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Feller semigroups and invariant measures

Abstract. This paper is devoted to present both classical and recent results on Markov semigroups associated with elliptic second order operators with (possibly unbounded) coefficients, using analytic methods. We consider second order elliptic operators like $A = \sum_{i,j=1}^{N} a_{ij} D_{ij} u + \sum_{i=1}^{N} b_i D_i u + cu$ defined in $\mathbb{R}^N$; we study the invariant measures associated with the semigroup $\{T(t)\}$ generated by A in $C_b(\mathbb{R}^N)$ and the regularity properties of $\{T(t)\}$ in L^p-spaces related to this measure. A concrete example of an elliptic operator with unbounded coefficients in $\mathbb{R}^N$ is given by the Ornstein-Uhlenbeck operator whose main properties are also presented.

Keywords. Markov processes, elliptic operators with unbounded coefficients, invariant measures.

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Contents

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1 - Introduction

This survey is an extended version of a 10 hours course of the second author during the Workshop on Harmonic Analysis and Evolution Equations held in Parma, 4-8 February, 2008.

All the topics here contained have been treated during the lectures, except for some details of the proofs.

We present some aspects of the theory of semigroups generated by second order elliptic operators with unbounded coefficients and the associated resolvent equations, starting from a probabilistic motivation. This is done in Section 2 where we use Kolmogorov's approach and introduce invariant measures, i.e. stationary distributions of the underlying Markov process.

The prototype of elliptic operators with unbounded coefficients is the Ornstein-Uhlenbeck operator studied in Section 3 and defined by

$$L=\sum_{i,j=1}^N q_{ij}D_{ij}+\sum_{i,j=1}^N b_{ij}x_jD_i=\mathrm{tr}(QD^2)+\langle Bx,\nabla\rangle,\qquad x\in\mathbb{R}^N,$$

where $Q=(q_{ij})_{i,j=1,\dots,N}$ is a real symmetric and positive definite matrix and $B=(b_{ij})_{i,j=1,\dots,N}$ is a non-zero real matrix. We first show that the generated semigroup $T(t)$ admits an explicit representation formula (due to Kolmogorov) which allows the study of its main properties. We then focus on the spectral properties of the Ornstein Uhlenbeck operators in $L^p(\mathbb{R}^N)$ with respect to the Lebesgue measure, by stating a characterization theorem for its spectrum.

In Section 4, assuming that the spectrum of B is contained in the left halfplane, we show that the semigroup $T(t)$ admits a unique invariant measure μ given by a Gaussian density. Moreover the semigroup extends to a strongly continuous semigroup of positive contractions in $L^p(\mathbb{R}^N, d\mu)$, $1 \leq p < \infty$, where further nice properties hold. One can prove for example that the semigroup is analytic and compact in $L^p(\mathbb{R}^N, d\mu)$ for $1 < p < \infty$. A characterization of the eigenfunctions and a description of the spectrum of L in $L^p(\mathbb{R}^N, d\mu)$ are also given. We do not deal with hypercontractivity properties of the semigroup and log-Sobolev inequalities for its generator. For all these questions we refer the reader to the original paper by Gross or to [**3**].

The next section is devoted to the study of more general elliptic operators A with unbounded coefficients in spaces of continuous and bounded functions on $\mathbb{R}^N$ and the associated Markov semigroup. Our main interest is in the existence of bounded (in space) solutions of the parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) \ = f(x) & x \in \mathbb{R}^N \end{cases}$$

with initial datum $f \in C_b(\mathbb{R}^N)$. Since the coefficients can be unbounded, the classical theory does not apply and existence and uniqueness for the solution of the problem above are not standard. However, when the coefficients are sufficiently smooth, existence is not a problem. Indeed we show how to construct in a direct way a semigroup $T(t)$ generated by the operator as limit of solutions of parabolic problems in the cylinders $]0, \infty[\times B_\rho$. On the other hand, uniqueness is not generally true (see Example 5.1) but it is consequence of the existence of a Lyapunov function for the operator A.

We also deduce for $T(t)$ an integral representation formula and we discuss some qualitative properties such as continuity at $t = 0$, irreducibility and strong Feller property. Even though the semigroup is not strongly continuous, a weak notion of generator can be also defined by considering the Laplace transform of the semigroup. In addition, in this section we prove Has'minskii's Theorem that provides a condition under which an invariant measure does exist.

Compacteness and preservation of $C_0(\mathbb{R}^N)$ properties are also analyzed.

In Section 6 we describe some regularity properties enjoyed by the invariant measure of the semigroup when the drift of the operator A belongs to $L^p(\mathbb{R}^N, d\mu)$ for a suitable p. Furthermore, by assuming growth conditions on the drift term, we also provide some pointwise estimates for the density of the invariant measure.

In the last section we characterize the domain $D_p^\mu(A)$ of the operator $A = \Delta - \nabla\phi \cdot \nabla$ in $L^p(\mathbb{R}^N, d\mu)$ for $1<p<\infty$. Under suitable conditions on ϕ, we show that $D_p^\mu(A) = W_\mu^{2,p}(\mathbb{R}^N)$.

2 - A short probabilistic motivation

Let $\Omega \subseteq \mathbb{R}^N$ be an open set, (X, P) a probability space. We consider a stochastic process (ξ_t) in Ω i.e. $\xi_{(\cdot)}(\,\cdot\,) : [0,\infty) \times X \to \Omega$ such that $\xi_t(\,\cdot\,)$ is measurable in Ω for every $t > 0$.

Given $\Gamma \subseteq \Omega$ measurable, we consider the transition probabilities defined by

$$p(t, x, \Gamma) = P(\xi_t \in \Gamma | \xi_0 = x)$$

which satisfy the following time-independence property

$$p(t, x, \Gamma) = P(\xi_{t+s} \in \Gamma | \xi_s = x). \tag{1}$$

The equality (1) expresses the lack of memory typical of Markov processes for which

$$P(\xi_t \in \Gamma | (\xi_\tau)_{\tau \le s}) = P(\xi_t \in \Gamma | \xi_s), \qquad 0 \le s < t,$$

holds. This means that the future depends on the past only through the present.

Markov processes satisfy the Chapman-Kolmogorov equation

$$p(t+s, x, \Gamma) = \int_\Omega p(t, y, \Gamma) p(s, x, dy). \tag{2}$$

If we denote the initial distribution by $\mu(\Gamma) = P(\xi_0 \in \Gamma)$, then

$$P(\xi_t \in \Gamma) = \int_\Omega p(t, x, \Gamma)\, d\mu = \mu_t(\Gamma).$$

From now on we assume that the probability measure $p(t, x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure and we write $p(t, x, dy) = p(t, x, y)dy$.

Given the initial distribution μ of $(\xi_{(t)})$, it is sufficient to determine the family of measures $p(t, x, \cdot)$ in order to reconstruct completely the process. This fact leads to an equation satisfied by $p(t, x, \cdot)$. Such an equation actually exists and it is known as Kolmogorov "backward equation". Now we briefly describe this approach. If $f \in C_b(\Omega)$, define

$$T(t)f(x) = \int_\Omega p(t, x, dy) f(y), \quad T(0)f = f.$$

The Chapman-Kolmogorov equation shows that

$$T(t+s) = T(t)T(s).$$

Kolmogorov's idea is to recover $p(t, x, \cdot)$ from $T(t)f(x)$ under the following assumptions.

For all $\varepsilon > 0$ and $x \in \Omega$,

(H_1) $p(t, x, \Omega \setminus B_\varepsilon(x)) = o(t), \quad t \to 0$

(H_2) $p(t, x, \Omega) - 1 = tc(x) + o(t), \quad t \to 0$

(H_3) $\displaystyle\int_{B(x,\varepsilon)} (y_i - x_i)p(t, x, dy) = tb_i(x) + o(t), \quad t \to 0$

(H_4) $\displaystyle\int_{B(x,\varepsilon)} (y_i - x_i)(y_j - x_j)p(t, x, dy) = ta_{ij}(x) + o(t), \quad t \to 0.$

Then, if $u \in C_b^2(\Omega)$, $x \in \Omega$,

$$\lim_{t \to 0} \frac{T(t)u(x) - u(x)}{t} = \frac{1}{2} \sum_{i,j=1}^N a_{ij} D_{ij} u + \sum_{i=1}^N b_i D_i u + cu = Au. \tag{3}$$

Remark 2.1. (i) *Assumption* (H_1) *is the so called "Dynkin-Kinney" condition and ensures the "continuity of the paths" that is* $\xi_t(\omega)$ *is continuous in t for a.e.* ω*;*

(ii) $c(x) \le 0$ *is the absorption coefficient at* x*;*

(iii) *if* $c \equiv 0$*, the expectation* $E(\xi_i(t, x) - x_i)$ *is equal to* $tb_i(x) + o(t)$ *where* $b = (b_1, \ldots, b_N)$ *is called the drift term;*

(iv) *if* $c \le 0$*,* $E[(\xi_i - x_i)(\xi_j - x_j)] = ta_{ij}(x) + o(t)$ *where* $a_{ij} = a_{ji}$ *are the diffusion coefficients and satisfy* $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \ge 0$.

Under the previous assumptions it turns out that the generator of $T(t)$ is a second order differential operator with (possibly) unbounded coefficients. If $f \in D(A)$, $u(t, x)$ solves the parabolic Cauchy problem

$$\begin{cases} u_t = Au \\ u(0) = f. \end{cases} \tag{4}$$

Recalling that $T(t)f(x) = \int_\Omega p(t, x, y)f(y)\, dy$, if p is regular then

$$\begin{cases} p_t = A_x p \\ p(0, x, y) = \delta_y(x). \end{cases} \tag{5}$$

Moreover, if $f \in D(A)$,

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af$$

i.e.

$$\int_{\Omega} p_t(t,x,y)f(y)\ dy = \int_{\Omega} p(t,x,y)Af(y)\ dy = \int_{\Omega} A^*_y p(t,x,y)f(y)\ dy$$

hence p solves also the Fokker-Planck problem

$$\begin{cases} p_t = A^*_y p \\ p(0,x,y) = \delta_x(y). \end{cases} \tag{6}$$

Here A^*_y is the formal adjoint of A, when this last is considered as a differential operator acting in the y-variable. Let observe that $T(t)1 = \int_{\Omega} p(t,x,y)\ dy \le 1$ and that $f \ge 0$ implies $T(t)f \ge 0$. These facts yield that $T(t)$ is a contractive semigroup, that is $\|T(t)f\|_\infty \le \|f\|_\infty$.

If $\Omega \ne \mathbb{R}^N$ and there exists $\bar{t} > 0$ such that $\xi(\bar{t},x) \in \partial\Omega$, we need boundary conditions to determine the process. In fact, both the backward and the Fokker-Planck equation hold on the interior of Ω and say nothing on $\partial\Omega$.

Let $\bar{x} \in \partial\Omega$.

Dirichlet boundary conditions. They consist in requiring that when $\xi(\bar{t},x) = \bar{x}$ the process "dies", i.e., $p(t,\bar{x},\cdot) = 0$ and

$$u(t,\bar{x}) = \int_{\Omega} p(t,\bar{x},dy)f(y) = 0.$$

Ventcel boundary conditions. In this case, if $\xi(\bar{t},\bar{x}) = \bar{x}$, then $\xi(\bar{t},\bar{x}) = \bar{x}$ for $t \ge \bar{t}$, i.e. $p(t,\bar{x},\cdot) = \delta_{\bar{x}}$ and

$$u(t,\bar{x}) = \int_{\Omega} p(t,\bar{x},dy)f(y)\ dy = f(\bar{x})$$

i.e. $\dfrac{d}{dt}u(t,\bar{x}) = Au(t,\bar{x}) = 0$.

Let $u \in D(A)$. If we impose Dirichlet boundary conditions, then $u = 0$ in $\partial\Omega$ whereas under Ventcel boundary conditions we have $Au = 0$ in $\partial\Omega$.

Other boundary conditions such as the Neumann or the Robin conditions can be imposed. The conservation of probabilities ($T(t)1 = 1$) holds if $c \equiv 0$ under Ventcel, Neumann but not Dirichlet boundary conditions.

Example 2.1. *We consider the Cauchy problem*

$$\begin{cases} u_t(t,x) = Au(t,x) & t \in (0,+\infty),\ x \in \mathbb{R}^N \\ u(0,x) \ = u_0(x) & x \in \mathbb{R}^N. \end{cases} \tag{7}$$

1) *If* $Au = -cu$ *with* $c = c(x) \geq 0$ (*c is the absorption coefficient*), *then the solution of* (7) *is given by* $u(t,x) = e^{-tc(x)}u_0(x)$ *and* $p(t,x,\cdot) = e^{-tc(x)}\delta_x(\,\cdot\,)$.

2) *If* $Au = b \cdot \nabla u$ *with* $b = b(x)$, *then the solution of* (7) *is given by* $u(t,x) = u_0(\phi(t,x))$ *where* $\phi(t,x)$ *solves the ODE*

$$\begin{cases} \phi_t = b(\phi) \\ \phi(0) = x. \end{cases} \tag{8}$$

Thus $p(t,x,\cdot) = \delta_{\phi(t,x)}(\,\cdot\,)$. *If* $|\phi(\bar{t},x)| = \infty$ *for some* $\bar{t} > 0$, $x \in \mathbb{R}^N$, *boundary condition are needed even when* $\Omega = \mathbb{R}^N$.
Now, suppose $N = 1$. *If b is constant, then* $\phi(t,x) = x + bt$.
When $b(x) = b \cdot x$ (*Ornstein- Uhlenbeck operator*), $\phi(t,x) = e^{tb}x$.
For $b(x) = -x^3$, $\phi(t,x) = \dfrac{x}{\sqrt{1+2tx^2}}$. *Therefore* $|\phi(t,x)| \leq \dfrac{1}{\sqrt{2t}}$ *for all* $x \in \mathbb{R}^N$.
Finally, if $b(x) = x^3$, $\phi(t,x) = \dfrac{x}{\sqrt{1-2tx^2}}$. *Therefore, for* $t = \dfrac{1}{2x^2}$, $|\phi(t,x)|$ *blows up.*

2.1 - *Invariant measures*

Suppose $\Omega = \mathbb{R}^N$. Recall that the initial distribution is denoted by $\mu(\Gamma) = P(\xi_0 \in \Gamma)$ and

$$P(\xi_t \in \Gamma) = \int_{\mathbb{R}^N} p(t,x,\Gamma)d\mu(x) = \mu_t(\Gamma).$$

We say that μ is an invariant measure if

$$\mu_t(\Gamma) = P(\xi_t \in \Gamma) = \mu(\Gamma), \quad t \geq 0,\ \Gamma \subset \mathbb{R}^N.$$

In this case $d\mu(\,\cdot\,) = \int_{\mathbb{R}^N} p(t,x,\cdot)d\mu(x)$.

On the other hand

$$\begin{aligned} \int_{\mathbb{R}^N} f(y)d\mu(y) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(y)p(t,x,dy)d\mu(x) \\ &= \int_{\mathbb{R}^N} d\mu(x)\Bigg[\int_{\mathbb{R}^N} p(t,x,dy)f(y)\Bigg]. \end{aligned}$$

By definition, a probability measure μ is an invariant measure for $\{T(t)\}$ if and only if

$$\int_{\mathbb{R}^N} T(t) f d\mu = \int_{\mathbb{R}^N} f d\mu, \qquad t > 0$$

for all $f \in C_b(\mathbb{R}^N)$. A characterization of invariant measures for $\{T(t)\}$ can be given in terms of the generator A of the semigroup, indeed, (see [**4**, Proposition 8.1.2]) a Borel probability measure μ is an invariant measure for $\{T(t)\}$ if and only if

$$\int_{\mathbb{R}^N} A f d\mu = 0, \qquad f \in D(A).$$

Example 2.2. *Consider the problem*

$$\begin{cases} u_t = b \cdot \nabla u \\ u(0) = u_0. \end{cases}$$

Then $u(t,x) = u_0(\phi(t,x))$ *and* $p(t,x,\cdot) = \delta_{\phi(t,x)}(\,\cdot\,)$. *In this case* μ *is an invariant measure if and only if*

$$\mu_t(\Gamma) = \int_{\mathbb{R}^N} \delta_{\phi(t,x)}(\Gamma) d\mu(x) = \mu(\Gamma)$$

i.e. $\mu(\phi(t,\cdot)^{-1}(\Gamma)) = \mu(\Gamma)$ *for all* $t > 0$ *i.e.* $\mu(\Gamma) = \mu(\phi(t,\cdot)(\Gamma))$ *for all* $t > 0$, $\Gamma \subset \mathbb{R}^N$. *If* $b(\overline{x}) = 0$ *then* $\phi(t,\overline{x}) = \overline{x}$ *and* $d\mu = \delta_{\overline{x}}$ *is invariant.*

Example 2.3. *Suppose* $N = 2$, $b(x,y) = (-y, x)$. *Then*

$$\phi(t,x,y) = \begin{pmatrix} \cos t & \operatorname{sen} t \\ -\operatorname{sen} t & \cos t \end{pmatrix}$$

and $(0,0)$ *is the unique fixed point. Any measure invariant under rotations is invariant.*

Remark 2.2. *If an invariant measure* μ *exists, then* $c \equiv 0$ *a.e. with respect to* μ. *In fact, since* $c \leq 0$ *(see Remark* 2.1), $T(t)\mathbf{1} \leq \mathbf{1}$ *but also* $\int_{\mathbb{R}^N} T(t)\mathbf{1}\, d\mu = \int_{\mathbb{R}^N} \mathbf{1}\, d\mu$. *Then* $T(t)\mathbf{1} = \mathbf{1}$, $c = A\mathbf{1} = 0$ μ*-almost everywhere.*

Definition 2.1. *A semigroup* $(T(t))_{t\geq 0}$ *in* $B_b(\mathbb{R}^N)$ *is* irreducible *if for any nonempty open set* $U \subset \mathbb{R}^N$, $T(t)\chi_U(x) > 0$ *for every* $t > 0$ *and* $x \in \mathbb{R}^N$.

Definition 2.2. *We say that* $(T(t))_{t\geq 0}$ *satisfies the* strong Feller property *if* $T(t)f \in C_b(\mathbb{R}^N)$ *for any bounded Borel function* f.

Theorem 2.1 (Doob). *Let μ be an invariant measure. If the irreducibility and the strong Feller property are satisfied, then*

(i) *all the measures $p(t, x, \cdot)$, $t > 0$, $x \in \mathbb{R}^N$ are equivalent;*
(ii) *μ is equivalent to $p(t, x, \cdot)$;*
(iii) *μ is the unique invariant measure.*

We recall that two measures μ, ν are equivalent if they have the same null-sets, that is if each of them is absolutely continuous with respect to the other. We refer to [**12**, Theorem 4.2.1] for a proof of the above result.

Remark 2.3. *$(T(t))_{t\geq 0}$ acts in $C_b(\mathbb{R}^N)$ but in general it does not preserve either $C_0(\mathbb{R}^N)$ or $L^p(\mathbb{R}^N)$. Consider the operator $Au = -x^3u'$. The solution of the parabolic problem associated with A with initial datum f is*

$$u(t,x) = f\left(\frac{x}{\sqrt{1+2x^2t}}\right).$$

Obviously we have

$$\lim_{x\to\infty} u(t,x) = f\left(\frac{1}{\sqrt{2t}}\right),$$

which is not always zero.

