))^m$$
$$= (f \circ' g)^m = \rho'(f \circ' g),$$

hence ρ' is also endomorphism of $[B^*;\circ']$.

Now, we consider $[A;+,\cdot]$, φ, R as defined before these examples, in this section, and, from now on, we denote by ρ a fixed (on the other hand, arbitrary) element of A^*_φ.

We introduce a structure $R_1 = [A \times A;+,\circ_1]$ in the following way. Let (x_1,x_2), $(y_1,y_2) \in A \times A$. We define $(x_1,x_2)+(y_1,y_2) = (x_1+x_2, y_1+y_2)$. Moreover, if $x_1 \neq 0$, then put

$$(x_1,x_2) \circ_1 (y_1,y_2) = (x_1 \circ y_1, x_2\varphi_{x_1}(y_1) + \rho(x_1)\varphi_{x_1}(y_2)),$$

and, if $x_1 = 0$, then put $(0,x_2) \circ_1 (y_1,y_2) = (0, x_2 \circ y_1)$. Let $J = \{0\} \times A$. We observe that, since $\rho \in A^*_\varphi$, for $x_1 \in A^*$ we have

$$\rho(x_1)\varphi_{x_1}(y_2) = \rho(x_1)\varphi_{\rho(x_1)}(y_2) = \rho(x_1) \circ y_2.$$

Theorem 2.1. *The structure* R_1 *is a (zero-symmetric) near-ring having identity* $(1,0)$*, in which* J *is an ideal with* $\frac{R_1}{J}$ *isomorphic to* R*. Furhermore,* J *coincides with the set of nilpotent elements of* R_1*, and* $J \circ_1 J = \{(0,0)\}$.

Proof. We verify the associativity of $\circ_1$. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in A \times A$. Suppose $x_1 \neq 0, y_1 \neq 0$. From the definition of $\circ_1$, we can write

$$((x_1,x_2) \circ_1 (y_1,y_2)) \circ_1 (z_1,z_2) = (x_1 \circ y_1, x_2\varphi_{x_1}(y_1) + \rho(x_1)\varphi_{x_1}(y_2)) \circ_1 (z_1,z_2)$$

$$= (x_1 \circ y_1 \circ z_1, (x_2\varphi_{x_1}(y_1) + \rho(x_1)\varphi_{x_1}(y_2))\varphi_{x_1\circ y_1}(z_1) + \rho(x_1 \circ y_1)\varphi_{x_1\circ y_1}(z_2)).$$

We recall that, since $\rho \in A^*_\varphi$, we have $\rho(x_1 \circ y_1) = \rho(x_1) \circ \rho(y_1)$. Therefore the last ordered pair is equal to

$$(x_1 \circ y_1 \circ z_1, x_2\varphi_{x_1}(y_1)\varphi_{x_1\circ y_1}(z_1) + \rho(x_1)\varphi_{x_1}(y_2)\varphi_{x_1\circ y_1}(z_1) + (\rho(x_1) \circ \rho(y_1))\varphi_{x_1\circ y_1}(z_2)).$$

We bear in mind that (1) φ_{x_1} is an endomorphism of $[A;+,\cdot]$, (2) $\varphi_{x_1} = \varphi_{\rho(x_1)}$, (3) $y_1 \circ z_1 = y_1\varphi_{y_1}(z_1)$; we then have

