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Lyapunov exponent of a rational map and multipliers of repelling cycles

Abstract. We establish an approximation property for the Lyapunov exponent of a rational map with respect to its maximal entropy measure. This result is usefull in the study of bifurcation currents of holomorphic families of rational maps.

Keywords. Lyapunov exponent, maximal entropy measure.

Mathematics Subject Classification (2000): 37F99.

The aim of this short note is to prove the following

Theorem 0.1. *Let $f : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ be a rational map of degree $d \geq 2$ and L the Lyapunov exponent of f with respect to its maximal entropy measure. Then:*

$$L = \lim_n \frac{d^{-n}}{n} \sum_{p \in R_n^*} \ln |(f^n)'(p)|$$

where $R_n^* := \{p \in \boldsymbol{P}^1 / \, p$ has exact period n and $|(f^n)'(p)| > 1\}$.

This property is of special interest for investigating the structure of the bifurcation locus of an holomorphic family f_λ since it supports the current $dd^c L(f_\lambda)$ (see [**8**], [**1**]). The papers [**2**], [**3**] exploit this approach. Although this result has been proved by Szpiro and Tucker [**11**] (Corollary 6.1) for rational maps with coefficients in a number field and in [4] (Corollary 1.6) for holomorphic endomorphisms of $\boldsymbol{P}^k$, no

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simple proof was available so far in the one dimensional holomorphic setting. The one we present simplifies that of [**4**]. It consists in reproving the equidistribution of repelling cycles - a theorem due to Lyubich [**9**] - by carefully estimating the multipliers of the cycles which are exhibited. In dimension one, the approach of Briend and Duval [**6**], which we shall follow step by step, offers a clear access to this estimate. Recently, Okuyama [**10**] has shown how to deduce this result from Lyubich's theorem via a potential theoretic argument.

We now start to prove the theorem. For the simplicity of notations we consider polynomials and therefore work on $\boldsymbol{C}$ with the euclidean metric. We shall denote $D(x,r)$ the open disc centered at $x \in \boldsymbol{C}$ and radius $r > 0$. From now on, f is a degree $d \geq 2$ polynomial whose Julia set is denoted J and whose maximal entropy measure is denoted μ. We denote by C_f the set of critical points of f. As it follows from the Margulis-Ruelle inequality, the Lyapunov exponent $L := \int \ln|f'|\, d\mu$ is strictly positive. We refer to [**5**] (Chapitre 8) for basic properties of μ.

- *The natural extension $(\hat{J}, \hat{f}, \hat{\mu})$ and the contraction of inverse branches.* The set $\hat{J} := \{\hat{x} := (x_n)_{n\in\mathbb{Z}} \;/\; x_n \in J,\; f(x_n) = x_{n+1}\}$ is the space of orbits, $\hat{f}$ is the shift defined by $\hat{f}(\hat{x}) := (x_{n+1})_{n\in\mathbb{Z}}$ and $\hat{\mu}$ is a $\hat{f}$-invariant probability measure on $\hat{J}$ which is characterized by the identity $\pi_*(\hat{\mu}) = \mu$ where $\pi : \hat{J} \to J$ is the canonical projection given by $\pi(\hat{x}) = x_0$. Let us observe that $\pi \circ \hat{f} = f \circ \pi$ and that the system $(\hat{J}, \hat{f}, \hat{\mu})$ is invertible. Setting $\tau := \hat{f}^{-1}$ one has $\pi \circ \tau^k(\hat{x}) = x_{-k}$. It is important to stress that the measure $\hat{\mu}$ inherits from μ the property of being mixing (see [**7**] Chapter 10 for this construction).

Let $\widehat{X} := \{\hat{x} \in \hat{J} \;/\; x_n \notin C_f \;;\; \forall n \in \mathbb{Z}\}$. As $\hat{\mu}$ is $\hat{f}$-invariant and μ does not give mass to points, one sees that $\hat{\mu}(\widehat{X}) = 1$. For every $\hat{x} \in \widehat{X}$ and every $p \in \mathbb{Z}$, we denote by f_{x_p} the injective map induced by f on some neighbourhood of x_p. Its inverse is defined on some neighbourhood of x_{p+1} and will be denoted $f_{x_p}^{-1}$. We then define an "iterated inverse branch of f along $\hat{x}$ and of depth n" by settting $f_{\hat{x}}^{-n} := f_{x_{-n}}^{-1} \circ \cdots \circ f_{x_{-1}}^{-1}$. The following lemma is crucial, in particular the estimate on the Lipschitz's constant $\operatorname{Lip} f_{\hat{x}}^{-n}$ will allow to control the multipliers of the repelling cycles produced in the last step.