Proposition 2.1. *If an invariant measure exists, then $T(t)$ in contractive in $L^p_\mu(\mathbb{R}^N)$ for $1 \leq p < \infty$.*

Proof. Indeed, since $\int_{\mathbb{R}^N} p(t,x,dy) = 1$, we have

$$|T(t)f(x)|^p \leq \int_{\mathbb{R}^N} |f(y)|^p p(t,x,dy) = T(t)|f|^p(x),$$

whence

$$\int_{\mathbb{R}^N} |T(t)f|^p d\mu \leq \int_{\mathbb{R}^N} T(t)|f|^p d\mu = \int_{\mathbb{R}^N} |f|^p d\mu.$$

Example 2.4. *Consider the symmetric operator $A = \Delta - \nabla\phi \cdot \nabla$ with $\phi \in C^1(\mathbb{R}^N)$, $e^{-\phi} \in L^1(\mathbb{R}^N)$. The operator A can be written as $e^{\phi}\mathrm{div}(e^{-\phi}\nabla)$. Let $d\mu = e^{-\phi}dx$. If $u \in C_c^\infty(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} Au\,d\mu = \int_{\mathbb{R}^N} \mathrm{div}(e^{-\phi}\nabla u)dx = 0$$

that is μ is an invariant measure for A. In order to prove this, we consider the sesquilinear form a in $L^2_\mu(\mathbb{R}^N)$ given by

$$a(u,v)=\int_{\mathbb{R}^N}\nabla u\overline{\nabla v}d\mu$$

on the form domain

$$D(a)=H^1_\mu(\mathbb{R}^N)=W^{1,2}_\mu(\mathbb{R}^N)=\{u:\ u,\ \nabla u\in L^2_\mu(\mathbb{R}^N)\}.$$

It is possible to prove that a is a bilinear, coercive, symmetric and positive form and it is associated with the operator $\mathcal{A}$ so defined

$$D(\mathcal{A})=\left\{u\in H^1_\mu(\mathbb{R}^N):\ \exists f\in L^2_\mu(\mathbb{R}^N):\ a(u,v)=-\int_{\mathbb{R}^N}f\overline{v}d\mu,\ v\in C_c^\infty(\mathbb{R}^N)\right\}$$

and $\mathcal{A}u=f,\ u\in D(\mathcal{A})$. (In the definition of $D(\mathcal{A})$ we can consider $v\in H^1_\mu(\mathbb{R}^N)$ by density). Since $\nabla\phi\in L^\infty_{loc}(\mathbb{R}^N)$, by local elliptic L^2-regularity

$$D(\mathcal{A})=\left\{u\in H^1_\mu(\mathbb{R}^N)\cap H^2_{loc,\mu}(\mathbb{R}^N):\ \Delta u-\nabla\phi\cdot\nabla u\in L^2_\mu(\mathbb{R}^N)\right\}$$

and $\mathcal{A}u=Au,\ u\in D(\mathcal{A})$. Observe that $\mathbf{1}\in D(\mathcal{A}),\ A\mathbf{1}=0$. Then, if $u\in D(\mathcal{A})$,

$$\int_{\mathbb{R}^N}Au\,d\mu=(Au,\mathbf{1})_{L^2_\mu(\mathbb{R}^N)}=(u,A\mathbf{1})_{L^2_\mu(\mathbb{R}^N)}=0.$$

In particular, if $A=\Delta-x\cdot\nabla$ then $d\mu=e^{-\frac{|x|^2}{2}}dx$ is the invariant measure. In the one-dimensional case, if $A=D^2-x^3D$, $d\mu=e^{-\frac{x^4}{4}}dx$ is the invariant measure.

3 - The Ornstein-Uhlenbeck semigroup in $L^p(\mathbb{R}^N)$

Here we consider the Ornstein-Uhlenbeck operator

$$L=\sum_{i,j=1}^N q_{ij}D_{ij}+\sum_{i,j=1}^N b_{ij}x_jD_i=\mathrm{tr}(QD^2)+\langle Bx,\nabla\rangle,\qquad x\in\mathbb{R}^N,\tag{9}$$

where $Q=(q_{ij})_{i,j=1,\dots,N}$ is a real symmetric and positive definite matrix, and $B=(b_{ij})_{i,j=1,\dots,N}$ is a non-zero real matrix. In this section we consider L acting in $L^p(\mathbb{R}^N)$ with respect to the Lebesgue measure. In the next section we deal with the same operator but in L^p with respect to the invariant measure.

An elementary change of variable allows us to assume $Q=I$. Indeed, setting $u(x)=v(Mx)$, with M a real matrix we get

$$\nabla u(x)=M^*\nabla v(Mx)$$

and

$$D^2u(x) = M^*D^2v(Mx)M.$$

Hence

$$\sum_{i,j=1}^{N} q_{ij}(x)D_{ij}u(x) = \mathrm{tr}\,(QD^2u(x)) = \mathrm{tr}\,(QM^*D^2v(Mx)M)$$
$$= \mathrm{tr}\,(MQM^*D^2v(Mx)).$$

Choosing M real such that $MQM^* = I$ we may restrict to study operators of the form

$$L = \Delta + Bx\cdot\nabla \tag{10}$$

(where we still denote by B the matrix MBM^{-1}).

The explicit representation of the semigroup generated by L in the form (10) is due to Kolmogorov whose heuristic argument we illustrate below. Let consider the following parabolic initial value problem

$$\begin{cases} u_t = \Delta u + Bx\cdot\nabla u \\ u(0,x) = f(x). \end{cases} \tag{11}$$

Problem (11) can be simplified getting rid of the drift term $Bx\cdot\nabla u$ using the flow generated by Bx

$$\begin{cases} \dot\xi = B\xi \\ \xi(0) = x \end{cases} \tag{12}$$

whose solution is given by $\xi(t,x) = e^{tB}x$. Thus, setting $u(t,x) = v(t, e^{tB}x)$,

$$u_t(t,x) = v_t(t,e^{tB}x) + \langle Be^{tB}x, \nabla v(t,e^{tB}x)\rangle$$
$$\nabla u(t,x) = e^{tB^*}\nabla v(t,e^{tB}x)$$
$$\Delta u(t,x) = \mathrm{tr}\,(e^{tB^*}D^2v(t,e^{tB}x)e^{tB})$$

and

$$u_t - \Delta u - Bx\cdot\nabla u = v_t - \ \mathrm{tr}\,(e^{tB}e^{tB^*}D^2v).$$

Therefore $u(t,x)$ is solution of (11) if and only if $v(t,e^{tB}x)$ is solution of the following non autonomous problem

$$\begin{cases} v_t = \mathrm{tr}\,(C(t)D^2v) = A(t)v \\ v(0) = f \end{cases}$$

where $C(t) = e^{tB}e^{tB^*}$ and $A(t) = \mathrm{tr}\,(C(t)D^2)$. Since the coefficients of $A(t)$ depend only on t whereas D^2 acts in the space variables one easily sees that $A(t)A(s) = A(s)A(t)$. The solution can be written in the following form

$$v(t,x) = e^{\int_0^t A(s)ds} f(x) = e^{\int_0^t \mathrm{tr}\,(C(s)D^2)ds} f(x) = e^{\mathrm{tr}\,(Q_t D^2)} f(x)$$

where $Q_t = \int_0^t e^{sB} e^{sB^*}\, ds$. Thus $v(t,\cdot)$ coincides with $w(1,\cdot)$ where w solves

$$\begin{cases} v_s = \mathrm{tr}\,(Q_t D^2 w) \\ w(0) = f \end{cases}$$

and hence

$$v(t,x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\frac{\langle Q_t^{-1}y,y\rangle}{4}} f(x-y)dy, \qquad f \in C_b(\mathbb{R}^N).$$

Finally the solution of (11) is given by

$$T(t)f(x) := u(t,x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\frac{\langle Q_t^{-1}y,y\rangle}{4}} f(e^{tB}x - y)dy. \tag{13}$$

3.1 - *Properties of* $(T(t))_{t\geq 0}$

In this section we collect some classical results for $(T(t))_{t\geq 0}$. Smoothing properties of $(T(t))_{t\geq 0}$ are established in [**9**], in spaces of continuous functions and in [**20**], in $L^p(\mathbb{R}^N)$. We start recalling that the semigroup $(T(t))_{t\geq 0}$ is strongly continuous on $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$ and on $C_0(\mathbb{R}^N)$.

One can show that L, with a suitable domain, is the generator of $(T(t))_{t\geq 0}$. For $1 < p < \infty$ we define

$$D_p(L) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N) : \, Lu \in L^p(\mathbb{R}^N)\}$$

and for $p = \infty$

$$D_\infty(L) = \{u \in C_0(\mathbb{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N) \ \forall p > N : \, Lu \in L^\infty(\mathbb{R}^N)\}.$$

From now on, we denote by L_p the realization of L in $L^p(\mathbb{R}^N)$, that is $(L, D_p(L))$. The following result is contained in [**21**].

Proposition 3.1. *If $1 < p \leq \infty$, the generator of $(T(t))_{t\geq 0}$ in L^p coincides with L_p and $C_c^\infty(\mathbb{R}^N)$ is a core for L_p. For $p = 1$ the generator is the closure of L_p on $C_c^\infty(\mathbb{R}^N)$.*

Remark 3.1 (Contractivity of $T(t)$). *If we denote by*

$$g_t(y) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}}\; e^{-\frac{\langle Q_t^{-1}y,y\rangle}{4}}, \tag{14}$$

then $\|g_t\|_1 = 1$ *and*

$$T(t)f(x) = (g_t * f)(e^{tB}x). \tag{15}$$

Young's inequality for convolutions and the identity $\det(e^{-tB}) = e^{-t\mathrm{tr}(B)}$ *prove that*

$$\|T(t)f\|_p \le e^{-\frac{t\mathrm{tr}(B)}{p}}\|g_t\|_1\|f\|_p = e^{-\frac{t\mathrm{tr}(B)}{p}}\|f\|_p. \tag{16}$$

Remark 3.2. (i) *$T(t)$ is not strongly continuous in $BUC(\mathbb{R}^N)$ endowed with the sup norm $\|\cdot\|_\infty$. Indeed*

$$\lim_{t\to 0^+}\|T(t)f - f\|_\infty = 0$$

if and only if

$$f \in \{h : \mathbb{R}^N \to \mathbb{R} :\ \lim_{t\to 0}[\, h(e^{tB}x) - h(x)\,] = 0 \text{ uniformly for } x \in \mathbb{R}^N\}$$

(*see* [**9**, Lemma 3.2] *for details*). *For $N = 1$ and $B = 1$ a counterexample is thus provided by $f(x) = \sin x \in BUC(\mathbb{R})$. In fact, $f(e^t x)$ does not converge uniformly to $f(x)$ as $t \to 0$.*

(ii) *$T(t)$ is not analytic in $L^p(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N)$* (*see* [**20**] *and* [**29**]).

If f is a smooth function, from (15) we get

$$\nabla T(t)f(x) = e^{tB^*}(g_t * \nabla f)(e^{tB}x) = e^{tB^*}T(t)\nabla f(x), \tag{17}$$

whereas for a generic f

$$\nabla T(t)f(x) = e^{tB^*}(\nabla g_t * f)(e^{tB}x). \tag{18}$$

Since $\nabla g_t(y) = g_t(y)\Big(-\frac{1}{2}Q_t^{-1}y\Big)$,

$$\nabla T(t)f(x) = -\frac{1}{2}\int_{\mathbb{R}^N} e^{tB^*}Q_t^{-1}y\, g_t(y)f(e^{tB}x - y)dy = (g_t^{(1)} * f)(e^{tB}x)$$

where $g_t^{(1)}(y) = -\frac{1}{2}e^{tB^*}Q_t^{-1}y\, g_t(y)$. We estimate $\|g_t^{(1)}\|_1$ as follows

$$\|g_t^{(1)}\|_1 \le \frac{1}{2}e^{t\|B\|}\int_{\mathbb{R}^N}|Q_t^{-1}y|\frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}}e^{-\frac{1}{4}\langle Q_t^{-1}y,y\rangle}dy$$

$$\frac{1}{2(4\pi)^{N/2}}e^{t\|B\|}\int_{\mathbb{R}^N}e^{-\frac{|z|^2}{4}}|Q_t^{-1/2}z|dz \le \frac{c}{\sqrt{t}} \qquad 0<t\le 1,$$

where in the last step we have used the inequality $|Q_t^{-1/2}z| \leq \frac{c}{\sqrt{t}}|z|$ (which easily follows from $Q_t/t \to I$ as $t \to 0$). Young's inequality again yields

$$\|\nabla T(t)f\|_p \leq e^{-\frac{t\mathrm{tr}B}{p}}\|g_t^{(1)}\|_1\|f\|_p \leq \frac{c}{\sqrt{t}}\|f\|_p \qquad 0<t \leq 1.$$

A simple iteration procedure shows that, in general,

$$\|D^\alpha T(t)f\|_p \leq \frac{c}{t^{|\alpha|/2}}\|f\|_p \qquad 0<t \leq 1. \tag{19}$$

For instance, if $|\alpha| = 2$

$$\begin{aligned}\|\nabla D_i T(t)f\|_p &= \|D_i \nabla T(t/2)T(t/2)f\|_p = \|D_i e^{\frac{t}{2}B^*} T(t/2)\nabla T(t/2)f\|_p \\ &\leq \frac{c}{\sqrt{t}}\|\nabla T(t/2)f\|_p \leq \frac{c}{t}\|f\|_p \qquad 0<t \leq 1.\end{aligned}$$

The previous estimates prove that $T(t)$ has good smoothing properties in all L^p spaces. If $f \in L^p(\mathbb{R}^N)$ then $T(t)f \in W^{k,p}(\mathbb{R}^N)$ for every $k \in \mathbb{N}$ with classical estimates. However $Bx \cdot \nabla T(t)f \notin L^p(\mathbb{R}^N)$, see the next section.

We study now spectral properties of L in $L^p(\mathbb{R}^N)$. We point out that L is the sum of the diffusion term $\mathrm{tr}(QD^2)$ and the drift term $\langle Bx, \nabla\rangle$. Whereas the spectral properties of the diffusion term are quite obvious, being an elliptic operator with constant coefficients, those of the drift term are more interesting and depend both on p and the matrix B.

3.2 - *Spectrum of the drift*

Let $B = (b_{ij})$ be a real $N \times N$ matrix. We consider the drift operator

$$\mathcal{L} = \sum_{i,j=1}^{N} b_{ij}x_j D_i = Bx \cdot \nabla$$

and its realization $\mathcal{L}_p$ in $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, that is $(\mathcal{L}, D_p(\mathcal{L}))$ with domain

$$D_p(\mathcal{L}) = \{u \in L^p(\mathbb{R}^N) : \mathcal{L}u \in L^p(\mathbb{R}^N)\} \quad 1 \leq p < \infty,$$

and

$$D_\infty(\mathcal{L}) = \{u \in C_0(\mathbb{R}^N) : \mathcal{L}u \in C_0(\mathbb{R}^N)\}$$

where $\mathcal{L}u$ is understood in the sense of distributions.

Lemma 3.1. *The operator $(\mathcal{L}, D_p(\mathcal{L}))$ is closed in $L^p(\mathbb{R}^N)$.*

Proof. Let $(u_j)_j \in D_p(\mathcal{L})$ and suppose it converges to u and $\mathcal{L}u_j$ converges to v in $L^p(\mathbb{R}^N)$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $\mathcal{L}^*$ be the formal adjoint of $\mathcal{L}$. Then

$$\int_{\mathbb{R}^N} u\,\mathcal{L}^*\varphi\,dx = \lim_{j\to\infty}\int_{\mathbb{R}^N} u_j\mathcal{L}^*\varphi\,dx = \lim_{j\to\infty}\int_{\mathbb{R}^N}(\mathcal{L}u_j)\varphi\,dx = \int_{\mathbb{R}^N} v\,\varphi\,dx.$$

Hence $u \in D_p(\mathcal{L})$ and $\mathcal{L}u = v$.

Proposition 3.2. *The operator $(\mathcal{L}, D_p(\mathcal{L}))$ is the generator of the C_0- group*

$$S(t)\,f(x) = f(e^{tB}x)$$

for $f \in L^p(\mathbb{R}^N), t \in \mathbb{R}$. Moreover $C_c^\infty(\mathbb{R}^N)$ is a core of $(\mathcal{L}, D_p(\mathcal{L}))$ and

$$\|S(t)\,f\|_p = e^{-\frac{t\mathrm{tr}\,(B)}{p}}\|f\|_p \tag{20}$$

for every $f \in L^p(\mathbb{R}^N)$.

Proof. A simple change of variable, together with the equality $\det e^{-tB} = e^{-t\mathrm{tr}(B)}$ proves that (20) holds. If f is continuous with compact support then $S(t)f \to f$ in $L^p(\mathbb{R}^N)$ as $t \to 0$. By density and (20) we deduce the strong continuity of $(S(t))_{t\in\mathbb{R}}$ in $L^p(\mathbb{R}^N)$. Since the group law is clear, we have only to prove that $(\mathcal{L}, D_p(\mathcal{L}))$ is the generator of $(S(t))_{t\in\mathbb{R}}$. Let (A_p, D_p) be its generator in $L^p(\mathbb{R}^N)$ and take $f \in C_c^\infty(\mathbb{R}^N)$. A straightforward computation shows that

$$\lim_{t\to 0}\frac{S(t)f - f}{t} = \mathcal{L}f$$

in $L^p(\mathbb{R}^N)$ and hence $C_c^\infty(\mathbb{R}^N) \subset D_p$ and $A_p f = \mathcal{L}f$ if $f \in C_c^\infty(\mathbb{R}^N)$. Moreover, since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ and $S(t)$-invariant, by [**13**, Proposition 1.7] we deduce that it is a core for (A_p, D_p). The closedness of $(\mathcal{L}, D_p(\mathcal{L}))$ implies that $D_p \subset D_p(\mathcal{L})$ and that $A_p f = \mathcal{L}f$ if $f \in D_p$. Let $\mathcal{L}^* = -\mathcal{L} - \mathrm{tr}(B)$ be the formal adjoint of $\mathcal{L}$ and note that $\mathcal{L}^* = -A_{p'} - \mathrm{tr}(B)$ on $D_{p'}$, $1/p + 1/p' = 1$. If $u \in D_p(\mathcal{L})$, then the equality

$$\int_{\mathbb{R}^N}\mathcal{L}u\phi\,dx = \int_{\mathbb{R}^N} u\mathcal{L}^*\phi\,dx \tag{21}$$

holds for all $\phi \in D_{p'}$, by the density of $C_c^\infty(\mathbb{R}^N)$ in $D_{p'}$ with respect to the graph norm induced by $\mathcal{L}^*$.

For λ large, take $v \in D_p$ such that $\lambda v - A_p v = \lambda u - \mathcal{L}u$. Then $w = v - u \in D_p(\mathcal{L})$ satisfies $\lambda w - \mathcal{L}w = 0$ and from (21) we deduce that

$$0 = \int_{\mathbb{R}^N}(\lambda w - \mathcal{L}w)\,\phi dx = \int_{\mathbb{R}^N} w\,(\lambda\phi - \mathcal{L}^*\phi)\,dx,$$

for all $\phi \in D_{p'}$. Since $(\lambda - \mathcal{L}^*)(D_{p'}) = (\lambda + \mathrm{tr}(B) + A_{p'})(D_{p'}) = L^{p'}$ (for λ large), we deduce that $w = 0$ and that $u \in D_p$.

The following two theorems characterize completely the spectrum of $\mathcal{L}_p$. In the first we consider the case where $\mathrm{tr}(B) \neq 0$ and use an argument from [**2**, Section 3].

Theorem 3.1. *If* $\mathrm{tr}(B) \neq 0$ *then* $\sigma(\mathcal{L}_p) = -\dfrac{\mathrm{tr}(B)}{p} + i\mathbb{R}$.

Proof. Suppose for example that $\mathrm{tr}(B) < 0$ and let $1 \le p < q \le \infty$; then (20) implies $\sigma(\mathcal{L}_p) \subseteq -\mathrm{tr}(B)/p + i\mathbb{R}$ and $\sigma(\mathcal{L}_q) \subseteq -\mathrm{tr}(B)/q + i\mathbb{R}$. If $\mu \in \mathbb{R}$, $-\mathrm{tr}(B)/q < \mu < -\mathrm{tr}(B)/p$ and $f \in C_c^\infty(\mathbb{R}^N), f \ge 0, f \neq 0$ we have

$$R(\mu, \mathcal{L}_q)f = \int_0^\infty e^{-\mu t} S(t) f dt \ge 0$$

whereas

$$R(\mu, \mathcal{L}_p)f = -R(-\mu, -\mathcal{L}_p) = -\int_0^\infty e^{\mu t} S(-t) f dt \le 0$$

so that for these values of μ the resolvent operators are not consistent. Using [**2**, Proposition 2.2] we obtain that the resolvent operator does not coincide for $-\mathrm{tr}(B)/q < \mathrm{Re}\,\mu < -\mathrm{tr}(B)/p$ and that $\sigma(\mathcal{L}_p) = -\mathrm{tr}(B)/p + i\mathbb{R}$, $\sigma(\mathcal{L}_q) = -\mathrm{tr}(B)/q + i\mathbb{R}$. A similar argument can be applied if $\mathrm{tr}(B) < 0$.