$$\begin{aligned}
&(x_1,x_2) \circ_1 ((y_1,y_2) \circ_1 (z_1,z_2)) = (x_1,x_2) \circ_1 (y_1 \circ z_1, y_2\varphi_{y_1}(z_1) + \rho(y_1)\varphi_{y_1}(z_2))\\
&= (x_1 \circ y_1 \circ z_1, x_2\varphi_{x_1}(y_1 \circ z_1) + \rho(x_1)\varphi_{x_1}(y_2\varphi_{y_1}(z_1) + \rho(y_1)\varphi_{y_1}(z_2)))\\
&= (x_1 \circ y_1 \circ z_1, x_2\varphi_{x_1}(y_1\varphi_{y_1}(z_1)) + \rho(x_1)\varphi_{x_1}(y_2)(\varphi_{x_1} \circ \varphi_{y_1})(z_1)\\
&\qquad + \rho(x_1)\varphi_{x_1}(\rho(y_1))(\varphi_{x_1} \circ \varphi_{y_1})(z_2))\\
&= (x_1 \circ y_1 \circ z_1, x_2\varphi_{x_1}(y_1)(\varphi_{x_1} \circ \varphi_{y_1})(z_1)\\
&\qquad + \rho(x_1)\varphi_{x_1}(y_2)(\varphi_{x_1} \circ \varphi_{y_1})(z_1) + \rho(x_1)\varphi_{\rho(x_1)}(\rho(y_1))(\varphi_{x_1} \circ \varphi_{y_1})(z_2)).
\end{aligned}$$

In view of the identities $\varphi_{x_1} \circ \varphi_{y_1} = \varphi_{x_1\circ y_1}$, $\rho(x_1)\varphi_{\rho(x_1)}(\rho(y_1)) = \rho(x_1) \circ \rho(y_1)$, we now have

$$((x_1,x_2) \circ_1 (y_1,y_2)) \circ_1 (z_1,z_2) = (x_1,x_2) \circ_1 ((y_1,y_2) \circ_1 (z_1,z_2)).$$

If $x_1 \neq 0, y_1 = 0$, our argument is analogous (here, we recall the equalities before the statement); remaining cases are simpler. Hence we have the associativity of $\circ_1$. Since any φ_x is an endomorphism of $[A;+]$ and since $\circ$ is left distributive with respect to the sum onto A, the operation $\circ_1$ is distributive on the left with respect to the sum defined onto $A \times A$. Further, we see that the function $A \times A \to A$, $(x_1,x_2) \to x_1$ is an epimorphism from the near-ring R_1 to the near-ring R, whose kernel is J.

Finally, in consideration of the equality $\rho(1) = 1$, and since 1 is identity in R, $(1,0)$ is identity in R_1. The rest is clear. □

Theorem 2.2. *If* R *is a local near-ring (in particular, if* R *is a near-field), then* R_1 *is local also.*

Proof. Suppose that R is a local near-ring. We denote by L the set of elements of A without right inverses with respect to $\circ$, and by U the set $A \setminus L$. We have $\varphi(U) \subseteq Aut\,[A;+,\cdot]$[4]. We note that $L \times A$ is a subgroup of $[A \times A;+]$, since L is a subgroup of $[A;+]$. We assert that $L \times A$ consists of elements without right inverses with respect to $\circ_1$. Indeed suppose, on the contrary, that, for a $(u,v) \in L \times A$, there is a $(t,w) \in A \times A$ such that $(u,v) \circ_1 (t,w) = (1,0)$. Then $u \circ t = 1$, which contradicts the fact that u is in L. Hence, it remains to show that each element of

$$U \times A = (A \times A) \setminus (L \times A)$$

is endowed with right inverse with respect to $\circ_1$ (see also Theorem 2.3 of [**5**]). We take (x_1, x_2) in $U \times A$. We demonstrate that there is an $(y_1, y_2) \in A \times A$ such that $(x_1, x_2) \circ_1 (y_1, y_2) = (1,0)$. This last condition is equivalent to the system

$$\begin{cases} x_1 \circ y_1 = 1, \\ x_2 \varphi_{x_1}(y_1) + \rho(x_1)\varphi_{x_1}(y_2) = 0. \end{cases}$$

Here y_1 is uniquely determined as the two-sides inverse of x_1 in $[A;\circ]$ (Lemma 2.4 of [**5**]). From $x_1 \circ y_1 = y_1 \circ x_1 = 1$, we have $\rho(x_1) \circ \rho(y_1) = \rho(y_1) \circ \rho(x_1) = \rho(1) = 1$, that is $\rho(x_1)\varphi_{\rho(x_1)}(\rho(y_1)) = \rho(y_1)\varphi_{\rho(y_1)}(\rho(x_1)) = 1$. Then $\rho(x_1), \rho(y_1)$ are in U; thus $\varphi_{\rho(x_1)}, \varphi_{\rho(y_1)}$ are automorphisms of $[A;+,\cdot]$, and consequently $\rho(x_1)$ is invertible in $[A;\cdot]$. Since $x_1 \in U$, we have that φ_{x_1} is an automorphism of $[A;+,\cdot]$ too. Therefore, from the second equality of the previous system, the element y_2 is uniquely determined also. □

If $[N;+,\cdot]$ is a near-ring, for every $a \in N$ define $A_l(a) = \{z \in N | za = 0\}$. In [**4**], a near-ring N is said to be an S-near-ring if, $\forall a \in N$, the relation S_a defined onto N by $xS_ay \Leftrightarrow xa = ya$ is a congruence of N. From [**3**], [**4**] it is clear that a zero-symmetric near-ring N is S-near-ring if and only if, $\forall a \in N$, $A_l(a)$ is an ideal, such that $\forall x \in N$ $[x]_{S_a} = x + A_l(a)$.

Theorem 2.3. *The near-ring R_1 is S-near-ring if and only if:*
1. *ρ is injective,*
2. *each non-null element of A is cancelable with respect to $\circ$.*

Proof. We remember that every non-null element of A is left cancelable with respect to $\circ$. Suppose that 1., 2. holds. Let $a = (0, y_2) \in A \times A$, $y_2 \neq 0$. It is

[4] In fact, by Lemma 2.4 of [**5**], every element of U is invertible in $[A;\circ]$. Then, for a generic $a \in U$, there is $b \in U$ such that $a \circ b = b \circ a = 1$; thus $\varphi_{a \circ b} = \varphi_{b \circ a} = \varphi_1$, which implies $\varphi_a \circ \varphi_b = \varphi_b \circ \varphi_a = id_A$, so φ_a is an automorphism of $[A;+,\cdot]$.

immediate that $A_l(a) = J$[5]. We demonstrate that two elements of $A \times A$ are equivalent with respect to S_a[6] if and only if they belongs to the same coset of J. Suppose $(x_1, x_2)S_a(x'_1, x'_2)$ with $x_1 \neq 0$ (hence also $x'_1 \neq 0$). We have $(x_1, x_2) \circ_1 (0, y_2) = (x'_1, x'_2) \circ_1 (0, y_2)$, that is $(0, \rho(x_1) \circ y_2) = (0, \rho(x'_1) \circ y_2)$. Because of 2., $\rho(x_1) = \rho(x'_1)$; therefore, by 1., $x_1 = x'_1$, so $(x_1, x_2) + J = (x'_1, x'_2) + J$. If $x_1 = 0$ then also $x'_1 = 0$, hence $(x_1, x_2) + J = J = (x'_1, x'_2) + J$. Conversely, by a direct verification we see that, for arbitrary $x_1, x_2 \in A$, any two elements of $(x_1, x_2) + J = (x_1, 0) + J$ are equivalent with respect to S_a. For all $b \notin J$ we demonstrate that S_b is the discrete relation, so is congruence. We take $b = (y_1, y_2)$, $y_1 \neq 0$. Let $(x_1, x_2)S_b(x'_1, x'_2)$. If $x_1 \neq 0$, then $x'_1 \neq 0$, and $x_1 \circ y_1 = x'_1 \circ y_1$. From 2., $x_1 = x'_1$. Therefore,

$$x_2\varphi_{x_1}(y_1) + \rho(x_1)\varphi_{x_1}(y_2) = x'_2\varphi_{x_1}(y_1) + \rho(x_1)\varphi_{x_1}(y_2);$$

since $y_1 \neq 0$, this implies $x_2 = x'_2$. If $x_1 = 0$, then $x'_1 = 0$, and $(0, x_2 \circ y_1) = (0, x'_2 \circ y_1)$, thus, by 2., $x_2 = x'_2$. Consequently, R_1 is an S-near-ring.