The following theorem, whose proof can be found in [**21**, Theorems 2.5 and 2.6], characterizes the spectrum of $\mathcal{L}_p$ when $\mathrm{tr}(B) = 0$.

Theorem 3.2. *If* $\mathrm{tr}(B) = 0$, *then* $\sigma(\mathcal{L}_p)$ *is an unbounded subgroup of* $i\mathbb{R}$ (*independent of* p). *It coincides with* $i\mathbb{R}$ *if* B *is not similar to a diagonal matrix with purely imaginary eigenvalues.*

3.3 - *Spectrum of Ornstein-Uhlenbeck operators*

Now we come back to the Ornstein-Uhlenbeck operator defined in (9) and the associated semigroup $(T(t))_{t \ge 0}$ given in (13). The main result of this section is stated in the following theorem.

Theorem 3.3. *The spectrum of* L_p *contains the spectrum of the drift* $\mathcal{L}_p$ *for any* $p \in [1, +\infty]$.

Proof. For every $k \in \mathbb{N}$, let V_k be the isometry of L^p defined by

$$V_k u(x) = k^{-N/p} u(k^{-1}x).$$

Then

$$V_k^{-1} L V_k u = \frac{1}{k^2} \Delta u(x) + \langle Bx, Du \rangle$$

and hence $V_k^{-1} L V_k u \to \mathcal{L} u$ in L^p, as $k \to \infty$ for every $u \in C_c^\infty(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is a core for $(\mathcal{L}, D_p(\mathcal{L}))$, by using the Trotter-Kato Theorem (see [**13**, Section III.4]) we obtain that

$$R(\lambda, V_k^{-1} L V_k) f \overset{k\to\infty}{\longrightarrow} R(\lambda, \mathcal{L}) f$$

for every $f \in L^p$ and for every λ with $\operatorname{Re}\lambda > -\operatorname{tr}(B)/p$. Therefore

$$\begin{aligned} \|R(\lambda,\mathcal{L})\|_{\mathcal{L}(L^p)} &\le \liminf_{k\to\infty} \|R(\lambda, V_k^{-1} L V_k)\|_{\mathcal{L}(L^p)} \\ &= \liminf_{k\to\infty} \|V_k^{-1} R(\lambda, L) V_k\|_{\mathcal{L}(L^p)} = \|R(\lambda, L)\|_{\mathcal{L}(L^p)}. \end{aligned} \tag{22}$$

Now let $\omega \in \sigma(\mathcal{L}_p)$, if $\lambda \to \omega$ then $\|R(\lambda,\mathcal{L})\| \overset{\lambda\to\omega}{\longrightarrow} \infty$ and by (22) we get $\|R(\lambda, L)\| \overset{\lambda\to\omega}{\longrightarrow} \infty$ hence $\omega \in \sigma(L_p)$.

As a consequence of the previous theorem we get that $\sigma(L_p)$ contains a vertical line or a subgroup of $i\mathbb{R}$ then the semigroup $(T(t))_{\ge 0}$ is not norm continuous (see [**13**, Theorem 4.18]) and hence not analytic, nor differentiable. This explains why the term $Bx \cdot \nabla T(t) f$ does not always belong to $L^p(\mathbb{R}^N)$.

The last theorem of this section characterizes the L^p-spectrum of Ornstein-Uhlenbeck operators when either the spectrum of B is contained in $\mathbb{C}^-$ or in $\mathbb{C}^+$ or finally when B is symmetric. Its proof is contained in [**21** Section 4, Section 5].

Theorem 3.4. $\sigma(L_p) = \left\{ \lambda \in \mathbb{C} : \ \operatorname{Re}\lambda \le -\frac{\operatorname{tr}(B)}{p} \right\}$ *if either*

(i) $B = B^*$ *or*

(ii) $\sigma(B) \subseteq \mathbb{C}^-$ *or*

(iii) $\sigma(B) \subseteq \mathbb{C}^+$.

4 - The Ornstein-Uhlenbeck semigroup in $L^p_\mu(\mathbb{R}^N)$

Let us consider the Ornstein-Uhlenbeck semigroup given by Kolmogorov's formula

$$(T(t)f)(x) = \frac{1}{\sqrt{(4\pi)^N \det Q_t}} \int_{\mathbb{R}^N} e^{-\frac14 \langle Q_t^{-1} y, y\rangle} f(e^{tB}x - y)\, dy, \quad t > 0,\ x \in \mathbb{R}^N \tag{23}$$

with $f \in C_b(\mathbb{R}^N)$. Here $0 \neq B$ is a real $N \times N$ matrix and Q_t is given by

$$Q_t = \int_0^t e^{sB} e^{sB^*} \, ds.$$

The generator of $(T(t))_{t \geq 0}$ is the Ornstein-Uhlenbeck operator

$$L = \Delta + Bx \cdot \nabla.$$

In this section we assume that the spectrum of B, $\sigma(B)$ is contained in the open left half plane $\mathbb{C}_-$. This assumption, as proved in [**11**], is equivalent to the existence of an invariant measure μ for $(T(t))_{t \geq 0}$, i.e., a probability measure μ such that

$$\int_{\mathbb{R}^N} T(t) f \, d\mu = \int_{\mathbb{R}^N} f \, d\mu$$

for every $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$.

We observe that $Q_t = \int_0^t e^{sB} e^{sB^*} \, ds$ converges increasing to $Q_\infty = \int_0^\infty e^{sB} e^{sB^*} ds$ and that e^{tB} converges to 0 as $t \to \infty$, thus

$$T(t)f(x) \overset{t \to \infty}{\to} \frac{1}{\sqrt{(4\pi)^N \det Q_\infty}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}\langle Q_\infty^{-1} y, y \rangle} f(y) \, dy$$

pointwise. As regards the invariant measure μ, one can check that it is given by a Gaussian density, $g(x)$, i.e.,

$$d\mu(x) = \frac{1}{\sqrt{(4\pi)^N \det Q_\infty}} e^{-\frac{1}{4}\langle Q_\infty^{-1} x, x \rangle} \, dx = g(x) \, dx. \tag{24}$$

By a direct computation one can verify that $L^* g = 0$ where L^* is the formal adjoint operator of L. Then, if $f \in C_c^\infty(\mathbb{R}^N)$,

$$\frac{d}{dt} \int_{\mathbb{R}^N} T(t) f(x) \, d\mu = \int_{\mathbb{R}^N} L T(t) f(x) \, d\mu = \int_{\mathbb{R}^N} T(t) f(x) L^* g(x) \, dx = 0,$$

therefore

$$\int_{\mathbb{R}^N} T(t) f(x) \, d\mu(x) = \int_{\mathbb{R}^N} f(x) \, d\mu(x)$$

and $g(x)dx$ is an invariant measure ($C_c^\infty(\mathbb{R}^N)$ is a core for the generator, see for example Section 7).

As we have seen in the first Section, $(T(t))_{t\geq 0}$ extends to a strongly continuous semigroup of positive contractions in $L^p_\mu(\mathbb{R}^N)$ for every $1 \leq p < \infty$. We denote by $D^\mu_p(L)$ the domain of its generator. Remark that, since $Q_t < Q_\infty$ in the sense of quadratic forms, the integral in (23) converges for every $f \in L^p_\mu(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, so that the extension of $(T(t))_{t\geq 0}$ to $L^p_\mu(\mathbb{R}^N)$ is still given by (23). Observe also that $D^\mu_p(L) \subset D^\mu_q(L)$ if $p \geq q$ and $L_p u = L_q u$ for $u \in D^\mu_p(L)$. If $1 < p < \infty$, we will prove in Section 7, that the domain $D^\mu_p(L)$ is nothing but the weighted Sobolev space $W^{2,p}_\mu(\mathbb{R}^N)$.

Lemma 4.1. *Let $1 < p \leq \infty$; then, for every $t > 0$, $T(t)$ maps $L^p_\mu(\mathbb{R}^N)$ into $C^\infty(\mathbb{R}^N) \cap W^{k,p}_\mu(\mathbb{R}^N)$ for every $k \in \mathbb{N}$. Moreover, there exists $C = C(k,p) > 0$ such that for every $f \in L^p_\mu(\mathbb{R}^N)$ the inequality*

$$\|D^\alpha T(t) f\|_{L^p_\mu(\mathbb{R}^N)} \leq \frac{C}{t^{|\alpha|/2}} \|f\|_{L^p_\mu(\mathbb{R}^N)}, \qquad t \in (0,1)$$

holds for every multiindex α with $|\alpha| = k$.

Proof. Let $f \in L^p_\mu(\mathbb{R}^N)$. Differentiating under the integral sign in (23) we get

$$\big(DT(t)f\big)(x) = -\frac{1}{2} \int_{\mathbb{R}^N} e^{tB^*} Q_t^{-1} y f\big(e^{tB}x - y\big) g_t(y)\, dy$$

where g_t is defined in (14) and B^* denotes the adjoint matrix of B. By Hölder inequality and

$$\|Q_t^{-1/2}\| \leq \frac{C}{t^{1/2}}, \qquad t \in (0,1],$$

we can estimate

$$\begin{aligned}
|(\nabla T(t) f)(x)| &\leq C \|Q_t^{-1/2}\| \int_{\mathbb{R}^N} |Q_t^{-1/2} y|\; |f(e^{tB}x - y)| g_t(y)\, dy \\
&\leq C \|Q_t^{-1/2}\| \left(\int_{\mathbb{R}^N} |Q_t^{-1/2} y|^{p'} g_t(y)\, dy \right)^{1/p'} \left(\int_{\mathbb{R}^N} |f(e^{tB}x - y)|^p g_t(y)\, dy \right)^{1/p} \\
&\leq C_p t^{-1/2} \big((T(t)|f|^p)(x)\big)^{1/p}.
\end{aligned}$$

Raising to the power p and integrating the above inequality with respect to μ, we deduce

$$\int_{\mathbb{R}^N} |\nabla T(t) f|^p d\mu \leq \frac{C_p}{t^{p/2}} \int_{\mathbb{R}^N} T(t)|f|^p d\mu = \frac{C_p}{t^{p/2}} \int_{\mathbb{R}^N} |f|^p d\mu$$

which is the thesis for $k = 1$. Using the equality $\nabla T(t)f = e^{tB^*}T(t)\nabla f$, which holds for every $f \in W^{1,p}_\mu(\mathbb{R}^N)$ and a simple iteration procedure one can prove the claim for $k > 1$. For example for $k = 2$, $i, j = 1, \ldots, N$ and $t \in (0, 1)$ we have

$$\begin{aligned}\|D_{ij}T(t)f\|_{L^p_\mu(\mathbb{R}^N)} &= \|D_i(D_jT(t/2)T(t/2)f)\|_{L^p_\mu(\mathbb{R}^N)}\\ &= \|D_i(e^{tB^*/2}T(t/2)DT(t/2)f)_j\|_{L^p_\mu(\mathbb{R}^N)}\\ &\le c\|\nabla T(t/2)f\|_{L^p_\mu(\mathbb{R}^N)}\|\nabla T(t/2)\|_{\mathcal{L}(L^p_\mu(\mathbb{R}^N))} \le \frac{c_p}{t}\|f\|_{L^p_\mu(\mathbb{R}^N)}.\end{aligned}$$

Lemma 4.2. *If $1 < p < \infty$, the map $u \to |x|u$ is continuous from $W^{1,p}_\mu(\mathbb{R}^N)$ to $L^p_\mu(\mathbb{R}^N)$.*

Proof. It suffices to show that there is a constant K_p such that for every $u \in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |x_h u(x)|^p\, d\mu(x) \le K_p \int_{\mathbb{R}^N} \big(|u(x)|^p + |Du(x)|^p\big)\, d\mu(x). \tag{25}$$

By a linear change of variables we may assume that Q_∞ is diagonal with eigenvalues $\mu_1, \ldots, \mu_N$ and hence that

$$g(x) = \frac{1}{(4\pi)^{N/2}(\mu_1 \cdots \mu_N)^{1/2}} \exp\Big\{-\sum_{i=1}^N x_i^2/(4\mu_i)\Big\}.$$

As a first case, assume $p \ge 2$. If $u \in C_c^\infty(\mathbb{R}^N)$, then one has, for $C = 2\max\{\mu_1, \ldots, \mu_N\}$:

$$\begin{aligned}&\int_{\mathbb{R}^N} |x_h u(x)|^p\, d\mu(x) \le -C \int_{\mathbb{R}^N} |u(x)|^p |x_h|^{p-2} x_h \cdot D_h g(x)\, dx\\ &= C \int_{\mathbb{R}^N} \big(p x_h u(x) |x_h u(x)|^{p-2} D_h u(x) + (p-1)|x_h|^{p-2}|u(x)|^p\big)\, d\mu(x)\\ &\le C_1 \int_{\mathbb{R}^N} |x_h|^{p-2}|u(x)|^p\, d\mu(x)\\ &+ C_2 \Big(\int_{\mathbb{R}^N} |x_h u(x)|^p\, d\mu(x)\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N} |D_h u(x)|^p\, d\mu(x)\Big)^{\frac{1}{p}}\\ &\le \varepsilon \int_{\mathbb{R}^N} |x_h u(x)|^p\, d\mu(x) + C_\varepsilon \int_{\mathbb{R}^N} (|u(x)|^p + |D_h u(x)|^p)\, d\mu(x),\end{aligned}$$

for every $\varepsilon > 0$, with a suitable C_ε (in the last line we have used Young's inequality and the estimate $|x_h|^{p-2} \le \varepsilon |x_h|^p + C_\varepsilon$). Choosing $\varepsilon < 1$ we deduce (25).

Let us deal with the case $1 < p < 2$. We proceed as before but we have to estimate in a different way the term

$$\int_{\mathbb{R}^N} |x_h|^{p-2} |u(x)|^p \, d\mu(x).$$

To simplify the notation, take $h = N$ and write $x' = (x_1, \ldots, x_{N-1})$, $g(x) = g'(x') \dfrac{e^{-x_N^2/4\mu_N}}{(4\pi\mu_N)^{1/2}}$, and $d\mu' = g'(x')dx'$, $d\mu'' = (4\pi\mu_N)^{-1/2} \exp\{-x_N^2/\mu_N\} dx_N$, so that

$$\begin{aligned}
\int_{\mathbb{R}^N} |x_N|^{p-2} |u(x)|^p \, d\mu(x) &= \int_{\mathbb{R}^{N-1}} d\mu'(x') \int_{\mathbb{R}} |x_N|^{p-2} |u(x', x_N)|^p \, d\mu''(x_N) \\
&= \int_{\mathbb{R}^{N-1}} d\mu'(x') \int_{|x_N| \ge 1} |x_N|^{p-2} |u(x', x_N)|^p \, d\mu''(x_N) \\
&+ \int_{\mathbb{R}^{N-1}} d\mu'(x') \int_{-1}^{1} |x_N|^{p-2} |u(x', x_N)|^p \, d\mu''(x_N) \\
&= J_1 + J_2.
\end{aligned}$$

Clearly, $J_1 \le \int_{\mathbb{R}^N} |u(x)|^p \, d\mu(x)$. Let us estimate J_2. For every $x' \in \mathbb{R}^{N-1}$ we have, by the Sobolev embedding $W^{1,p}(-1,1) \hookrightarrow L^\infty(-1,1)$,

$$\begin{aligned}
\int_{-1}^{1} |x_N|^{p-2} |u(x', x_N)|^p \, d\mu''(x_N) &\le C \Big(\sup_{|x_N| \le 1} |u(x', x_N)| \Big)^p \int_{-1}^{1} |x_N|^{p-2} \, dx_N \\
&\le C_1 \int_{-1}^{1} \big(|u(x', x_N)|^p + |D_N u(x', x_N)|^p \big) \, dx_N \\
&\le C_2 \int_{\mathbb{R}} \big(|u(x', x_N)|^p + |D_N u(x', x_N)|^p \big) \, d\mu''(x_N)
\end{aligned}$$

whence, integrating on $\mathbb{R}^{N-1}$,

$$J_2 \le C_2 \int_{\mathbb{R}^N} (|u(x)|^p + |Du(x)|^p) \, d\mu(x),$$

and this completes the proof.

It follows, in particular, that the map $\mathcal{L}u = \langle Bx, Du\rangle$ is bounded from $W^{2,p}_\mu(\mathbb{R}^N)$ into $L^p_\mu(\mathbb{R}^N)$ for $1<p<\infty$, in fact

$$\| \, |x| \, \nabla u\|_{L^p_\mu} \le c\|u\|_{W^{2,p}_\mu}, \qquad u \in W^{2,p}_\mu(\mathbb{R}^N). \tag{26}$$

We observe that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}_\mu(\mathbb{R}^N)$, $1 \le p<\infty$. Indeed, a simple truncation argument shows that the set of $W^{k,p}_\mu$-functions with compact support is dense and, given $u \in W^{k,p}_\mu(\mathbb{R}^N)$ with compact support, the usual approximating functions $\phi_\varepsilon * u$ converge to u, as $\varepsilon \to 0$, in $W^{k,p}(\mathbb{R}^N)$ and hence in $W^{k,p}_\mu(\mathbb{R}^N)$ (here $\phi_\varepsilon(x) = \varepsilon^{-N}\phi(x/\varepsilon)$ where $\phi \in C_c^\infty(\mathbb{R}^N)$ is positive with integral 1).

Corollary 4.1. *For $1<p<\infty$ the semigroup $T(t)$ is analytic in $L^p_\mu(\mathbb{R}^N)$.*

Proof. If $f \in \mathcal{S}(\mathbb{R}^N)$ then $T(t)f \in \mathcal{S}(\mathbb{R}^N) \subset D^\mu_p$. From Lemmas 4.1, 4.2 it follows that

$$\|\Delta T(t)f\|_{L^p_\mu} \le \frac{C}{t}\|f\|_{L^p_\mu}, \quad 0<t\le 1,$$

and

$$\|Bx \cdot \nabla T(t)f\|_{L^p_\mu} \le c\|x \cdot \nabla T(t)f\|_{L^p_\mu} \le c\|T(t)f\|_{W^{2,p}_\mu} \le \frac{c}{t}\|f\|_{L^p_\mu}.$$

Hence summing up we get

$$\|LT(t)f\|_{L^p_\mu} \le \frac{c}{t}\|f\|_{L^p_\mu}$$

and the thesis follows.

Lemma 4.3. *$T(t)$ is compact in $L^p_\mu(\mathbb{R}^N)$, $1<p<\infty$.*

The proof of the compactness of $T(t)$ relies on the fact that $T(t)$ maps $L^p_\mu(\mathbb{R}^N)$ into $W^{1,p}_\mu(\mathbb{R}^N)$ and on the compactness of embedding of $W^{1,p}_\mu(\mathbb{R}^N)$ in $L^p_\mu(\mathbb{R}^N)$ as the following lemma asserts.

Lemma 4.4. *The embedding $W^{1,p}_\mu(\mathbb{R}^N) \hookrightarrow L^p_\mu(\mathbb{R}^N)$ is compact for $1<p<\infty$.*

Proof. As already seen in Lemma 4.2, $u \to |x|u$ is bounded from $W^{1,p}_\mu$ to L^p_μ. We consider the unit ball $\mathcal{B}$ in $W^{1,p}_\mu(\mathbb{R}^N)$

$$\mathcal{B} = \{u \in W^{1,p}_\mu(\mathbb{R}^N) : \ \|u\|_{W^{1,p}_\mu} \le 1\}.$$

By the previous result we get $\| \, |x| \, u\|_{L^p_\mu} \le C \quad \forall u \in \mathcal{B}$, i.e.

$$\int_{\mathbb{R}^N} |x|^p |u|^p \, d\mu(x) \le C^p.$$

Fix $\varepsilon > 0$ and choose $R > 0$ large enough such that

$$\int_{|x|\geq R} |u|^p \, d\mu(x) \leq \int_{|x|\geq R} \frac{|x|^p}{R^p} |u|^p \, d\mu(x) \leq \frac{C^p}{R^p} \leq \varepsilon \quad \forall u \in \mathcal{B}.$$

The compactness of $\mathcal{B}_{|B_R}$ in $L^p_\mu(B_R) = L^p(B_R)$ and above estimate show the existence of a finite ε-net for $\mathcal{B}$, which concludes the proof.