We assume now R_1 to be S-near-ring. Consider elements $x_1, x_2 \in A^*$ such that $x_1 \neq x_2$. We prove that $\rho(x_1) \neq \rho(x_2)$. Suppose on the contrary that $\rho(x_1) = \rho(x_2)$. Then there are two elements of $A \times A$ equivalent with respect to $S_{(0,1)}$, belonging to two distinct cosets of $J = A_l((0, 1))$, that is the elements $(x_1, 0)$, $(x_2, 0)$:

$$(x_1, 0) \circ_1 (0, 1) = (0, \rho(x_1)) = (0, \rho(x_2)) = (x_2, 0) \circ_1 (0, 1).$$

This contradicts the fact that R_1 is S-near-ring. Let $x_1, x'_1 \in A$, $x \in A^*$, $x_1 \circ x = x'_1 \circ x$. We assert that $x_1 = x'_1$. Assume $x_1 \neq x'_1$. We then have

$$(x_1, 0) \circ_1 (x, 0) = (x'_1, 0) \circ_1 (x, 0)$$

with $(x_1, 0), (x'_1, 0)$ belonging to two distinct cosets of $A_l((x, 0)) = \{(0, 0)\}$, a contradiction. Thus, 1.,2. holds. □

3 - Equivalence

Throughout this section, in addition to the previous ρ and R_1, we consider another (arbitrary) element γ of A^*_φ, and, if f is any automorphism of $[A \times A; +]$ (in which $+$ is the componentwise operation as above), then we will note $f_1 = \pi_1 \circ f \circ i_1$, $f_2 = \pi_2 \circ f \circ i_2, f_3 = \pi_2 \circ f \circ i_1$, where $i_1, i_2 : A \to A \times A$ are the canonical injections, while $\pi_1, \pi_2 : A \times A \to A$ are the canonical projections. The f_i are endomorphisms of $[A; +]$. Furthermore, let $R_2 = [A \times A; +, \circ_2]$ be the near-ring derived from R and

[5] Obviously, here $A_l(a) = \{(z, t) \in A \times A | (z, t) \circ_1 a = (0, 0)\}$.

[6] Defined here onto $A \times A$ by $(z, t)S_a(u, v) \Leftrightarrow (z, t) \circ_1 a = (u, v) \circ_1 a$.

from γ, on the analogy of R_1. Explicitly, the addition in R_2 is again componentwise operation, while the operation $\circ_2$ is such that, for (x_1, x_2), $(y_1, y_2) \in A \times A$, if $x_1 \neq 0$ then $(x_1, x_2) \circ_2 (y_1, y_2) = (x_1 \circ y_1, x_2\varphi_{x_1}(y_1) + \gamma(x_1)\varphi_{x_1}(y_2))$, if $x_1 = 0$ then $(0, x_2) \circ_2 (y_1, y_2) = (0, x_2 \circ y_1)$.

ρ will be called equivalent to γ if R_1 is isomorphic to R_2. More precisely, we say that ρ is equivalent to γ by f, if f is an isomorphism from R_1 to R_2, and we remark that, in this case, we have in particular $f \in Aut\,[A \times A; +]$.

Theorem 3.1. *If ρ is equivalent to γ by f, then $f(J) = J$, $f_1 \in Aut R$, $f_2 \in Aut[A; +]$ and $f_3(1) = 0$.*

Proof. Since J is the ideal of nilpotent elements of R_1, and, at the same time, of nilpotent elements of R_2, and f is isomorphism, we have $f(J) = J$. Therefore, for all $y \in A$, $f((0, y)) = (0, f_2(y))$; hence f_2 is bijective.

Let $z \in A$. Since f is surjective, there is an $(x, y) \in A \times A$ such that $f((x, y)) = (z, 0)$. Then $(z, 0) = f((x, 0) + (0, y)) = f((x, 0)) + f((0, y)) = (f_1(x), f_3(x)) + (0, f_2(y)) = (f_1(x), f_3(x) + f_2(y))$. Hence $z = f_1(x)$, so f_1 is surjective. We state that f_1 is also injective. Suppose, on the contrary, the existence of an $x \in A^*$, $x \in Ker f_1$. Then $f((x, 0)) = (f_1(x), f_3(x)) = (0, f_3(x)) \in J$, so the non-nilpotent element $(x, 0)$ of R_1 is sended by f to a nilpotent element of R_2, a contradiction.