The compactness of the semigrouop implies that of the resolvent and gives, as an important consequence, the discreteness of the spectrum of $(L, D^\mu_p(L))$, in contrast with the results of the preceeding section where the Lebesgue measure was considered, instead of the invariant measure. We examine now in detail the spectrum of $(L, D^\mu_p(L))$.

4.1 - *Eigenfunctions*

In this section we assume that $1 < p < \infty$. The following estimate is the main step to show that the eigenfunctions of $(L, D^\mu_p(L))$ are polynomials. We define $s(B) = \sup\{\mathrm{Re}\,\lambda : \ \lambda \in \sigma(B)\} < 0$.

Lemma 4.5. *Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be such that $s(B) + \varepsilon < 0$. Then there exists $C = C(k, \varepsilon)$ such that for every $u \in W^{k,p}_\mu(\mathbb{R}^N)$*

$$\sum_{|\alpha|=k} \|D^\alpha T(t)u\|_p \leq C e^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p, \quad t \geq 0. \tag{27}$$

Proof. Let $C_1 = C_1(\varepsilon)$ be such that $\|e^{tB^*}\| \leq C_1 e^{t(s(B)+\varepsilon)}$ for any $t > 0$ and recall that $\nabla T(t)u = e^{tB^*} T(t)\nabla u$ for every $u \in W^{1,p}_\mu(\mathbb{R}^N)$. Since $T(t)$ is contractive in $L^p_\mu(\mathbb{R}^N)$ we have

$$\begin{aligned} \|\nabla T(t)u\|_{L^p_\mu} &= \|e^{tB^*} T(t)\nabla u\|_{L^p_\mu} \\ &\leq C_\varepsilon e^{t(s(B)+\varepsilon)} \|T(t)\nabla u\|_{L^p_\mu} \\ &\leq C_\varepsilon e^{t(s(B)+\varepsilon)} \|T(t)\nabla u\|_{L^p_\mu} \end{aligned}$$

and the statement is proved for $k = 1$. Suppose that the statement is true for k with a suitable constant C_k and consider $u \in W^{k+1,p}_\mu(\mathbb{R}^N)$. Then, if $|\alpha| = k$,

$$\begin{aligned} \|DD^\alpha T(t)u\|_p = \|D^\alpha \nabla T(t)u\|_p &= \|D^\alpha e^{tB^*} T(t)\nabla u\|_p \\ &\leq C_1 e^{t(s(B)+\varepsilon)} \|D^\alpha T(t)\nabla u\|_p \\ &\leq C_1 C_k e^{t(k+1)(s(B)+\varepsilon)} \|DD^\alpha u\|_p \end{aligned}$$

and the claim follows.

Proposition 4.1. *The eigenfunctions of L in $L^p_\mu(\mathbb{R}^N)$ are polynomials.*

Proof. Let $u \in D^\mu_p(L)$ be an eigenfunction, i.e. there exists $\lambda \in \mathbb{C}$ such that $\lambda u = Lu$. Since $T(t)u = e^{\lambda t}u$, from Lemma 4.1 we deduce that $u \in W^{k,p}_\mu(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, for every k. Clearly $D^\alpha T(t)u = e^{\lambda t}D^\alpha u$ for every multiindex α. Then

$$e^{(\operatorname{Re}\lambda)t}\sum_{|\alpha|=k}\|D^\alpha u\|_{L^p_\mu} = \sum_{|\alpha|=k}\|D^\alpha T(t)u\|_{L^p_\mu} \le C(k,\varepsilon)e^{tk(s(B)+\varepsilon)}\sum_{|\alpha|=k}\|D^\alpha u\|_{L^p_\mu}$$

and finally

$$\sum_{|\alpha|=k}\|D^\alpha u\|_{L^p_\mu} \le C(k,\varepsilon)e^{t[(k(s(B)+\varepsilon)-\operatorname{Re}\lambda]}\sum_{|\alpha|=k}\|D^\alpha u\|_{L^p_\mu}.$$

If $ks(B) < \operatorname{Re}\lambda$ then $\|D^\alpha u\|_{L^p_\mu} = 0$. Hence $D^\alpha u = 0$ if $k > \dfrac{|\operatorname{Re}\lambda|}{s(B)}$ and u is a polynomial of degree less than or equal to $\dfrac{|\operatorname{Re}\lambda|}{s(B)}$.

4.2 - *Spectrum of L in $L^p_\mu(\mathbb{R}^N)$*

As before, we denote by

$$\mathcal{L}u = \langle Bx, Du\rangle$$

the drift term in (10) and we reduce the computation of the spectrum of L to that of $\mathcal{L}$.

Lemma 4.6. *The following statements are equivalent.*

(i) $\lambda \in \sigma((L, D^\mu_p(L)))$.

(ii) *There exists a homogeneous polynomial $u \neq 0$ such that $\mathcal{L}u = \lambda u$.*

(iii) *There exists a homogeneous polynomial $u \neq 0$ such that*

$$S(t)u(x) = e^{\lambda t}u(x).$$

Remark 4.1. *Consider the equation $\lambda u - \mathcal{L}u = 0$ with u polynomial, $\lambda \in \mathbb{C}$. If $B = -I$ this is the well-known Euler equation satisfied by all homogenuous functions of degree $(-\lambda)$. If we require that u is a polynomial, we obtain $(-\lambda) \in \mathbb{N}$, hence all negative integers are eigenvalues of $\mathcal{L}$ and, for every $n \in \mathbb{N}$, all homogeneous polynomials of degree n are eigenfunctions.*

The equation with a general B is much more complicated and we shall not characterize all polynomial solutions but only the values of λ for which such a solution exists. Observe that a polynomial u satisfies $\lambda u - \mathcal{L}u = 0$ if and only if

$$u(e^{tB}x) = e^{t\lambda}u(x) \qquad t \ge 0,\ x \in \mathbb{R}^N \tag{28}$$

or, equivalently, by analytic continuation

$$u(e^{tB}z) = e^{t\lambda}u(z) \qquad t \geq 0,\ z \in \mathbb{C}^N.$$

This suggests that the general case can be treated by decomposing B into Jordan blocks. Assume, for example, that $B = \gamma I + R$ consists of only one Jordan block (hence R is nilpotent) and that u is an n-homogeneous polynomial satisfying

$$e^{t\lambda}u(z) = u(e^{tB}z) = u(e^{\gamma t}e^{tR}z) = e^{n\gamma t}u(e^{tR}z).$$

Since e^{tR} is a polynomial in t, R being nilpotent, comparing the growth as $t \to \infty$ we deduce $\lambda = n\gamma$ and hence the spectrum of L is contained in $\gamma\mathbb{N}$ (the opposite inclusion is easily proved by considering the functions $u(z) = z_N^n$).

Arguing similarly in the general case, one proves the following result which describes the spectrum of L in $L^p_\mu(\mathbb{R}^N)$ in terms of the spectrum of B, see [**22**].

Theorem 4.1. *Let $\lambda_1, \dots, \lambda_r$ be the (distinct) eigenvalues of B . Then*

$$\sigma((L, D_p^\mu(L))) = \left\{\lambda = \sum_{j=1}^r n_j\lambda_j : n_j \in \mathbb{N} \cup \{0\}\right\}, \qquad 1<p<\infty.$$

Moreover, the linear span of the generalized eigenfunctions of L is dense in $L^p_\mu(\mathbb{R}^N)$.

4.3 - *Angle of analyticity of $T(t)$*

The standard theory of analytic semigroups and the above result imply that the angle of sectoriality θ_p of $(T(t))_{t\geq 0}$, satisfies the inequality $\theta_p \leq \pi/2 - \theta$ where θ is the spectral angle of $(L, D_p^\mu(L))$ that coincides with the spectral angle of B. Surprisingly enough, there are situations where $\theta_2 < \pi/2 - \theta$. In these cases, the angle of sectoriality is not determined by the spectral angle of L or, equivalently, by the spectral angle of B.

For every $\theta \in (0, \pi]$ we define the open sector Σ_θ by

$$\Sigma_\theta := \{z \in \mathbb{C} : |\arg z| < \theta\}.$$

The following result is proved in [8, Theorem 2].

Theorem 4.2. *Let $(T(t))_{t\geq 0}$ be the Ornstein-Uhlenbeck semigroup on $L^p_\mu(\mathbb{R}^N)$, $1<p<\infty$. Let $\theta_p \in \left(0, \frac{\pi}{2}\right]$ be defined by*

$$\cot\theta_p = \frac{\sqrt{(p-2)^2 + p^2\gamma^2}}{2\sqrt{p-1}},$$

where $\gamma := 2\|\frac{1}{2}I + Q_\infty B^*\|$. *Then the following assertions are true:*

(i) $(T(t))_{t\geq 0}$ *extends to an analytic contraction semigroup on the sector* Σ_{θ_p}.

(ii) *If* $(T(t))_{t\geq 0}$ *extends to an analytic semigroup on the sector* $\Sigma_{\theta'}$ *for some* $\theta' \in \left(0, \frac{\pi}{2}\right]$, *then* $\theta' \leq \theta_p$, *i.e., the angle* θ_p *is optimal.*

In the selfadjoint case we obtain $\cot\theta_2 = 0$ and $\cot\theta_p = \dfrac{|p-2|}{2\sqrt{p-1}}$.

5 - More general operators

In this section we introduce elliptic operators with unbounded coefficients and we study the Markov semigroups associated with them.

We follow the approach of [**24**, Section 4]. We consider second order elliptic partial differential operators

$$Au(x) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^N F_i(x)D_iu(x), \qquad x \in \mathbb{R}^N$$

under the following hypotheses which will be kept in the whole section

(H1) $a_{ij} = a_{ji}$, F_i real-valued locally Hölder continuous functions of exponent $0<\alpha<1$;

(H2) the ellipticity condition:

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

for every $x,\ \xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$.

The operator so defined is locally uniformly elliptic, that is uniformly elliptic on every compact subset of $\mathbb{R}^N$.

We consider A endowed with its maximal domain in $C_b(\mathbb{R}^N)$

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad p<\infty :\ Au \in C_b(\mathbb{R}^N)\}.$$

Our main interest is in the existence of (spatial) bounded solutions to the parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N,\ t>0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{29}$$

with initial datum $f \in C_b(\mathbb{R}^N)$. The unbounded interval $[0,\infty[$ can be changed to a finite interval to any bounded $[0,T]$ without affecting the results. Since the coefficients can be unbounded, the classical theory does not apply and existence and uniqueness for (29) are not clear. Quite surprisingly, existence is never a problem, but uniqueness is, as the following example shows. More precisely, in what follows we prove non uniqueness in $D_{max}(A)$ for the corresponding non homogeneous elliptic equation. This, however, implies non uniqueness for the parabolic problem (see for example [**24**, Section 5]).

Example 5.1. *Let* $Au(x) = u''(x) + 4x^3u'$. *Then* $I - A$ *is not injective on* $D_{max}(A)$.

We prove that all the solutions of the equation

$$u = u'' + 4x^3u' \tag{30}$$

are bounded. Equation (30) *can be written as* $u = e^{-x^4}\dfrac{d}{dx}(e^{x^4}u')$ *or equivalently* $e^{x^4}u = \dfrac{d}{dx}(e^{x^4}u')$. *Let* $\alpha > 0$. *Integrating from* $\alpha > 0$ *to* x *we deduce*

$$e^{x^4}u'(x) = e^{\alpha^4}u'(\alpha) + \int_\alpha^x e^{t^4}u(t)dt,$$

then

$$u'(x) = e^{\alpha^4 - x^4}u'(\alpha) + \int_\alpha^x e^{-x^4+t^4}u(t)dt.$$

Let now $s > 0$. *Integrating from* α *to* s, *we obtain*

$$u(s) = u(\alpha) + \int_\alpha^s e^{\alpha^4 - x^4}u'(\alpha)dx + \int_\alpha^s dx \int_\alpha^x e^{-x^4+t^4}u(t)dt.$$

Let $\beta > \alpha$ *and set* $v(\beta) = \max\{|u(s)| : \alpha \le s \le \beta\}$. *Then*

$$|u(s)| \le c_1(\alpha) + c_2(\alpha) + v(\beta)\int_\alpha^s dx \int_\alpha^x e^{-x^4+t^4}dt$$

with $c_1(\alpha) = |u(\alpha)|$, $c_2(\alpha) = \sqrt{\pi}\,|u'(\alpha)|e^{\alpha^4}$. *Now we observe that there exists a positive constant* c *not depending on* α *such that, for positive* x,

$$\int_\alpha^x e^{t^4}dt \le \int_0^x e^{t^4}dt \le c\frac{e^{x^4}}{1+x^3}.$$

Then, for $\alpha \le s \le \beta$,

$$|u(s)| \le c_1(\alpha) + c_2(\alpha) + cv(\beta) \int_\alpha^s \frac{1}{1+x^3}\,dx$$

$$\le c_1(\alpha) + c_2(\alpha) + cv(\beta) \int_\alpha^\infty \frac{1}{x^3}\,dx$$

$$\le c_1(\alpha) + c_2(\alpha) + c\frac{v(\beta)}{\alpha^2}.$$

If α *is large enough such that* $\frac{c}{\alpha^2} \le \frac{1}{2}$, *then it follows that* $v(\beta) \le 2c_1(\alpha) + 2c_2(\alpha)$ *for all* $\beta > \alpha$.

Uniqueness is implied by the existence of Lyapunov functions for the operator A.

Definition 5.1. *We say that* V *is a* Lyapunov function for A *if* $V \in C^2(\mathbb{R}^N)$, $V \ge 0$, V *goes to infinity as* $|x| \to \infty$ *and* $\lambda V - AV \ge 0$ *for some* $\lambda > 0$.

Proposition 5.1. *Suppose that* V *is a Lyapunov function for the operator* A. *Then problem* (29) *admits at most one bounded solution.*

Proof. We prove that, if u solves

$$\begin{cases} u_t = Au \\ u(0) = 0, \end{cases}$$

then $u \le 0$. Consider the function $z(t,x) = e^{-\lambda t}u(t,x)$ where λ is as in the definition of Lyapunov functions. Then z satisfies $z_t = (A - \lambda)z$. For every $\varepsilon > 0$, introduce a second auxiliary function $w(t,x) = z(t,x) - \varepsilon V(x)$. Then $w_t - (A-\lambda)w = z_t - (A-\lambda)z + \varepsilon(A-\lambda)V = \varepsilon(A-\lambda)V \le 0$ and

$$\begin{cases} w_t - (A-\lambda)w \le 0 \\ w(0,x) = -\varepsilon V(x) \le 0. \end{cases}$$

Moreover $w(t,x) \to -\infty$ as $|x| \to \infty$ uniformly with respect to t on the compact intervals of the form $[0,T]$. Suppose that $w > 0$ somewhere. Then there exists (t_0,x_0) maximum point for w with $w(t_0,x_0) > 0$. Then $t_0 > 0$, $w_t(t_0,x_0) \ge 0$, $Aw(t_0,x_0) \le 0$ (see [**25**, Appendix 8]) and obviously $\lambda w(t_0,x_0) > 0$. This is a contradiction. We conclude that $w(t,x) \le 0$ i.e. $z(t,x) \le \varepsilon V(x)$ for every $\varepsilon > 0$. Letting ε go to 0, the claim follows.

Example 5.2. *Suppose that the coefficients of the operator A satisfy*

$$\sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} F_i(x)x_i \leq \lambda |x|^2$$

for $|x|$ large and for a suitable $\lambda > 0$, then $V(x) = \frac{|x|^2}{2}$ is a Lyapunov function for A.
If the coefficients satisfy

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\frac{x_i x_j}{|x|^2} + \sum_{i=1}^{N} F_i(x)\,\frac{x_i}{|x|^2} \leq \lambda |x|^2 \log |x|$$

for $|x|$ large and for a suitable $\lambda > 0$, then $V(x) = \log |x|$ is a Lyapunov function for A.

The existence of a Lyapunov function will be always assumed from now on, even though it not necessary for the existence part.

Remark 5.1. *If there exists a Lyapunov function for A, then the constant function $\mathbf{1}$ solves the parabolic problem with initial datum $f = \mathbf{1}$. By uniqueness $T(t)\mathbf{1} = \mathbf{1}$ and, by the representation formula,*

$$\mathbf{1} = T(t)\mathbf{1} = \int_{\mathbb{R}^N} p(t, x, y)\, dy.$$

5.1 - *Existence*

We will prove the following theorem.

Theorem 5.1. *There exists a positive semigroup $(T(t))_{t\geq 0}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, $u(t,x) = T(t)f(x)$ belongs to the space $C^{1+\frac{\alpha}{2}, 2+\alpha}_{loc}((0, +\infty) \times \mathbb{R}^N)$, is a bounded solution of the following differential equation*

$$u_t(t,x) = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u(t,x) + \sum_{i=1}^{N} F_i(x) D_i u(t,x)$$

and satisfies

$$\lim_{t\to 0} u(t,x) = f(x)$$

pointwise.

Let us fix a ball $B_\rho = B_\rho(0)$ in $\mathbb{R}^N$ and consider the problem

$$\begin{cases} u_t(t,x) = Au(t,x) & t>0, \;\; x\in B_\rho, \\ u(t,x) = 0 & t>0, \;\; x \in \partial B_\rho, \\ u(0,x) = f(x) & x \in \mathbb{R}^N. \end{cases} \tag{31}$$

Since the operator A is uniformly elliptic and the coefficients are bounded in B_ρ, there exists a unique solution u_ρ of problem (31). The operator $A_\rho = (A, D_\rho(A))$ with

$$D_\rho(A) = \{u \in C_0(B_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p<\infty : Au \in C(\overline{B}_\rho)\}$$

generates an analytic semigroup $(T_\rho(t))_{t\geq 0}$ in the space $C(\overline{B}_\rho)$ and the function defined by $T_\rho(t)f(x) = u_\rho(t,x)$ solves (31).

Since the domain $D_\rho(A)$ is not dense in $C(\overline{B}_\rho)$, the semigroup is not strongly continuous at 0; indeed one can prove that $T_\rho(t)f$ converges uniformly to f in $\overline{B}_\rho$ as $t\to 0$ if and only if $f \in C_0(B_\rho)$. However the convergence is uniform in compact sets $\overline{B}_\sigma$ for every $\sigma<\rho$ and hence pointwise on B_ρ. The operators $T_\rho(t)$ are also bounded in $L^p(B_\rho)$ for every $1\leq p<\infty$. We refer to [**18**, Chapter 3] and [**15**, Chapter 3, Section 7] for a detailed description of the results mentioned above.

Now we let ρ go to infinity in order to define the semigroup associated with A in $\mathbb{R}^N$. To this aim we need an easy consequence of the parabolic maximum principle.

Lemma 5.1. *Let $0\leq f \in C_b(\mathbb{R}^N)$ and let $\rho<\rho_1<\rho_2$. Then for every $t\geq 0$ and $x\in B_\rho$ we have $0 \leq T_{\rho_1}(t)f(x) \leq T_{\rho_2}(t)f(x)$.*

Proof. First suppose that $f\equiv 0$ on the boundary ∂B_{ρ_1}. Then, since $T_{\rho_1}(t)f$ converges uniformly to f in $\overline{B}_{\rho_1}$ as $t\to 0$ if and only if $f\in C_0(B_{\rho_1})$, $w(t,x) = T_{\rho_2}(t)f(x) - T_{\rho_1}(t)f(x)$ is continuous on $[0,\infty)\times \overline{B}_{\rho_1}$, vanishes for $t=0$, is nonnegative for $x\in\partial B_{\rho_1}$ and solves $w_t(t,x) = 0$ for $x\in B_{\rho_1}$, $t>0$. By the maximum principle $w(t,x)\geq 0$ in $[0,\infty)\times\overline{B}_{\rho_1}$. In general, if $f\in C_b(\mathbb{R}^N)$, we approximate it in the $L^2(B_{\rho_2})$ norm with continuous functions vanishing on ∂B_{ρ_1}. Using the first part of the proof and the boundedness of $T_{\rho_i}(t)$ in $L^2(B(\rho_i))$, $i=1, 2$, the claim follows.

Proof. (Part of Theorem 5.1). If $f\in C_b(\mathbb{R}^N)$, $x\in\mathbb{R}^N$ we set

$$T(t)f(x) := \lim_{\rho\to\infty} T_\rho(t)f(x).$$

We know that this limit exists if $f\geq 0$ by monotonicity, and in the general case by writing $f = f^+ - f^-$. $T(t)$ are positive operators and $\|T(t)f\|_\infty \leq \|f\|_\infty$, since this is true for all operators $T_\rho(t)$. Let us prove that the operators so defined satisfy the semigroup law. Consider $f\geq 0$. Let $t,s>0$. Then

$$T(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t)T_\rho(s)f(x) \leq T(t)T(s)f(x).$$

On the other hand, for every $\rho_1 > 0$ we have

$$T(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t)T_\rho(s)f(x) \geq \lim_{\rho\to\infty} T_{\rho_1}(t)T_\rho(s)f(x) = T_{\rho_1}(t)T(s)f(x)$$

and, letting $\rho_1 \to \infty$, it follows that $T(t+s)f(x) \geq T(t)T(s)f(x)$. Hence the semigroup law is proved for positive function. The general case follows by linearity, as above.

Set $u(t,x) = T(t)f(x)$, $u_\rho(t,x) = T_\rho(t)f(x)$ for $t \geq 0$ and $x \in \mathbb{R}^N$. Fix positive numbers ε, τ, σ with $0<\varepsilon<\tau$. By the interior Schauder estimates ([**15**, Chapter 3, Section 2]) there exists a positive constant C such that for $\rho > \sigma$

$$\|u_\rho\|_{C^{1+\frac{\alpha}{2},2+\alpha}([\varepsilon,\tau]\times\overline{B}_\sigma)} \leq C\|u_\rho\|_\infty \leq C\|f\|_\infty.$$

So by Ascoli's Theorem it follows that u_ρ converges to u uniformly in $[\varepsilon,\tau]\times\overline{B}_\sigma$. Fix now $\sigma_1<\sigma$, $\varepsilon<\varepsilon_1<\tau_1<\tau$ and apply again Schauder estimates. For $\rho_2 > \rho_1 > \sigma > \sigma_1$ we have

$$\|u_{\rho_2} - u_{\rho_1}\|_{C^{1+\frac{\alpha}{2},2+\alpha}([\varepsilon_1,\tau_1]\times\overline{B}_{\sigma_1})} \leq C\|u_{\rho_2} - u_{\rho_1}\|_{L^\infty([\varepsilon,\tau]\times\overline{B}_\sigma)}.$$

Then $u \in C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,\infty)\times\mathbb{R}^N)$ and, letting $\rho\to\infty$ in the equation satisfied by u_ρ, it follows that $\partial_t u = Au$.

To complete the proof of Theorem 5.1 we have to show that the initial value f is taken with continuity. We observe that we need to prove much less than the strong continuity of $T(t)$ (which fails even for the Ornstein-Uhlenbeck semigroup), but we start by proving strong continuity on functions vanishing at infinity.

Proposition 5.2. *For every* $f \in C_0(\mathbb{R}^N)$

$$\lim_{t\to 0} T(t)f = f$$

uniformly on $\mathbb{R}^N$.

Proof. Consider first $f \in C^2(\mathbb{R}^N)$ with support contained in B_σ and let $\rho > \sigma$. Then, for $x \in B_\rho$, since $f \in D_\rho(A)$

$$T_\rho(t)f(x) - f(x) = \int_0^t T_\rho(s)Af(x)\,ds$$

and, letting $\rho\to\infty$ by dominated convergence,

$$T(t)f(x) - f(x) = \int_0^t T(s)Af(x)\,ds.$$

By the arbitrarity of ρ, the equality above holds for every $x \in \mathbb{R}^N$ and, taking the supremum over $x \in \mathbb{R}^N$, we get

$$\|T(t)f - f\|_\infty \le t\|Af\|_\infty.$$

This implies that $T(t)f$ converges to f uniformly as $t \to 0$. The claim follows by approximating a function $f \in C_0(\mathbb{R}^N)$ with functions as above, and using the contractivity of $T(t)$.

Remark 5.2. *By the previous proposition we cannot deduce that $(T(t))_{t\ge 0}$ restricted to $C_0(\mathbb{R}^N)$ is strongly continuous since the invariance property of $C_0(\mathbb{R}^N)$ under the semigroup is not guaranteed (see Section 5.5).*

In order to deal with arbitrary continuous and bounded functions we need to prove an integral representation of $T(t)$. Let $f \in C_c(\mathbb{R}^N)$, the space of continuous functions having compact support in $\mathbb{R}^N$. By the Riesz representation theorem, for every $t > 0, x \in \mathbb{R}^N$ we can find a Borel measure $p(t, x, dy)$ such that

$$T(t)f(x) = \int_{\mathbb{R}^N} p(t, x, y)f(y)\, dy.$$

Since $T(t)$ is positive and contractive, $p(t, x, \cdot)$ is a positive measure having total mass less than or equal to 1.

Lemma 5.2. *For every $t > 0, x \in \mathbb{R}^N$, $p(t, x, dy)$ is a probability measure.*

Proof. By uniqueness $1 = T(t)1 = \lim_{\rho\to\infty} u_\rho$ where u_ρ solves (31) with $f = 1$. Let $\phi_\rho \in C_c(\mathbb{R}^N)$ be such that $0 \le \phi_\rho \le 1$ and $\phi_\rho = 1$ on B_ρ. Then u_ρ solves also (31) with $f = \phi_\rho$. It follows that

$$u_\rho(t, x) \le T(t)\phi_\rho \le 1$$

in B_ρ hence

$$u_\rho(t, x) \le \int_{\mathbb{R}^N} p(t, x, dy)\phi_\rho(y) \le 1.$$

Letting $\rho \to \infty$ the thesis follows by dominated convergence.

We can now prove the representation formula for arbitrary $f \in C_b(\mathbb{R}^N)$.

Theorem 5.2. *The following representation formula for $T(t)$ holds*

$$T(t)f(x) = \int_{\mathbb{R}^N} p(t, x, dy)f(y)$$

for $f \in C_b(\mathbb{R}^N)$.

Proof. We may assume that $0 \le f \le 1$. Let $0 \le f_n \le f$ be such that $f_n \in C_c(\mathbb{R}^N)$ and $f_n \to f$ pointwise. Then

$$T(t)f_n = \int_{\mathbb{R}^N} p(t,x,dy)f_n(y)$$

and $T(t)f_n \le T(t)f$. By dominated convergence

$$\int_{\mathbb{R}^N} p(t,x,dy)f(y) \le T(t)f(x).$$

Changing f with $1-f$ and using the fact that the measure $p(t,x,\cdot)$ is a probability measure, one obtains the opposite inequality.

Remark 5.3. *By using the integral representation formula, we can extend the semigroup to the space of the bounded measurable functions. If $f \in B_b(\mathbb{R}^N)$, with $T(t)f(x)$ we mean $\int_{\mathbb{R}^N} p(t,x,y)f(y)\,dy$.*

We now show the continuity up to $t=0$ of $u(t,x) = T(t)f(x)$ thus completing the proof of Theorem 5.1. For any measurable set $E \subset \mathbb{R}^N$, we set

$$p(t,x,E) = \int_E p(t,x,y)\,dy.$$

Theorem 5.3. *Let $f \in C_b(\mathbb{R}^N)$. Then $T(t)f$ converges to f as $t \to 0$ uniformly on compact subsets of $\mathbb{R}^N$.*

Proof. Let $\rho > 0$ and $f_1, f_2 \in C_0(\mathbb{R}^N)$ be such that $0 \le \chi_{B_\rho} \le f_1 \le \chi_{B_{2\rho}} \le f_2 \le 1$. By the positivity of $T(t)$,

$$T(t)f_1(x) \le p(t,x,B_{2\rho}) \le T(t)f_2(x)$$

for all $x \in \mathbb{R}^N$. By Proposition 5.2, $T(t)f_1 \to f_1$, $T(t)f_2 \to f_2$ uniformly on $\overline{B}_\rho$ as $t \to 0$. We observe that $f_1 = f_2 \equiv 1$ on $\overline{B}_\rho$. It follows that $p(t,x,B_{2\rho}) \to 1$ on $\overline{B}_\rho$ as $t \to 0$. Then

$$0 \le p(t,x,\mathbb{R}^N \setminus B_{2\rho}) = p(t,x,\mathbb{R}^N) - p(t,x,B_{2\rho}) \le 1 - p(t,x,B_{2\rho}) \to 0 \tag{32}$$

as $t \to 0$ uniformly on $\overline{B}_\rho$.

Let now $f \in C_b(\mathbb{R}^N)$ and $\eta \in C_0(\mathbb{R}^N)$ be such that $0 \le \eta \le 1$, $\eta = 1$ on $B_{2\rho}$, $\mathrm{supp}(\eta) \in B_{3\rho}$. Then

$$T(t)f - f = T(t)f - T(t)(\eta f) + T(t)(\eta f) - \eta f$$

on B_ρ. By Proposition 5.2, $\|T(t)(\eta f) - \eta f\|_\infty \to 0$ as $t \to 0$. Concerning the remaining terms, by (32) we have

$$\begin{aligned}|T(t)f(x) - T(t)(\eta f)(x)| &= T(t)((1-\eta)f)(x)\\ &= \int_{\mathbb{R}^N} p(t,x,y)((1-\eta(y))f(y))\,dy\\ &\le p(t,x,\mathbb{R}^N \setminus B_{2\rho})\|f\|_\infty \to 0\end{aligned}$$

uniformly on $\overline{B}_\rho$. We conclude therefore that $T(t)f \to f$ uniformly on $\overline{B}_\rho$ and by the arbitrarity of ρ the claim follows.

Now we prove some properties of the operators $T(t)$ in $C_b(\mathbb{R}^N)$.

Proposition 5.3. *Let (g_n) be a bounded sequence in $C_b(\mathbb{R}^N)$, $g \in C_b(\mathbb{R}^N)$ and suppose that $g_n(x) \to g(x)$ for every $x \in \mathbb{R}^N$. Then, for every $0<\varepsilon<\tau$ and $\sigma > 0$, $T(t)g_n(x) \to T(t)g(x)$ uniformly for $(t,x) \in [\varepsilon,\tau] \times \overline{B}_\sigma$.*

Proof. Using the integral representation of $T(t)g$ and the Lebesgue dominated convergence Theorem, we immediately deduce that $T(t)g_n(x) \to T(t)g(x)$ pointwise in $\mathbb{R}^N$. Let $K > 0$ be such that $\|g_n\|_\infty \le K$ for every $n \in \mathbb{N}$. Then $\|T(t)g_n\|_\infty \le K$ for every $n \in \mathbb{N}$ and, by the Schauder estimates, for every $0<\varepsilon<\tau$ and $\sigma > 0$ there exists $C > 0$ such that

$$\sup_n \|T(\,\cdot\,)g_n(\,\cdot\,)\|_{C^1([\varepsilon,\tau]\times\overline{B}_\sigma)} \le C.$$

By Ascoli's Theorem we deduce that the convergence of $T(\,\cdot\,)g_n(\,\cdot\,)$ is uniform in $[\varepsilon,\tau] \times \overline{B}_\sigma$.

As consequence of the continuity result just proved, we deduce that $(T(t))_{t\ge0}$ satisfies the strong Feller property.

Proposition 5.4. *The semigroup $(T(t))_{t\ge0}$ is irreducible (see Definition 2.1) and has the strong Feller property.*

Proof. The irreducibility follows from the integral representation since the kernel p is strictly positive. We do not prove here this result which depends on Harnack's inequality and refer to [**25**]. We prove that the semigroup has the strong Feller property. Let f be a bounded Borel function and let $(f_n) \in C_b(\mathbb{R}^N)$ be a bounded sequence such that $f_n(x) \to f(x)$ for almost every $x \in \mathbb{R}^N$. By dominated convergence (using the integral representation), $T(t)f_n \to T(t)f$ pointwise in $\mathbb{R}^N$. Using the interior Schauder estimates, as in Proposition 5.3, we deduce that $T(t)f_n \to T(t)f$ uniformly on compact sets and then the limit $T(t)f \in C_b(\mathbb{R}^N)$.

5.2 - *The generator of* $T(t)$

Even though $(T(t))_{t\geq 0}$ is not strongly continuous one can define its generator in a weak sense, as done in [**28**].

Lemma 5.3. *The operator* A *with domain* $D_{max}(A)$ *is closed.*

Proof. Let (u_n) be a sequence in $D_{max}(A)$, assume that u_n converges to $u \in C_b(\mathbb{R}^N)$ and Au_n to $g \in C_b(\mathbb{R}^N)$ uniformly in $\mathbb{R}^N$. For any pair of bounded sets $\Omega \subset\subset \Omega' \subset \mathbb{R}^N$ the estimate

$$\|u_n - u_k\|_{W^{2,p}(\Omega)} \leq C[\|Au_n - Au_k\|_{L^p(\Omega')} + \|u_n - u_k\|_{L^p(\Omega')}] < \infty$$

holds for every $1 < p < \infty$ and any $n,\ k \in \mathbb{N}$, with $C = C(p, \Omega, \Omega', A)$ (see e.g. [**16**, Theorem 9.11]). Hence, by the arbitrariness of Ω, the function u belongs to $W^{2,p}_{loc}(\mathbb{R}^N)$. Finally, by the continuity of A from $W^{2,p}_{loc}(\mathbb{R}^N)$ into $L^p_{loc}(\mathbb{R}^N)$, we infer that $g = Au$.

For $\lambda > 0$ consider the (pointwise) Laplace transform of the semigroup given by

$$R(\lambda)f(x) = \int_0^\infty e^{-\lambda t} T(t) f(x) dt, \qquad x \in \mathbb{R}^N.$$

Proposition 5.5. *For every* $f \in C_b(\mathbb{R}^N)$, $u = R(\lambda) f \in D_{max}(A)$ *and* $\lambda u - Au = f$.

Proof. For every $n \in \mathbb{N}$, set $u_n(x) = \int_{\frac{1}{n}}^{n} e^{-\lambda t} T(t) f(x) dt$. Then $u_n \in C^2(\mathbb{R}^N)$ and

$$\begin{aligned} Au_n(x) &= \int_{\frac{1}{n}}^{n} e^{-\lambda t} A T(t) f(x) dt = \int_{\frac{1}{n}}^{n} e^{-\lambda t} \frac{d}{dt} T(t) f(x) dt \\ &= [e^{-\lambda t} T(t) f(x)]_{\frac{1}{n}}^{n} + \lambda \int_{\frac{1}{n}}^{n} e^{-\lambda t} T(t) f(x) dt, \end{aligned}$$

hence

$$(\lambda - A) u_n(x) = -e^{-\lambda n} T(n) f(x) + e^{-\frac{\lambda}{n}} T\left(\frac{1}{n}\right) f(x)$$

and, letting $n \to \infty$, $(\lambda - A) u_n \to f$ uniformly on compact sets of $\mathbb{R}^N$. On the other hand $u_n \to u$ in $C_b(\mathbb{R}^N)$. As in the proof of Lemma 5.3, we deduce $u \in D_{max}(A)$ and $\lambda u - Au = f$.

Theorem 5.4. *The operator $\lambda - A$ is bijective from $D_{max}(A)$ to $C_b(\mathbb{R}^N)$. Moreover $(\lambda - A)^{-1} = R(\lambda)$ and $\|(\lambda - A)^{-1}\| \le \dfrac{1}{\lambda}$.*

Proof. By Proposition 5.5, $\lambda - A$ is surjective. Let us prove the injectivity.

Let $0 \not\equiv u_0 \in D_{max}(A)$ be such that $\lambda u_0 - Au_0 = 0$. It is easy to check that $u(t,x) = e^{\lambda t} u_0(x)$ solves the problem

$$\begin{cases} u_t = Au \\ u(\cdot, 0) = u_0. \end{cases}$$

By uniqueness $T(t)u_0 = e^{\lambda t} u_0$ (see Proposition 5.1) and, since the solution must be bounded in $[0,\infty) \times \mathbb{R}^N$, $u_0 = 0$. The resolvent estimate immediately follows by the expression of $R(\lambda)$.

In the next two propositions we characterize $D_{max}(A)$ as for strongly continuous semigroups. Observe however that the convergence in the sup-norm topology is replaced by pointwise and dominated convergence. Similarly, all time derivatives involved should be understood as pointwise derivatives rather than respect to the norm-topology.

Proposition 5.6. *If $u \in D_{max}(A)$, then*

$$\sup_{t \in (0,1]} \left\| \frac{T(t)u - u}{t} \right\|_\infty \le C$$

for some positive constant C and, for every $x \in \mathbb{R}^N$,

$$\lim_{t \to 0} \frac{T(t)u(x) - u(x)}{t} = Au(x).$$

Moreover $T(t)u \in D_{max}(A)$ and

$$\frac{d}{dt} T(t)u = AT(t)u = T(t)Au.$$

Proof. Given $\lambda > 0$, $u \in D_{max}(A)$, set $f = \lambda u - Au$. We know that $u(x) = R(\lambda)f(x) = \int_0^\infty e^{-\lambda s} T(s) f(x) ds$. Since $T(t)$ is continuous with respect to the dominated convergence (in the sense of Proposition 5.3), $T(t)u$ can be computed moving $T(t)$ inside the integral defining u and so

$$\begin{aligned}T(t)u(x)-u(x)&=\int_0^\infty e^{-\lambda s}T(t+s)f(x)ds-\int_0^\infty e^{-\lambda s}T(s)f(x)ds\\&=\int_t^\infty e^{-\lambda(s-t)}T(s)f(x)ds-\int_0^\infty e^{-\lambda s}T(s)f(x)ds\\&=e^{\lambda t}\int_t^\infty e^{-\lambda s}T(s)f(x)ds-\int_0^t e^{-\lambda s}T(s)f(x)ds.\end{aligned}$$

Then

$$\sup_{t\in(0,1]}\left|\frac{T(t)u(x)-u(x)}{t}\right|\le C$$

for some positive constant C, for all $x\in\mathbb{R}^N$ and

$$\lim_{t\to 0}\frac{T(t)u(x)-u(x)}{t}=\lambda\int_0^\infty e^{-\lambda s}T(s)f(x)ds-f(x)=\lambda u(x)-f(x)=Au(x).$$

Moreover, by Proposition 5.5,

$$\begin{aligned}T(t)u(x)&=T(t)\int_0^\infty e^{-\lambda s}T(s)f(x)ds=\int_0^\infty e^{-\lambda s}T(t)T(s)f(x)ds\\&=R(\lambda)T(t)f(x)\in D_{max}(A)\end{aligned}$$

and

$$\frac{d}{dt}T(t)u=AT(t)u=\lim_{h\to 0}\frac{T(h)T(t)u-T(t)u}{h}=T(t)\lim_{h\to 0}\frac{T(h)u-u}{h}=T(t)Au$$

pointwise in $\mathbb{R}^N$, since $t^{-1}(T(h)u-u)$ is bounded and converges pointwise to Au pointwise.

Conversely, we have

Proposition 5.7. *Let $u\in C_b(\mathbb{R}^N)$ be such that*

$$\sup_{t\in(0,1]}\left\|\frac{T(t)u-u}{t}\right\|_\infty\le C$$

for some positive constant C and

$$\lim_{t\to 0}\frac{T(t)u(x)-u(x)}{t}=f(x)$$

for every $x\in\mathbb{R}^N$ and some $f\in C_b(\mathbb{R}^N)$. Then $u\in D_{max}(A)$ and $Au=f$.

Proof. Let $u \in C_b(\mathbb{R}^N)$. For $s > 0$, consider $u_s = T(s)u$. Then, since $T(t)$ is continuous with respect to the dominated convergence,

$$Au_s = \frac{d}{ds}T(s)u = \lim_{t\to 0}\frac{T(t+s)u - T(s)u}{t} = T(s)\lim_{t\to 0}\frac{T(t)u - u}{t} = T(s)f \to f$$

for $s \to 0$ boundedly and locally uniformly in $\mathbb{R}^N$. Therefore $u_s \in D_{max}(A)$, $u_s \to u$, $Au_s \to f$ for $s \to 0$ boundedly and locally uniformly. We conclude that $u \in D_{max}(A)$ and $Au = f$ (see the proof of Lemma 5.3).

As in the case of strongly continuous semigroups the domain $D_{max}(A)$ has certain density properties in $C_b(\mathbb{R}^N)$.