For $x, y \in A$ the equality $f_1(x \circ y) = f_1(x) \circ f_1(y)$ is immediate if $x = 0$. If $x \neq 0$, we bear in mind that $f_1((x, 0) \circ_1 (y, 0)) = f((x, 0)) \circ_2 f((y, 0))$, namely $f((x \circ y, 0)) = (f_1(x), f_3(x)) \circ_2 (f_1(y), f_3(y))$ which implies

$$(f_1(x \circ y), f_3(x \circ y)) = (f_1(x) \circ f_1(y), s)$$

for a suitable $s \in A$. Hence, $f_1(x \circ y) = f_1(x) \circ f_1(y)$, so $f_1 \in Aut R$.

We recall now that $(1, 0)$ is identity in R_1 and in R_2. Consequently, $(1, 0) = f((1, 0)) = (f_1(1), f_3(1)) = (1, f_3(1))$, therefore $f_3(1) = 0$. □

We underline that, under the hypothesis of Theorem 3.1, we can write (for every $x, y \in A$)

(1) $$f((x, y)) = (f_1(x), f_3(x) + f_2(y)).$$

Theorem 3.2. *ρ is equivalent to γ if and only if $\exists h$ $\exists a \in A^*$ such that*

1. $h \in Aut\,[A; +, \cdot] \cap Aut R$
2. *a is invertible in R*
3. $\varphi_a = id_A$
4. $\forall x \in A^* \quad a \circ h(\rho(x)) = (\gamma(h(x))) \circ a.$

Proof. Suppose ρ equivalent to γ by f. By Theorem 3.1, we have $f_1 \in Aut R$. Furthermore, for all $y \in A$, $f((0,1) \circ_1 (y,0)) = f((0,1)) \circ_2 f((y,0))$, which signifies $f((0,y)) = (0, f_2(1)) \circ_2 (f_1(y), f_3(y))$, that is

$$(0, f_2(y)) = (0, f_2(1) \circ f_1(y)).$$

Hence

$$\forall y \in A \quad f_2(y) = f_2(1) \circ f_1(y). \tag{2}$$

We show that $f_2(1)$ is invertible in R. Since f_2 is bijective by Theorem 3.1, there is a $t \in A$ such that $f_2(t) = 1$. So, by (2) we have $f_2(1) \circ f_1(t) = 1$. Therefore

$$f_2(1) \circ f_1(t) \circ f_2(1) = 1 \circ f_2(1) = f_2(1) \circ 1.$$

Since $f_2(1)$ is left-cancellable in $[A; \circ]$, the equality $f_1(t) \circ f_2(1) = 1$ is then also true.

For every $x \in A^*$, $f((x,0) \circ_1 (0,1)) = f((x,0)) \circ_2 f((0,1))$, i.e. $f(0, \rho(x)) = (f_1(x), f_3(x)) \circ_2 (0, f_2(1))$, namely $(0, f_2(\rho(x))) = (0, (\gamma(f_1(x))) \circ f_2(1))$. Because of (2), it is possible to assert then that

$$\forall x \in A^* \quad f_2(1) \circ f_1(\rho(x)) = (\gamma(f_1(x))) \circ f_2(1).$$