Proposition 5.8. *Let $f \in C_b(\mathbb{R}^N)$, then there exists $(u_n) \subset D_{max}(A)$, a positive constant C such that $\|u_n\|_\infty \le C$ for every $n \in \mathbb{N}$ and $u_n \to f$ pointwise in $\mathbb{R}^N$.*

Proof. Let $f \in C_b(\mathbb{R}^N)$. Set $v(x) = \int_0^t T(s)f(x)ds$. Then

$$T(h)v(x) = \int_0^t T(s+h)f(x)ds = \int_h^{t+h} T(s)f(x)ds$$

and

$$\begin{aligned} T(h)v(x) - v(x) &= \int_h^{t+h} T(s)f(x)ds - \int_0^t T(s)f(x)ds \\ &= \int_t^{t+h} T(s)f(x)ds - \int_0^h T(s)f(x)ds. \end{aligned}$$

It follows that, for $h \to 0$,

$$\frac{T(h)v(x) - v(x)}{h} \to T(t)f(x) - f(x)$$

pointwise and boundedly in $\mathbb{R}^N$. By Proposition 5.7, this implies that $v \in D_{max}(A)$ and $Av = T(t)f - f$. Moreover

$$\frac{v(x)}{t} = \frac{1}{t}\int_0^t T(s)f(x)ds \to f(x)$$

for $t \to 0$ pointwise and boundedly in $\mathbb{R}^N$.

Remark 5.4. *In the previous proposition, we have proved also that, if* $f \in C_b(\mathbb{R}^N)$, *then*

$$T(t)f(x) - f(x) = A\int_0^t T(s)f(x)ds.$$

5.3 - *Existence of an invariant measure*

In this subsection we study the existence of an invariant measure for the semigroup. As stated in Proposition 5.4, the semigroup $(T(t))_{t\geq 0}$ is irreducible and satisfies the strong Feller property. Hence, from [**12**, Theorem 4.2.1], it follows that, if an invariant measure exists, it is unique and it is also absolutely continuous with respect to the Lebesgue measure. For these reasons, we investigate only the existence of an invariant measure.

Let us recall the following compactness result due to Prokhorov (see [**5**]). A family of probability measures $(\mu_j)_{j\in J}$ on $\mathbb{R}^N$ is (relatively) weakly compact, with respect to the duality induced by $C_b(\mathbb{R}^N)$, if and only if it is tight, i.e. it verifies the following condition: for every $\varepsilon > 0$ there exists $\rho > 0$ such that $\mu_j(B_\rho) \geq 1 - \varepsilon$ for every $j \in J$.

Theorem 5.5 (Has'minskii). *Suppose that there exists a function* $V \in C^2(\mathbb{R}^N)$ *such that* $V(x) \to \infty$, $Av \to -\infty$ *as* $|x| \to \infty$. *Then* $(T(t))_{t\geq 0}$ *has an invariant measure.*

Proof. First observe that, in particular, $V + C$ is a Lyapunov function for A if C is sufficiently large. Therefore, for every $\lambda > 0$, $\lambda - A$ is injective and $\int_{\mathbb{R}^N} p(t,x,y)dy = 1$.

By the assumption, there exists $K > 0$ such that $AV(x) \leq K$ for every $x \in \mathbb{R}^N$. For every $n \in \mathbb{N}$, we consider $\psi_n \in C^\infty(\mathbb{R}^N)$ such that $\psi_n(t) = t$ for $t \leq n$, ψ_n is constant in $[n+1,\infty)$, $\psi_n' \geq 0$, $\psi_n'' \leq 0$. Let $u_n(t,x) = T(t)(\psi_n \circ V)(x)$. Since $\psi_n \circ V \in D_{max}(A)$, we have

$$\begin{aligned}
\partial_t u_n(t,x) &= T(t)A(\psi_n \circ V) = \int_{\mathbb{R}^N} p(t,x,y)A(\psi_n \circ V)(y)\,dy \\
&= \int_{\mathbb{R}^N} p(t,x,y)\left[\psi_n'(V(y))AV(y) + \psi_n''(V(y))\sum_{i,j=1}^N a_{ij}(y)D_iV(y)D_jV(y)\right]dy \\
&\leq \int_{\mathbb{R}^N} p(t,x,y)[\psi_n'(V(y))AV(y)]\,dy.
\end{aligned}$$

Integrating this inequality we obtain

$$u_n(t,x) - \psi_n(V(x)) \le \int_0^t ds \int_{\mathbb{R}^N} p(s,x,y)[\psi_n'(V(y))AV(y)]\,dy.$$

Set $E = \{x \in \mathbb{R}^N : 0 \le AV(x) \le K\}$ where K is as above, then

$$\begin{aligned} u_n(t,x) - \psi_n(V(x)) \le & \int_0^t ds \int_{\mathbb{R}^N\setminus E} p(s,x,y)[\psi_n'(V(y))AV(y)]\,dy \\ & + \int_0^t ds \int_E p(s,x,y)[\psi_n'(V(y))AV(y)]\,dy. \end{aligned} \tag{33}$$

Observe now that ψ_n' is an increasing sequence which converges to 1 for n going to ∞ and, since AV is negative in $\mathbb{R}^N \setminus E$,

$$\int_{\mathbb{R}^N\setminus E} p(s,x,y)[\psi_n'(V(y))AV(y)]\,dy \to \int_{\mathbb{R}^N\setminus E} p(s,x,y)AV(y)\,dy$$

by monotone convergence and

$$\int_E p(s,x,y)[\psi_n'(V(y))AV(y)]\,dy \to \int_E p(s,x,y)AV(y)\,dy$$

by dominated convergence. Therefore, by monotone convergence again, letting $n \to \infty$ in (33), we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^N} p(t,x,y)V(y)\ dy - V(x) \\ & \le \int_0^t ds \int_{\mathbb{R}^N\setminus E} p(s,x,y)AV(y)\,dy + \int_0^t ds \int_E p(s,x,y)AV(y)\,dy \end{aligned}$$

and then, since $\int_{\mathbb{R}^N} p(t,x,y)V(y)\,dy$ is positive,

$$\begin{aligned} -\int_0^t ds \int_{\mathbb{R}^N\setminus E} p(s,x,y)AV(y)\,dy & \le Kt - \int_{\mathbb{R}^N} p(t,x,y)V(y)\,dy + V(x) \\ & \le Kt + V(x). \end{aligned}$$

Let $\varepsilon, \rho > 0$ be such that $-AV(y) \geq \frac{1}{\varepsilon}$ if $|y| \geq \rho$. Then

$$\frac{1}{\varepsilon}\int_0^t ds \int_{\mathbb{R}^N\setminus B_\rho} p(s,x,y)\,dy \leq -\int_0^t ds \int_{\mathbb{R}^N\setminus B_\rho} p(s,x,y)AV(y)\,dy$$

$$\leq -\int_0^t ds \int_{\mathbb{R}^N\setminus E} p(s,x,y)AV(y)\,dy$$

$$\leq Kt + V(x).$$

We have proved that

$$1 - \frac{1}{t}\int_0^t p(s,x,B_\rho)\,ds = \frac{1}{t}\int_0^t p(s,x,\mathbb{R}^N \setminus B_\rho)\,ds \leq \varepsilon\left(K + \frac{V(x)}{t}\right),$$

or equivalently,

$$\frac{1}{t}\int_0^t p(s,x,B_\rho)\,ds \geq 1 - \varepsilon\left(K + \frac{V(x)}{t}\right)$$

and, for $t \geq 1$,

$$\frac{1}{t}\int_0^t p(s,x,B_\rho)\,ds \geq 1 - \varepsilon(K + V(x)).$$

This implies that, for every fixed $x \in \mathbb{R}^N$, the family of probability measures $\left\{\frac{1}{t}\int_0^t p(s,x,\cdot)\,ds\right\}_{t\geq 1}$ is tight. Fix $x = 0$. By the Prokhorov Theorem there exist a measure μ and a sequence (t_n) diverging to infinity such that, for every $f \in C_b(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(y)d\mu(y) = \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} ds \int_{\mathbb{R}^N} p(s,0,y)f(y)\,dy$$

$$= \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} T(s)f(0)ds.$$

Then, for every $f \in C_b(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} T(t)f(y)d\mu(y) = \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} T(t+s)f(0)ds = \lim_{n\to\infty}\frac{1}{t_n}\int_t^{t+t_n} T(s)f(0)ds$$
$$= \lim_{n\to\infty}\frac{1}{t_n}\left[\int_0^{t_n} T(s)f(0)ds - \int_0^{t} T(s)f(0)ds + \int_{t_n}^{t+t_n} T(s)f(0)ds\right]$$
$$= \int_{\mathbb{R}^N} f(y)d\mu(y)$$

and μ is an invariant measure.

Example 5.3. *If $A = \Delta + F\cdot\nabla$ with $F\cdot x \to -\infty$ for $|x|\to\infty$, then A has an invariant measure. Indeed the function $V(x) = |x|^2$ satisfies the assumptions in Theorem* 5.5.

5.4 - *Preservation of $C_0(\mathbb{R}^N)$*

Proposition 5.9. *Suppose that $(T(t))_{t\geq 0}$ is the semigroup generated by the operator $(A, D_{max}(A))$ and suppose that there exists $W > 0$ such that $W\to 0$ as $|x|\to\infty$ and $AW \leq \lambda W$. Then $R(\lambda)$ (hence $T(t)$) preserves $C_0(\mathbb{R}^N)$.*

Proof. Let $0 \leq f \in C_c^\infty(\mathbb{R}^N)$ be such that $\operatorname{supp} f \subset B_\rho$ and let $r > \rho$. As in the parabolic case, given $\lambda > 0$, the solution of $\lambda u - Au = f$ in $D_{max}(A)$ is obtained as limit, as $r\to\infty$, of the solutions u_r of

$$\begin{cases} \lambda u_r(x) - Au_r(x) = f & x \in B_r \\ u_r(x) = 0 & |x| = r \end{cases}$$

(see [**24**, Theorem 3.4]). In $B_r \setminus B_\rho$ and for $c > 0$ such that $\frac{1}{\lambda}\|f\|_\infty - cW \leq 0$ on ∂B_ρ, we have

$$\begin{cases} (\lambda - A)(u_r - cW) = f - c(\lambda - A)W = -c(\lambda - A)W \leq 0 & \text{in } B_r\setminus B_\rho \\ u_r - cW = -cW \leq 0 & \text{on } \partial B_r \\ u_r - cW \leq \frac{1}{\lambda}\|f\|_\infty - cW \leq 0 & \text{on } \partial B_\rho. \end{cases}$$

By the maximum principle, $u_r \leq cW$ in $B_r\setminus B_\rho$ and, letting r go to infinity, $u \leq cW$ outside B_ρ and so $u = (\lambda - A)^{-1}f = R(\lambda)f \in C_0(\mathbb{R}^N)$. The case of a general $f \in C_0(\mathbb{R}^N)$ follows by approximation.

Example 5.4. *Suppose that* $A = \Delta + F \cdot \nabla$ *with* $|F(x) \cdot x| \leq c(1 + |x|^2)$ *for a suitable positive constant c, then the semigroup* $(T(t))_{t \geq 0}$ *is generated by* $(A, D_{max}(A))$ *and preserves* $C_0(\mathbb{R}^N)$.

It is sufficient to choose $V(x) = 1 + |x|^2$ *as Lyapunov function and* $W(x) = (\log(|x| + 1))^{-1}$ *in Proposition* 5.9.

5.5 - *Compactness in* $C_b(\mathbb{R}^N)$ *and non preservation of* $C_0(\mathbb{R}^N)$

In this subsection, following [**25**, Section 3], we state a sufficient condition, for the compactness of $T(t)$ in $C_b(\mathbb{R}^N)$. First we state a necessary and sufficient condition for the compactness.

Proposition 5.10. $(T(t))_{t \geq 0}$ *is compact in* $C_b(\mathbb{R}^N)$ *if and only if for all* $t,\ \varepsilon > 0$ *there exists* $R = R(t, \varepsilon)$ *such that* $p(t, x, B_R) \geq 1 - \varepsilon$ *for all* $x \in \mathbb{R}^N$.

Theorem 5.6. *Suppose that there exist* $V \geq 0$, $V \in C^2(\mathbb{R}^N)$ *such that* $V(x) \to \infty$ *as* $|x| \to \infty$ *and a convex function* $g : \mathbb{R} \to \mathbb{R}$ *integrable near* ∞ *such that* $AV \leq -g(V)$. *Then* $(T(t))_{t \geq 0}$ *is compact.*

Example 5.5. *If* $A = \Delta + F \cdot \nabla$ *with* $F(x) \cdot x \leq -c|x|^{2+\varepsilon}$ *for some positive* ε, *c, then* $(T(t))_{t \geq 0}$ *is compact. Indeed* $V(x) = |x|^2$ *and* $g(t) = c_1 - c_2 t^{1+\frac{\varepsilon}{2}}$, *for suitable positive constants* c_1 *and* c_2 *satisfy the assumptions in the previous theorem.*

Remark 5.5. *Let* $(T(t))_{t \geq 0}$ *be compact in* $C_b(\mathbb{R}^N)$. *By Proposition* 5.10, *it follows that for all* $t,\ \varepsilon > 0$ *there exists* $R = R(t, \varepsilon)$ *such that*

$$p(t, x, B_R) = T(t)\chi_{B_R}(x) \geq 1 - \varepsilon$$

for all $x \in \mathbb{R}^N$. *In particular* $T(t)\chi_{B_R}(x)$ *does not tend to zero as* $|x| \to \infty$ *and* $C_0(\mathbb{R}^N)$ *is not preserved. Similarly for* $L^p(\mathbb{R}^N)$.

Example 5.6. *We use polar coordinates* r, θ *in the plane, identifying the point* x *with the complex number* $re^{i\theta}$. *For* $\alpha > 0$, *let* S *be the angle* $S := \{r \geq 0,\ 0 \leq \theta \leq \alpha\}$, *let moreover* ϕ *be any smooth function such that* $0 \leq \phi \leq 1$, $\phi(\theta) = 0$ *for* $\theta \notin]0, \alpha[$ *and* $\phi(\theta) > 0$ *for* $\theta \in]0, \alpha[$ *and consider the operator*

$$Au = \Delta u - \phi(\theta) r^2 u_\theta - (1 - \phi(\theta)) r^2 u_r.$$

Then the semigroup generated by A is compact and $C_0(\mathbb{R}^2)$ *is not preserved* (*see* [**25,** Example 7.9]).

6 - Regularity of the invariant measure

In this section we study regularity properties and pointwise bounds of invariant measures associated with second order elliptic partial differential operators in $\mathbb{R}^N$.

We consider the operator A endowed with its maximal domain $D_{max}(A)$ in $C_b(\mathbb{R}^N)$ and the semigroup $(T(t))_{t\geq 0}$ generated by A. If μ is a Borel probability measure on $\mathbb{R}^N$, then

$$\int_{\mathbb{R}^N} T(t)f\,d\mu = \int_{\mathbb{R}^N} f\,d\mu \qquad \forall f \in C_b(\mathbb{R}^N) \tag{34}$$

if and only if

$$\int_{\mathbb{R}^N} Af\,d\mu = 0 \qquad \forall f \in D_{max}(A). \tag{35}$$

In particular $\int_{\mathbb{R}^N} Afd\mu = 0$ for every $f \in C_c^\infty(\mathbb{R}^N)$, i.e. $A^*\mu = 0$ in the sense of distributions where A^* is the formal adjoint operator of A.

We restrict our study to the simplest case of operators of the form

$$A = \Delta + F \cdot \nabla.$$

However more general operators are allowed: one can consider for example operators in divergence form

$$\sum_{i,j=1}^{N} D_i(a_{ij}D_j) + \sum_{i=1}^{N} F_i D_i$$

under suitable assumptions on the coefficients (see [**23**] for details). In order to describe regularity properties for the invariant measure we assume that the drift $F \in L^p_\mu(\mathbb{R}^N)$ for a suitable p. This assumption, though obscure at a first sight, is easily verified using the Lemma below (see also the Example), assuming suitable bounds on F.

In the following lemma we prove the integrability of certain unbounded functions with respect to μ via Lyapunov functions (see Definition 5.1) techniques. These results will be used to establish pointwise upper and lower bounds for the density ρ of the invariant measure μ.

Lemma 6.1. *Assume that there exists a C^2-function $V : \mathbb{R}^N \to \mathbb{R}$ such that $V(x) \to \infty$ as $|x| \to \infty$ and $AV(x) \to -\infty$ as $|x| \to \infty$. Then AV belongs to $L^1_\mu(\mathbb{R}^N)$.*

Proof. For every n, we consider $\psi_n \in C^\infty(\mathbb{R})$ such that $\psi_n(t) = t$ for $t \le n$, ψ_n is constant in $[n+1, \infty[$, $\psi_n' \ge 0$, $\psi_n'' \le 0$. Now, since $\psi_n \circ V \in D_{max}(A)$, (35) holds for $\psi_n \circ V$. Let B be a ball such that $AV(x) \le 0$ if $x \notin B$. Then

$$A(\psi_n \circ V) = (\psi_n' \circ V)AV + (\psi_n'' \circ V)\sum_{i,j=1}^{N} a_{ij} D_i V D_j V \le 0$$

outside B. Then, for large n

$$\int_{\mathbb{R}^N \setminus B} |A(\psi_n \circ V)|\, d\mu = -\int_{\mathbb{R}^N \setminus B} A(\psi_n \circ V)\, d\mu = \int_B AV\, d\mu \le C$$

and the statement follows letting $n \to \infty$ and using Fatou's lemma.

A simple consequence is that if $|F|^p \le c|AV|$ then $F \in L^p_\mu(\mathbb{R}^N)$.

Example 6.1. *Assume that*

$$\limsup_{|x|\to\infty}\left(|x|^{1-\beta} F(x) \cdot \frac{x}{|x|}\right) < -c < 0 \tag{36}$$

for some $c > 0$, $\beta > 0$. *Then* $V(x) = \exp\{\delta|x|^\beta\}$ *for* $|x| \ge 1$ *is a Lyapunov function for* $\delta < \beta^{-1}c$. *Moreover,* $\exp\{\delta|x|^\beta\}$ *is integrable with respect to* μ, *for* $\delta < \beta^{-1}c$.

The integrability of certain exponential functions will be important to derive upper bounds for the density of the invariant measure with respect to the Lebesgue measure.

Remark 6.1. *If* $F(x) = -c|x|^{\beta-1}\dfrac{x}{|x|} = -\dfrac{c}{\beta}\nabla|x|^\beta$, *then the invariant measure is given by* $d\mu = e^{-c/\beta|x|^\beta}dx$ *and* $e^{\delta|x|^\beta}$ *is Lyapunov function for* $\delta < \beta^{-1}c$.

We prove that under very weak conditions $d\mu = \rho\, dx$ with $\rho \in L^p(\mathbb{R}^N)$ for $p < N/(N-1)$.

Proposition 6.1. *If* $F \in L^1_\mu(\mathbb{R}^N)$, *then* $d\mu = \rho\, dx$ *with* $\rho \in L^p(\mathbb{R}^N)$, $1 \le p < N/(N-1)$.

Proof. The invariance of μ yields that for every $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\Delta\phi + F \cdot \nabla\phi)\, d\mu = 0,$$

hence

$$\int_{\mathbb{R}^N} (\phi - \Delta\phi)\, d\mu = \int_{\mathbb{R}^N} (\phi + F \cdot D\phi)\, d\mu.$$

Since $F \in L^1_\mu(\mathbb{R}^N)$,

$$\left| \int_{\mathbb{R}^N} (\phi - \Delta\phi)\, d\mu \right| \le C(1 + \|F\|_{L^1_\mu})\|\phi\|_{1,\infty} = c\|\phi\|_{1,\infty}. \tag{37}$$

Fix $1 < p < N/(N-1)$ and let $p' = p/(p-1)$ be the conjugate exponent of p. Clearly $p' > N$. Given $\psi \in C_c^\infty(\mathbb{R}^N)$ we consider w, the solution of $w - \Delta w = \psi$. Then $w \in \mathcal{S}(\mathbb{R}^N)$, w, $Dw \in C_0(\mathbb{R}^N)$ and $\|w\|_{2,p'} \le C_1\|\psi\|_{p'}$ with C_1 independent of ψ. Moreover, by Sobolev embedding $\|w\|_{1,\infty} \le C_2\|w\|_{2,p'}$.

In order to show that we can insert w in (37) we use a cut-off function. Let $\eta_n = \eta(x/n)$ where $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfies $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. Thus $w_n = \eta_n w \in C_c^\infty(\mathbb{R}^N)$, $w_n \to w$ in $C_b^2(\mathbb{R}^N)$ and $w_n - \Delta w_n \to w - \Delta w$ uniformly as $n \to \infty$.

Then, passing to the limit and using (37) we have

$$\begin{aligned}\left| \int_{\mathbb{R}^N} \psi d\mu \right| &= \lim_{n\to\infty} \left| \int_{\mathbb{R}^N} (w_n - \Delta w_n) d\mu \right| \\ &\le c \lim_{n\to\infty} \|w_n\|_{W^{1,\infty}} = c\|w\|_{W^{1,\infty}} \le \|\psi\|_{L^{p'}}\end{aligned}$$

for every $\psi \in C_c^\infty(\mathbb{R}^N)$. Then $\mu = \rho dx$ with $\rho \in L^p(\mathbb{R}^N)$.

Now, assuming $F \in L^k_\mu(\mathbb{R}^N)$ for some $k > N$, we prove global boundedness and Sobolev regularity for the density ρ.

Theorem 6.1. *If $F \in L^k_\mu(\mathbb{R}^N)$ for some $k > N$ then $\rho \in L^\infty(\mathbb{R}^N)$ and $\rho \in W^{1,k}(\mathbb{R}^N)$.*

Proof. As already seen $d\mu = \rho dx$ with $\rho \in L^p$ for every $1 \le p < N/(N-1)$. As in the proof of Proposition 6.1,

$$\int_{\mathbb{R}^N} (\phi - \Delta\phi)\, \rho\, dx = \int_{\mathbb{R}^N} (\phi + F \cdot D\phi)\, \rho\, dx$$

for all $\phi \in C_c^\infty(\mathbb{R}^N)$. Now let $1 < p_1 < N/(N-1)$ be such that $\frac{1}{r_1} = \frac{1}{k} + \frac{1}{p_1} < 1$, $(r_1 < p_1)$, then $\rho, |F|\rho \in L^{r_1}(\mathrm{supp}\phi)$, hence

$$\int_{\mathbb{R}^N} (\phi - \Delta\phi)\, \rho\, dx \le C\|\phi\|_{W^{1,r_1'}}$$

for all $\phi \in C_c^\infty(\mathbb{R}^N)$. Applying this estimate to the difference quotients

$$\tau_h \phi = \frac{\phi(x+h) - \phi(x)}{|h|}$$

we get

$$\left| \int_{\mathbb{R}^N} (\tau_h \phi - \Delta(\tau_h \phi))\rho dx \right| \leq C\|\phi\|_{W^{2,r_1'}}$$

or, equivalently,

$$\left| \int_{\mathbb{R}^N} (\phi - \Delta\phi)\tau_h \rho \, dx \right| \leq C\|\phi\|_{W^{2,r_1'}}. \tag{38}$$

As before, given $\psi \in C_c^\infty(\mathbb{R}^N)$ we can consider $\omega \in \mathcal{S}(\mathbb{R}^N)$ the solution of $w - \Delta w = \psi$ and insert it in (38), hence

$$\left| \int_{\mathbb{R}^N} \psi \tau_h \rho \, dx \right| \leq C\|\phi\|_{W^{2,r_1'}} \leq C\|\psi\|_{L^{r_1'}}. \tag{39}$$

Thus $\|\tau_h \rho\|_{L^{r_1}} \leq C$ and $\rho \in W^{1,r_1}(\mathbb{R}^N)$. If $r_1 > N$, then by Sobolev embedding we have $\rho \in L^\infty(\mathbb{R}^N)$ and the proof is complete. Otherwise, if $r_1 < N$, then $\rho \in L^{p_2}(\mathbb{R}^N)$ with $\dfrac{1}{p_2} = \dfrac{1}{r_1} - \dfrac{1}{N}$ and $p_2 > p_1$. As before we consider $\dfrac{1}{r_2} = \dfrac{1}{p_2} + \dfrac{1}{k}$ and notice that $\rho, |F|\rho \in L^{r_2}(\text{supp}\phi)$. Proceeding as before we get $\rho \in W^{1,r_2}$ with $r_2 > r_1$. If $r_2 > N$ we conclude, otherwise in the same way as before we can construct a sequence of p_n and r_n with

$$\frac{1}{p_{n+1}} = \frac{1}{r_n} - \frac{1}{N} \quad \text{and} \quad \frac{1}{r_n} = \frac{1}{p_n} + \frac{1}{k}.$$

After a finite number of steps $r_n > N$ and $\rho \in L^\infty(\mathbb{R}^N)$.

Then $\rho, |F|\rho \in L^k(\mathbb{R}^N)$, hence (38) implies

$$\left| \int (\phi - \Delta\phi)\tau_h \rho \, dx \right| \leq C\|\phi\|_{W^{2,k'}}$$

and as before $\rho \in W^{1,k}(\mathbb{R}^N)$.

Remark 6.2. *Suppose that $F = -D\Phi$ and $\Phi \in C^1(\mathbb{R}^N)$ satisfies $e^{-\Phi} \in L^1(\mathbb{R}^N)$. Then $\rho = e^{-\Phi}$ and the assumption $F \in L^k_\mu(\mathbb{R}^N)$ is equivalent to $e^{-\Phi/k} \in W^{1,k}(\mathbb{R}^N)$. The boundedness of ρ then follows by Sobolev embedding.*

6.1 - *Pointwise bounds*

Here we prove (pointwise) upper and lower bounds on the density ρ. For the upper bound, we assume that $V(x) = \exp\{\delta|x|^\beta\}$ is integrable with respect to μ for some $\delta, \beta > 0$, recalling that explicit estimates of δ, β follow from Example 6.1 under assumptions (36). We need the extra assumption that F does not grow more than some exponential, at infinity, in order to integrate $|F|^k$ with respect to μ for every k. Under these assumptions we show that ρ decays exponentially. For the lower bound we need more regularity on F and we confine ourselves to the case when F and its derivatives up to the second order have a polynomial growth.

6.2 - *Upper bounds*

Theorem 6.2. *Assume that*

$$\limsup_{|x|\to\infty}\left(|x|^{1-\beta}F(x)\cdot\frac{x}{|x|}\right) < -c < 0 \tag{40}$$

for some $c > 0$, $\beta > 0$. *Assume moreover that* $|F(x)| \le C\exp\{c|x|^\gamma\}$ *for some* $C, c > 0$ *and* $\gamma < \beta$. *Then there exist* $c_1, c_2 > 0$ *such that* $\rho(x) \le c_1 \exp\{-c_2|x|^\beta\}$.

Proof. We know that $V(x) = e^{\delta|x|^\beta}$ is a Lyapunov and a μ-integrable function for $\delta < c/\beta$. Moreover, since $|F(x)| \le C\exp\{|x|^\gamma\}$ for some $C > 0$ and $\gamma < \beta$, then by Theorem 6.1, $F \in L^k_\mu(\mathbb{R}^N)$ for every $k < \infty$ and $d\mu = \rho\,dx$ with $\rho \in L^\infty(\mathbb{R}^N)$. The invariance of μ yields

$$\int_{\mathbb{R}^N} (\Delta\phi)\rho dx = -\int_{\mathbb{R}^N} (F\cdot\nabla\phi)\rho dx$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Taking $\phi = w\psi$ with $\psi \in C_c^\infty(\mathbb{R}^N)$ and $w(x) = \exp\{c_2|x|^\beta\}$ for $|x| \ge 1$, we obtain

$$\int_{\mathbb{R}^N} (\Delta\psi)\rho w\,dx = -\int_{\mathbb{R}^N} \big(\psi\Delta w + 2\nabla\psi\nabla w + wF\cdot\nabla\psi + \psi F\cdot\nabla w\big)\rho\,dx. \tag{41}$$

Let us fix $q > p > N$ and choose $c_2 < \delta/q$. It is easy to see that $w, \nabla w, \Delta w$ belong to $L^q_\mu(\mathbb{R}^N)$. Moreover, since $1/p = 1/q + 1/k$ for some $k > 1$ and $F \in L^k_\mu(\mathbb{R}^N)$, it follows that $wF, |\nabla w||F| \in L^p_\mu(\mathbb{R}^N)$. Since $\rho \in L^\infty(\mathbb{R}^N)$ we deduce that all the functions $\rho\nabla w, \rho\Delta w, \rho wF$ belong to $L^p(\mathbb{R}^N)$. Then (41) yields

$$\left|\int_{\mathbb{R}^N} (\Delta\psi)\rho w dx\right| \le K\|\psi\|_{W^{1,p'}(\mathbb{R}^N)}$$

for a suitable K independent of ψ. Since also $\rho w \in L^p(\mathbb{R}^N)$ from Lemma 8.1 we get that ρw belongs to $W^{1,p}(\mathbb{R}^N)$, hence to $L^\infty(\mathbb{R}^N)$ since $p > N$ and the proof is complete.

Remark 6.3. *The constant c_2 can be made precise with more careful arguments. Actually for every $\varepsilon > 0$ one can find $c_\varepsilon > 0$ such that*

$$\rho(x) \le c_\varepsilon e^{-(c/\beta-\varepsilon)|x|^\beta}.$$

Simple examples show that such an estimate does not hold, in general, with $\varepsilon = 0$. See [**14**].

6.3 - *Lower bounds*

In order to get lower bounds on the density ρ we assume more regularity on F to simplify the exposition. However these extra-regularity assumptions have been removed in [**7**] by a careful analysis of the dependence of the constants in Moser's Harnack inequality.

In the next theorem we state a lower bound estimate on ρ when F and its derivatives up to the second order have a polynomial growth.

Theorem 6.3. *Assume that $F \in C^2(\mathbb{R}^N)$ satisfies*

$$|F(x)| + |\nabla F(x)| + |D^2 F(x)| \le C_1(1 + |x|^{\beta-1})$$

for some $\beta > 1$. Then

$$\rho(x) \ge \exp\{-K(1 + |x|^\beta)\},$$

where K depends only on C_1.

Proof. (Idea). Let $v = \log \rho$, then $\nabla v = \dfrac{\nabla \rho}{\rho}$, $\Delta v = \dfrac{\Delta \rho}{\rho} - \dfrac{|\nabla \rho|^2}{\rho^2}$. The invariance of μ yields $A^*\mu = 0$ in the sense of distribution, thus

$$\Delta\rho = \operatorname{div}(F\rho) = F \cdot \nabla\rho + \rho \operatorname{div} F \tag{42}$$

and

$$\Delta v + |\nabla v|^2 - F \cdot \nabla v = \operatorname{div} F.$$

Applying [**23**, Theorem 5.2] we obtain

$$|\nabla v(x)| \le C(1 + |x|^{\beta-1})$$

for $v = \log \rho$. Therefore $|v(x)| \le K(1 + |x|^\beta)$ and the statement follows.

The estimate of the logarithmic derivative of ρ in terms of F leads immediately to a quantitative Harnack inequality.

Proposition 6.2. *Assume that $F \in C^2(\mathbb{R}^N)$ satisfies*

$$|F(x)| + |\nabla F(x)| + |D^2F(x)| \le C_1(1 + |x|^{\beta-1})$$

for some $\beta > 1$. Then

$$\frac{\rho(y)}{\rho(x)} \le \exp\left\{K|x-y|\big(1 + |x|^{\beta-1} + |y|^{\beta-1}\big)\right\}, \tag{43}$$

where K depends only on C_1. Indeed, from Theorem 6.3

$$|\nabla v(x)| \le C(1 + |x|^{\beta-1}).$$

This yields $|v(y) - v(x)| \le C_1|x-y|\Big(1 + |x|^{\beta-1} + |y|^{\beta-1}\Big)$.

Remark 6.4. *Note that, if ρ decays exponentially and $|\nabla v(x)| \le c(1 + |x|^{\beta-1})$, then $\nabla\rho/\rho \in L^p_\mu(\mathbb{R}^N)$ for every $1 \le p < \infty$. Indeed*

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \left|\frac{\nabla\rho}{\rho}\right|^p \rho\, dx &= \int\limits_{\mathbb{R}^N} |\nabla v|^p \rho\, dx \\ &\le c\int\limits_{\mathbb{R}^N} (1 + |x|^{\beta-1})^p e^{-c_2|x|^\beta}\, dx < \infty.\end{aligned}$$

The case $p = 2$ is special and is the basis for more general considerations.

Proposition 6.3. *If $F \in L^2_\mu(\mathbb{R}^N)$, then*

$$\int\limits_{\mathbb{R}^N} \frac{|\nabla\rho|^2}{\rho}\, dx \le \int\limits_{\mathbb{R}^N} |F|^2 \rho\, dx. \tag{44}$$

Proof. (Idea). Since ρ is the density of the invariant measure μ we have

$$\Delta\rho = \operatorname{div}(F\rho).$$

Multiplying by $\log\rho$ and integrating by parts (of course everything here is only formal and all integrations by parts need arguments)

$$\int_{\mathbb{R}^N} \frac{|\nabla \rho|^2}{\rho}\,dx = \int_{\mathbb{R}^N} F\rho \frac{\nabla \rho}{\rho}\,dx$$

$$= \int_{\mathbb{R}^N} F\sqrt{\rho}\frac{\nabla \rho}{\sqrt{\rho}}\,dx$$

$$\leq \Bigg(\int_{\mathbb{R}^N} F^2\rho\Bigg)^{1/2}\Bigg(\int_{\mathbb{R}^N}\frac{|\nabla \rho|^2}{\rho}\,dx\Bigg)^{1/2}.$$

This estimate means that $\sqrt{\rho} \in H^1(\mathbb{R}^N)$.

Remark 6.5. *Estimate* (44) *holds also for more general operators like* $\sum_{ij} D_i(a_{ij}D_j) + F\cdot\nabla$ *with* a_{ij} *possibly unbounded provided that* $\sum_{ij} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$, *for some constant* $\nu > 0$. *In the case of unbounded coefficients the previous arguments do not work and one needs more refined tools such as for instance Moser's or De Giorgi's techniques.*

7 - Domain characterization

Suppose that $(T(t))_{t\geq 0}$ is the semigroup generated by $(A, D_{max}(A))$ in the space $C_b(\mathbb{R}^N)$ and has an invariant measure μ.

Proposition 7.1. *The semigroup* $(T(t))_{t\geq 0}$ *is strongly continuous in* $L^p_\mu(\mathbb{R}^N)$ *and* $D_{max}(A)$ *is a core for its generator* $(A, D^\mu_p(A))$.

Proof. Let $f \in C_b(\mathbb{R}^N)$. Then $T(t)f \to f$ as $t \to 0$ pointwise in $\mathbb{R}^N$ and $\|T(t)f\|_\infty \leq \|f\|_\infty$. By dominated convergence we deduce that $T(t)f \to f$ as $t \to 0$ in $L^p_\mu(\mathbb{R}^N)$. Since $C_b(\mathbb{R}^N)$ is dense in $L^p_\mu(\mathbb{R}^N)$ as it can be shown by standard approximation methods, $(T(t))_{t\geq 0}$ is strongly continuous in $L^p_\mu(\mathbb{R}^N)$ and the first statement follows.

Let now $D^\mu_p(A)$ be the domain of the generator of the semigroup in $L^p_\mu(\mathbb{R}^N)$. If $f \in D_{max}(A)$, by Proposition 5.6 $\dfrac{T(t)f - f}{t} \to Af$ pointwise as $t \to 0$ and $\left\|\dfrac{T(t)f - f}{t}\right\|_\infty \leq C$ for some positive constant C. Hence $\dfrac{T(t)f - f}{t} \to Af$ in $L^p_\mu(\mathbb{R}^N)$. This implies that $D_{max}(A) \subseteq D^\mu_p(A)$. By Proposition 5.8, $D_{max}(A)$ is pointwise and boundedly dense in $C_b(\mathbb{R}^N)$, hence it is dense in $L^p_\mu(\mathbb{R}^N)$. Since $D_{max}(A)$ is invariant under $T(t)$ (see Proposition 5.6), by the core theorem (see [**13**]) we deduce the claim.

We are now interested in characterizing the domain $D^\mu_p(A)$ for the symmetric operator

$$A = \Delta - \nabla\phi \cdot \nabla$$

on $L^p_\mu(\mathbb{R}^N)$ for $1<p<\infty$, where $\mu(dx) = e^{-\phi}dx$ and under the assumptions $\phi \in C^2(\mathbb{R}^N)$, $e^{-\phi} \in L^1(\mathbb{R}^N)$. We fix the following notation for the weighted Sobolev spaces

$$W^{k,p}_\mu(\mathbb{R}^N) = \{u \in W^{k,p}_{loc}(\mathbb{R}^N) : \ D^\alpha u \in L^p_\mu(\mathbb{R}^N) \text{ if } |\alpha| \le k\}$$

endowed with the usual norm.

Let us first consider the simplest case $p = 2$.

7.1 - *Characterization of the domain in* $L^2_\mu(\mathbb{R}^N)$

Here we follow the methods of [**10**]. We consider the sesquilinear form a in $L^2_\mu(\mathbb{R}^N)$ given by

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \overline{\nabla v} e^{-\phi} dx$$

on the form domain

$$D(a) = H^1_\mu(\mathbb{R}^N) = W^{1,2}_\mu(\mathbb{R}^N) = \{u : \ u, \nabla u \in L^2_\mu(\mathbb{R}^N)\}.$$

By using the quadratic forms method it is possible to construct a selfadjoint semigroup $(T(t))_{t\ge0}$ in $L^2_\mu(\mathbb{R}^N)$. Moreover, by the Berling-Deny conditions, it follows that $(T(t))_{t\ge0}$ is also positive and contractive (see [**27**]).

A natural question is whether $D^\mu_2(A) = H^2_\mu(\mathbb{R}^N) = W^{2,2}_\mu(\mathbb{R}^N)$. We state the following lemma whose proof is standard.

Lemma 7.1. *The space* $C_c^\infty(\mathbb{R}^N)$ *is dense in* $H^1_\mu(\mathbb{R}^N)$, $H^2_\mu(\mathbb{R}^N)$.

Lemma 7.2. *Suppose that* $\Delta\phi \le \eta|\nabla\phi|^2 + C_\eta$ *for some* $\eta<1$. *Then the map* $u \mapsto |\nabla\phi|u$ *is bounded from* $H^1_\mu(\mathbb{R}^N)$ *to* $L^2_\mu(\mathbb{R}^N)$ *and the map* $u \mapsto |\nabla\phi||\nabla u|$ *is bounded from* $H^2_\mu(\mathbb{R}^N)$ *to* $L^2_\mu(\mathbb{R}^N)$.

Proof. By density, it is sufficient to prove the boundedness for functions $u \in C_c^\infty(\mathbb{R}^N)$. We have

$$\int_{\mathbb{R}^N} |\nabla\phi|^2|u|^2 e^{-\phi}\,dx = \int_{\mathbb{R}^N} \nabla\phi\cdot\nabla\phi|u|^2 e^{-\phi}\,dx = -\int_{\mathbb{R}^N} \nabla\phi|u|^2\nabla(e^{-\phi})\,dx$$
$$= \int_{\mathbb{R}^N} \Delta\phi|u|^2 e^{-\phi} + 2\int_{\mathbb{R}^N} \nabla\phi\, u\cdot\nabla u\, e^{-\phi}\,dx$$

and hence

$$\int_{\mathbb{R}^N} |\nabla\phi|^2|u|^2 e^{-\phi}\,dx \le \int_{\mathbb{R}^N} \Delta\phi|u|^2 e^{-\phi}\,dx$$
$$+ 2\left(\int_{\mathbb{R}^N} |\nabla u|^2 e^{-\phi}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} |\nabla\phi|^2|u|^2 e^{-\phi}\,dx\right)^{\frac{1}{2}}$$
$$\le \int_{\mathbb{R}^N} \Delta\phi|u|^2 e^{-\phi}\,dx + \frac{1}{\varepsilon}\int_{\mathbb{R}^N} |\nabla u|^2 e^{-\phi}\,dx$$
$$+ \varepsilon\int_{\mathbb{R}^N} |\nabla\phi|^2|u|^2 e^{-\phi}\,dx.$$

The first statement follows by estimating $\Delta\phi$ as in the assumption and by choosing ε small enough. The second statement follows by the first one by considering $\nabla u \in H^1_\mu(\mathbb{R}^N)$ instead of u.