For all $x, y \in A$, the following relation is true

$$f((1,x) \circ_1 (y,0)) = f((1,x)) \circ_2 f((y,0))$$

and consequently, recalling (1) and that $f_3(1) = 0$ (Theorem 3.1),

$$f((y, xy)) = (1, f_2(x)) \circ_2 (f_1(y), f_3(y)).$$

So, on account of (1) we have $(f_1(y), f_3(y) + f_2(xy)) = (f_1(y), f_2(x) f_1(y) + f_3(y))$, which implies $f_2(xy) = f_2(x) f_1(y)$. Then, in consideration of (2) we can write

$$f_2(1) \circ f_1(xy) = (f_2(1) \circ f_1(x)) \cdot f_1(y). \tag{3}$$

For $x = 1$, (3) becomes $f_2(1) \circ f_1(y) = f_2(1) \cdot f_1(y)$; so $\varphi_{f_2(1)} = id_A$. Therefore, (3) assumes the form $f_2(1) \cdot f_1(xy) = f_2(1) \cdot f_1(x) \cdot f_1(y)$, which gives $f_1(xy) = f_1(x) f_1(y)$. We now have $f_1 \in Aut\,[A; +, \cdot] \cap Aut R$. The conditions $1., 2., 3., 4.$ of the assertion are then verified, with $h = f_1$, $a = f_2(1)$.

Conversely, we assume that there exists an h, and an $a \in A^*$, fulfilling the conditions $1., 2., 3., 4.$ of the statement. Let g be the automorphism of $[A \times A; +]$ defined by

$$g : (x, y) \to (h(x), a \circ h(y)).$$

We show that g is an isomorphism from R_1 to R_2.

For $m = (x_1, x_2)$, $n = (y_1, y_2)$ in $A \times A$, with $x_1 \neq 0$, we calculate, remembering that $h \in AutR$, and $\varphi_a = id_A$

$$g(m \circ_1 n) = g(x_1 \circ y_1, x_2\varphi_{x_1}(y_1) + \rho(x_1) \circ y_2) = (h(x_1) \circ h(y_1), c + d)$$

$$g(m) \circ_2 g(n) = (h(x_1), a \circ h(x_2)) \circ_2 (h(y_1), a \circ h(y_2)) = (h(x_1) \circ h(y_1), e + q)$$

where

$c = a \circ h(x_2\varphi_{x_1}(y_1))$
$d = a \circ h(\rho(x_1)) \circ h(y_2)$
$e = a \cdot h(x_2) \cdot \varphi_{h(x_1)}(h(y_1))$
$q = (\gamma(h(x_1))) \cdot \varphi_{h(x_1)}(a \circ h(y_2))$.

We remark now that q equals $(\gamma(h(x_1))) \circ a \circ h(y_2)$, since, through $\gamma \in A^*_\varphi$, we have $\varphi_{h(x_1)} = \varphi_{\gamma(h(x_1))}$.

Because of 1., the following steps are true

$$h(x_1 \circ y_1) = h(x_1\varphi_{x_1}(y_1)) = h(x_1) \cdot h(\varphi_{x_1}(y_1)) = h(x_1) \circ h(y_1) = h(x_1) \cdot \varphi_{h(x_1)}(h(y_1))$$

which implies $h(\varphi_{x_1}(y_1)) = \varphi_{h(x_1)}(h(y_1))$, so $c = e$. Moreover, by 4. we see that $d = q$.

Therefore, $g(m \circ_1 n) = g(m) \circ_2 g(n)$.

Furthermore, for arbitrary $x_2, y_1, y_2 \in A$, we have $g((0, x_2) \circ_1 (y_1, y_2)) = g((0, x_2 \circ y_1)) = (0, a \circ h(x_2 \circ y_1)) = (0, a \circ h(x_2) \circ h(y_1)) = (0, a \circ h(x_2)) \circ_2 (h(y_1), a \circ h(y_2)) = g((0, x_2)) \circ_2 g((y_1, y_2))$, which completes the proof. □

Corollary 3.1. *If the cardinality of $Im\rho$ is distinct from the cardinality of $Im\gamma$, then ρ is non-equivalent to γ.*

Proof. This follows from the condition 4. of Theorem 3.2. □

Remark. We advise that the content of the present article exists also in the preprint, of the same author Giordano Gallina, by the title "Sotto-quasi-anelli di quasi-anelli" (Quaderno n. 122 of the Dipartimento di Matematica dell'Università di Parma, December 1995).

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