Remark 7.1. *In particular, by the previous lemma, it follows that A is well defined on $H^2_\mu(\mathbb{R}^N)$.*

Theorem 7.1. *Suppose that ϕ is convex and $\Delta\phi \le \eta|\nabla\phi|^2 + C_\eta$ for some $\eta < 1$. Then $D_2^\mu(A) = H^2_\mu(\mathbb{R}^N)$.*

Proof. By definition

$$D_2^\mu(A) = \Big\{u \in H^1_\mu(\mathbb{R}^N) : \exists f \in L^2_\mu(\mathbb{R}^N) :$$
$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla v e^{-\phi}dx = -\int_{\mathbb{R}^N} f v e^{-\phi}dx\Big\}$$

for all $v \in C_c^\infty(\mathbb{R}^N)$ (or for all $v \in H^1_\mu(\mathbb{R}^N)$ by density).

An inclusion is easy. Indeed, if $u \in H^2_\mu(\mathbb{R}^N)$, $f = Au \in L^2_\mu(\mathbb{R}^N)$ by Lemma 7.2, integration by parts is allowed and one sees that $u \in D^\mu_2(A)$. Therefore $H^2_\mu(\mathbb{R}^N) \subseteq D^\mu_2(A)$. Now we show that the graph norm and the $H^2_\mu(\mathbb{R}^N)$ norm are equivalent on $C^\infty_c(\mathbb{R}^N)$. By Lemma 7.2, we have

$$\|u\|_{L^2_\mu} + \|Au\|_{L^2_\mu} \le C\|u\|_{H^2_\mu},$$

and we have to prove the converse. Let $u \in C^\infty_c(\mathbb{R}^N)$, $\lambda > 0$, set $f = \lambda u - \Delta u + \nabla\phi \cdot \nabla u$. Then $D_j f = \lambda D_j u - \Delta D_j u + \nabla\phi \cdot \nabla(D_j u) + \nabla(D_j\phi)\nabla u$. Multiplying by $D_j u$, integrating in $d\mu$ and summing over j we obtain

$$-\int_{\mathbb{R}^N} Aufd\mu = \lambda \int_{\mathbb{R}^N} |\nabla u|^2 d\mu + \int_{\mathbb{R}^N} |D^2 u|^2 d\mu + \int_{\mathbb{R}^N} \sum_{i,j} D_{ij}\phi D_i u D_j u.$$

By the convexity assumption the last term in the right hand side is positive. It follows that

$$\|D^2 u\|^2_{L^2_\mu} \le \|Au\|_{L^2_\mu}\|f\|_{L^2_\mu} = \|Au\|_{L^2_\mu}\|\lambda u - Au\|_{L^2_\mu}$$

and so $\|D^2 u\|_{L^2_\mu} \le C\|u\|_{D^\mu_2(A)}$ and the equivalence of the two norms, as stated.

Since $C^\infty_c(\mathbb{R}^N)$ is dense in $H^2_\mu(\mathbb{R}^N)$, to conclude the proof it suffices to prove that $H^2_\mu(\mathbb{R}^N)$ is a core for $(A, D^\mu_2(A))$. Let $u \in D^\mu_2(A)$. Then $u \in H^1_\mu(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$, by local elliptic regularity, and $Au \in L^2_\mu(\mathbb{R}^N)$. Let η be a cut-off function and set $u_n(x) = u(x)\eta\left(\frac{x}{n}\right) \in H^2_\mu(\mathbb{R}^N)$ (u_n is locally in $H^2(\mathbb{R}^N)$ and has compact support). Clearly $u_n \to u$ in $L^2_\mu(\mathbb{R}^N)$. Concerning Au_n, we have

$$\begin{aligned} Au_n &= \frac{1}{n^2}\Delta\eta\left(\frac{x}{n}\right)u + \frac{2}{n}\nabla u \cdot \nabla\eta\left(\frac{x}{n}\right) + \eta\left(\frac{x}{n}\right)\Delta u \\ &- \eta\left(\frac{x}{n}\right)\nabla\phi \cdot \nabla u - \frac{1}{n}u\nabla\eta\left(\frac{x}{n}\right) \cdot \nabla\phi \\ &= \eta\left(\frac{x}{n}\right)Au + \frac{1}{n^2}\Delta\eta\left(\frac{x}{n}\right)u + \frac{2}{n}\nabla u \cdot \nabla\eta\left(\frac{x}{n}\right) - \frac{1}{n}u\nabla\eta\left(\frac{x}{n}\right) \cdot \nabla\phi. \end{aligned}$$

Observe now that, for $n \to \infty$,

$$\left\|\frac{1}{n^2}\Delta\eta\left(\frac{x}{n}\right)u\right\|_{L^2_\mu} \to 0, \quad \left\|\frac{2}{n}\nabla u \cdot \nabla\eta\left(\frac{x}{n}\right)\right\|_{L^2_\mu} \to 0$$

and, by Lemma 7.2,

$$\left\|\frac{1}{n}u\nabla\eta\left(\frac{x}{n}\right) \cdot \nabla\phi\right\|_{L^2_\mu} \le \frac{C}{n}\|u\nabla\phi\|_{L^2_\mu} \le \frac{C'}{n}\|u\|_{H^2_\mu} \to 0$$

as $n \to \infty$. Therefore $\|Au_n - Au\|_{L^2_\mu} \to 0$.

Remark 7.2. *The semigroup $(T(t))_{t\geq 0}$ coincides on $C_b(\mathbb{R}^N)$ with the one constructed in Section* 5.1.

Proof. Let $0 \leq f \in C_c^\infty(\mathbb{R}^N) \subseteq L^2_\mu(\mathbb{R}^N)$. Then $u(t,x) = \lim_{\rho\to\infty} u_\rho(t,x)$ where $u_\rho \geq 0$ satisfies

$$\begin{cases} u_t(t,x) = Au(t,x) & t>0,\ x\in B_\rho, \\ u(t,x) = 0 & t>0,\ x\in \partial B_\rho, \\ u(0,x) = f(x) & x\in\mathbb{R}^N. \end{cases}$$

We have

$$\frac{d}{dt}\int_{B_\rho} u_\rho^2(t,x)e^{-\phi(x)}dx = \int_{B_\rho} \frac{d}{dt}u_\rho^2(t,x)e^{-\phi(x)}dx$$

$$= 2\int_{B_\rho} u_\rho(t,x)(\Delta u_\rho(t,x) - \nabla\phi(x)\cdot\nabla u_\rho(t,x))e^{-\phi(x)}dx$$

$$= 2\int_{B_\rho} u_\rho(t,x)\,\mathrm{div}\,(\nabla u_\rho(t,x)e^{-\phi(x)})dx = 2\int_{B_\rho} |\nabla u_\rho(t,x)|^2 e^{-\phi(x)}dx \leq 0.$$

This implies that $\int_{B_\rho} u_\rho^2(t,x)e^{-\phi(x)}dx \leq \int_{B_\rho} f^2(x)e^{-\phi(x)}dx$ and, letting ρ to infinity, $\int_{\mathbb{R}^N} u^2(t,x)e^{-\phi(x)}dx \leq \int_{\mathbb{R}^N} f^2(x)e^{-\phi(x)}dx$ and so the semigroup constructed by approximation in Section 5.1 extends to a strongly continuous semigroup in $L^2_\mu(\mathbb{R}^N)$ whose generator coincides with A on $C_c^\infty(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is a core for $(A, D_2^\mu(A))$ the semigroups coincide.

7.2 - *The general case:* $1<p<\infty$

Theorem 7.2. *Assume that for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that $\Delta\phi \leq \varepsilon|\nabla\phi|^2 + C_\varepsilon$. Then $D_p^\mu(A) = W_\mu^{2,p}(\mathbb{R}^N)$ for all $1<p<\infty$.*

Proof. (Idea). As in the case $p=2$, it is possible to prove that the map $u\mapsto|\nabla\phi|u$ is bounded from $W_\mu^{1,p}(\mathbb{R}^N)$ to $L^p_\mu(\mathbb{R}^N)$ and the maps $u\mapsto|\nabla\phi||\nabla u|$, $u\mapsto|\nabla\phi|^2|u|$ are bounded from $W_\mu^{2,p}(\mathbb{R}^N)$ to $L^p_\mu(\mathbb{R}^N)$. Therefore

$$\|u\|_{D_p^\mu(A)} \leq C\|u\|_{W_\mu^{2,p}}.$$

For $p\neq 2$, we cannot integrate by parts to prove the converse. We make a change of variable in order to work with an operator on $L^p(\mathbb{R}^N)$ instead of $L^p_\mu(\mathbb{R}^N)$. Namely we define the isometry

$$J : L^p_\mu(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad Ju = e^{-\frac{\phi}{p}}u.$$

A straightforward computation shows that

$$Au = J^{-1}BJu \quad \text{for } u \in C_c^\infty(\mathbb{R}^N)$$

where

$$Bv = \Delta v + \left(\frac{2}{p} - 1\right)\nabla\phi \cdot \nabla v - \frac{1}{p}\left[\left(1 - \frac{1}{p}\right)|\nabla\phi|^2 - \Delta\phi\right]v.$$

Setting $F = \left(\frac{2}{p} - 1\right)\nabla\phi$, $V = \frac{1}{p}\left[\left(1 - \frac{1}{p}\right)|\nabla\phi|^2 - \Delta\phi\right]$, B can be written as the complete second order operator $Bv = \Delta v + F \cdot \nabla v - Vv$ with drift F and potential V. In [**23**, Theorem 3.4], it is proved that, under the assumptions

(i) $|\nabla V| \leq \gamma V^{\frac{3}{2}} + C_\gamma$ for γ small enough;
(ii) $|F| \leq kV^{\frac{1}{2}}$ for some positive k;
(iii) $\theta\,\mathrm{div}F + V \geq 0$ for some $\theta > \frac{1}{p}$;

$(B, D(B))$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$, $1<p<\infty$, where $D(B) = W^{2,p}(\mathbb{R}^N) \cap D(V)$. One can prove that our assumptions on ϕ imply (i), (ii) and (iii) so that $A = J^{-1}BJ$ with domain $D(A) := \{u \in L^p_\mu(\mathbb{R}^N) : Ju \in W^{2,p}(\mathbb{R}^N) \cap D(V)\}$ generates an analytic semigroup on $L^p_\mu(\mathbb{R}^N)$. It is not hard to prove that $D^\mu_p(A) = W^{2,p}_\mu(\mathbb{R}^N)$ (see [**23**, Theorem 7.4]).

8 - Appendix

In order to make as self-contained as possible these notes we prove some L^p-estimates for second order elliptic operators often used in Section 6.1. We refer to [**1**] for further details.

Lemma 8.1. *Let $1<p<\infty$ and let $u \in L^p(\mathbb{R}^N)$ be such that*

$$\left|\int_{\mathbb{R}^N} u(\phi - \Delta\phi)\,dx\right| \leq C\|\phi\|_{W^{1,p'}(\mathbb{R}^N)} \tag{45}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Then $u \in W^{1,p}(\mathbb{R}^N)$.

Proof. By approximation one can extend (45) to all functions $\phi \in W^{2,p'}(\mathbb{R}^N)$. For $h \in \mathbb{R}^N$, set $\tau_h u = |h|^{-1}(u(\cdot + h) - u(\cdot))$. Thus, applying (45) to $\tau_{-h}\phi$ we deduce

$$\left|\int_{\mathbb{R}^N} (\tau_h u)(\phi - \Delta\phi)\,dx\right| \leq C_1\|\phi\|_{W^{2,p'}(\mathbb{R}^N)}.$$

Here C_1 is independent of ϕ and h. Given $f \in L^{p'}(\mathbb{R}^N)$ let ϕ be the solution of $\phi - \Delta\phi = f \in L^{p'}(\mathbb{R}^N)$. Then $\|\phi\|_{W^{2,p'}(\mathbb{R}^N)} \le C\|f\|_{L^{p'}}$, with C independent of f. Then we obtain

$$\left| \int_{\mathbb{R}^N} (\tau_h u) f \, dx \right| \le C_1 \|f\|_{L^{p'}(\mathbb{R}^N)}$$

hence,

$$\int_{\mathbb{R}^N} |\tau_h u|^p \, dx \le C_2$$

with C_2 independent of h. The boundedness of the L^p-norm of the difference quotients $\tau_h u$ implies that $u \in W^{1,p}(\mathbb{R}^N)$.

Lemma 8.2. *Let $1<p<\infty$ and let $u \in L^p(\mathbb{R}^N)$ be such that*

$$\left| \int_{\mathbb{R}^N} u(\phi - \Delta\phi) \, dx \right| \le C\|\phi\|_{L^{p'}(\mathbb{R}^N)} \tag{46}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Then $u \in W^{2,p}(\mathbb{R}^N)$.

Proof. By Lemma 8.1, $u \in W^{1,p}(\mathbb{R}^N)$. Applying (45) to $\tau_{-h}\phi$ we get

$$\left| \int_{\mathbb{R}^N} (\tau_h u)(\phi - \Delta\phi) \, dx \right| \le C\|\phi\|_{W^{1,p'}(\mathbb{R}^N)}$$

hence again by Lemma 8.1, $\|\tau_h u\|_{W^{1,p}} \le c$, i.e. $\|\tau_h \nabla u\|_{L^p} \le c$, thus $D^2 u \in L^p(\mathbb{R}^N)$.

List of symbols

$\mathbb{R}^N$	euclidean N-dimensional space.
$\mathbb{C}^-$	the space $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$.
$\mathbb{C}^+$	the space $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$.
f^+, f^-	$f \vee 0, f \wedge 0$.
$\mathrm{supp}\, f$	support of a given function f.
$\mathbf{1}$	function identically equal to 1 (everywere).
$B_b(\mathbb{R}^N)$	the space of Borel and bounded functions $f : \mathbb{R}^N \to \mathbb{R}$.
$C_b(\mathbb{R}^N)$	the space of continuous and bounded functions.
$BUC(\mathbb{R}^N)$	the space of uniformly continuous and bounded functions.

$C^\alpha(\mathbb{R}^N)$	space of α Hölder continuous functions.
$C^\alpha_{loc}(\mathbb{R}^N)$	space of α Hölder continuous functions in Ω for all bounded open set. $\Omega \subset \mathbb{R}^N$.
$C^{k+\alpha}(\mathbb{R}^N)$	space of functions such that the derivatives of order k are α-Hölder continuous.
$C^\infty_c(\mathbb{R}^N)$	space of test functions.
$L^p(\mathbb{R}^N)$	usual Lebesgue space.
$C_0(\mathbb{R}^N)$	space of continuous functions tending to 0 as $\lvert x\rvert$ tends to $+\infty$.
$C_0(B_\rho)$	space of continuous functions in B_ρ vanishing on the boundary.
$C^{1+\frac{\alpha}{2},2+\alpha}(\mathbb{R}^N)$	space of functions such that $\partial_t u$ and $D_{ij}u$ are α Hölder continuous with respect to the parabolic distance.
$\mathcal{S}(\mathbb{R}^N)$	Schwartz space.
$W^{k,j}(\mathbb{R}^N)$	space of functions $u \in L^k(\mathbb{R}^N)$ having weak space derivatives up to the order j in $L^k(\mathbb{R}^N)$.
$L^p_\mu(\mathbb{R}^N)$	$L^p(\mathbb{R}^N, d\mu)$.
$W^{k,p}_\mu(\mathbb{R}^N)$	the space $\{f \in L^p_\mu(\mathbb{R}^N) : D^\alpha f \in L^p_\mu(\mathbb{R}^N),\ \lvert\alpha\rvert \le k\}$, $k \in \mathbb{N}_0$, $1 \le p < \infty$.
I	the identity matrix.
$\det B$	the determinant of the matrix B.
$\mathrm{tr}\, B$	the trace of the matrix B.
B^*	the adjoint matrix of B.
$\sigma(B)$	the spectrum of B.
$(\cdot\vert\cdot)$	scalar product or, in general, duality.
$B_\rho(x)$	open ball for the euclidean distance with centre x and radius ρ.

References

[1] S. AGMON, *The L_p approach to the Dirichlet problem*, Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 405-448.

[2] W. ARENDT, *Gaussian estimates and interpolation of the spectrum in L^p*, Differential Integral Equations **7** (1994), 1153-1168.

[3] D. BAKRY, *L'hypercontractivité et son utilisation en théorie des semigroupes*, In *Lectures on probability theory*, Lecture Notes in Math., 1581, Springer, Berlin 1994, 1-114.

[4] M. BERTOLDI and L. LORENZI, *Analytical methods for Markov semigroups*, Chapman & Hall/CRC Press, Boca Raton, FL 2007.

[5] P. BILLINGSLEY, *Weak convergence of measures: Applications in probability*, CBMS-NSF Series 5, SIAM, Philadelphia 1971.

[6] V. I. BOGACHEV, N. KRYLOV and M. RÖCKNER, *On Regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Differential Equations **26** (2001), 2037-2080.

[7] V. I. BOGACHEV, M. RÖCKNER and S. V. SHAPOSHNIKOV, *Estimates of densities of stationary distributions and transition probabilities of diffusion processes*, Theory Probab. Appl. **52** (2008), 209-236.

[8] R. CHILL, E. FAŠANGOVÁ, G. METAFUNE and D. PALLARA, *The sector of analyticity of the Ornstein-Uhlenbeck semigroup on L^p spaces with respect to invariant measure*, J. London Math. Soc. (2) **71** (2005), 703-722.

[9] G. DA PRATO and A. LUNARDI, *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. **131** (1995), 94-114.

[10] G. DA PRATO and A. LUNARDI, *Elliptic operators with unbounded drift coefficients and Neumann boundary condition*, J. Differential Equations **198** (2004), 35-52.

[11] G. DA PRATO and J. ZABCZYK, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge 1992.

[12] G. DA PRATO and J. ZABCZYK, *Ergodicity for infinite-dimensional systems*, Cambridge University Press, Cambridge 1996.

[13] K.-J. ENGEL and R. NAGEL, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, 194, Springer-Verlag, New York 2000.

[14] S. FORNARO, N. FUSCO, G. METAFUNE and D. PALLARA, *Sharp upper bounds for the density of some invariant measures*, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), 1145-1161.

[15] A. FRIEDMAN, *Partial differential equations of parabolic type*, Prentice-Hall, New Jersey 1964.

[16] D. GILBARG and N. TRUDINGER, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Berlin 1983.

[17] G. M. LIEBERMAN, *Second order parabolic differential equations*, World Scientific, Singapore 1996.

[18] A. LUNARDI, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, 16, Birkhäuser Verlag, Basel 1995.

[19] A. LUNARDI, *On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures*, Trans. Amer. Math. Soc. **349** (1997), 155-169.

[20] A. LUNARDI and V. VESPRI, *Generation of strongly continuous semigroups by elliptic operators with unbounded coefficients in $L^p(\mathbb{R}^n)$*, Rend. Istit. Mat. Univ. Trieste **28** suppl. (1997), 251-279.

[21] G. METAFUNE, *L^p-spectrum of Ornstein-Uhlenbeck operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), 97-124.

[22] G. METAFUNE, D. PALLARA and E. PRIOLA, *Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures*, J. Funct. Anal. **196** (2002), 40-60.

[23] G. METAFUNE, D. PALLARA and A. RHANDI, *Global properties of invariant measures*, J. Funct. Anal. **223** (2005), 396-424.

[24] G. METAFUNE, D. PALLARA and M. WACKER, *Feller semigroups on $\mathbb{R}^N$*, Semigroup Forum **65** (2002), 159-205.

[25] G. METAFUNE, D. PALLARA and M. WACKER, *Compactness properties of Feller semigroups,* Studia Math. **153** (2002), 179-206.

[**26**] G. METAFUNE, J. PRÜSS, R. SCHNAUBELT and A. RHANDI, *L^p-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations **10** (2005), 1131-1164.

[**27**] E. M. OUHABAZ, *Analysis of heat equations on domains*, Princeton University Press, Princeton, NJ 2005.

[**28**] E. PRIOLA, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*, Studia Math. **136** (1999), 271-295.

[**29**] J. M. A. M. VAN NEERVEN and J. ZABCZYK, *Norm discontinuity of Ornstein-Uhlenbeck semigroups*, Semigroup Forum **59** (1999), 389-403.

[**30**] T. SEIDMAN, *How violent are fast controls?*, Math. Control Signals Systems **1** (1988), 89-95.

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