Lemma 0.2. *There exists $\varepsilon_0 > 0$ and, for $\varepsilon \in]0, \varepsilon_0]$, two functions $\eta_\varepsilon : \widehat{X} \to]0, 1]$ and $S_\varepsilon : \widehat{X} \to]1, +\infty]$ which are measurable and such that, for every $n \in \mathbb{N}$ and $\hat{\mu}$-almost every $\hat{x} \in \widehat{X}$, the map $f_{\hat{x}}^{-n}$ is defined on $D(x_0, \eta_\varepsilon(\hat{x}))$ and $\operatorname{Lip} f_{\hat{x}}^{-n} \leq S_\varepsilon(\hat{x}) e^{-n(L-\varepsilon)}$.*

Proof. We need the following quantitative version of the inverse mapping theorem (see [**6**] Lemme 2).

Fact: *Let* $\rho(x) := |f'(x)|$, $r(x) := \rho(x)^2$. *There exists* $\varepsilon_0 > 0$ *and, for* $\varepsilon \in]0, \varepsilon_0]$, $0 < C_1(\varepsilon), C_2(\varepsilon)$ *such that for every* $x \in J$:

1 - f *is one-to-one on* $D(x, C_1(\varepsilon)\rho(x))$,
2 - $D(f(x), C_2(\varepsilon)r(x)) \subset f\big(D(x, C_1(\varepsilon)\rho(x))\big)$,
3 - $\mathrm{Lip}\, f_x^{-1} \leq e^{\frac{\varepsilon}{3}}\rho(x)^{-1}$ *on* $D(f(x), C_2(\varepsilon)r(x))$.

We may assume that $0 < \varepsilon_0 < \dfrac{L}{2}$. Let us first build a function $\alpha_\varepsilon : \widehat{X} \to]0,1[$ satisfying $\alpha_\varepsilon(\tau(\hat{x})) \geq e^{-\varepsilon}\alpha_\varepsilon(\hat{x})$ and such that $f^{-1}_{x_{-k-1}}$ is defined on $D(x_{-k}, \alpha_\varepsilon(\tau^k(\hat{x})))$ for $\hat{\mu}$ a.e. $\hat{x} \in \widehat{X}$ and every $k \in \mathbb{Z}$.

Let us set $\beta_\varepsilon(\hat{x}) := \mathrm{Min}\,(1, C_2(\varepsilon)r(x_{-1}))$. According to the two first assertions of the Fact, $f^{-1}_{x_{-1}} = f^{-1}_{\hat{x}}$ is defined on $D(x_0, \beta_\varepsilon(\hat{x}))$ and, similarly, $f^{-1}_{x_{-k-1}} = f^{-1}_{\tau^k(\hat{x})}$ is defined on $D(x_{-k}, \beta_\varepsilon(\tau^k(\hat{x})))$. It clearly suffices to shape a function α_ε such that $0 < \alpha_\varepsilon < \beta_\varepsilon$ and $\alpha_\varepsilon(\tau(\hat{x})) \geq e^{-\varepsilon}\alpha_\varepsilon(\hat{x})$.

As μ admits continuous local potentials, the function $\ln \beta_\varepsilon$ is $\hat{\mu}$-integrable. Then, by Birkhoff ergodic theorem, $\int_{\hat{X}} \ln \beta_\varepsilon\, \hat{\mu} = \lim_{|n|\to+\infty} \dfrac{1}{|n|} \sum_{k=1}^{n} \ln \beta_\varepsilon(\tau^k(\hat{x}))$ and, in particular, $\lim_{|n|\to+\infty} \dfrac{1}{|n|} \ln \beta_\varepsilon(\tau^n(\hat{x})) = 0$ for $\hat{\mu}$-almost every $\hat{x} \in \widehat{X}$. In other words, for $\hat{\mu}$-a.e. $\hat{x} \in \widehat{X}$ there exists $n_0(\varepsilon, \hat{x}) \in \mathbb{N}$ such that $|n| \geq n_0(\varepsilon, \hat{x}) \Rightarrow \beta_\varepsilon(\tau^n(\hat{x})) \geq e^{-|n|\varepsilon}$. Setting then $V_\varepsilon := \inf_{|n|\leq n_0(\varepsilon,\hat{x})} \big(\beta_\varepsilon(\tau^n(\hat{x}))e^{|n|\varepsilon}\big)$ we obtain a measurable function $V_\varepsilon : \widehat{X} \to]0,1]$ such that: $\beta_\varepsilon(\tau^n(\hat{x})) \geq e^{-|n|\varepsilon}V_\varepsilon(\hat{x})$ for $\hat{\mu} - a.e.$ $\hat{x} \in \widehat{X}$ and every $n \in \mathbb{Z}$. As one may easily check, we may choose $\alpha_\varepsilon(\hat{x}) := \mathrm{Inf}_{n\in\mathbb{Z}} \{\beta_\varepsilon(\tau^n(\hat{x}))e^{|n|\varepsilon}\}$.

Since $f^{-n}_{\hat{x}} = f^{-1}_{x_{-n}} \circ \cdots \circ f^{-1}_{x_{-1}}$, the third assertion of the Fact yields $\ln \mathrm{Lip}\, f^{-n}_{\hat{x}} \leq n\dfrac{\varepsilon}{3} - \sum_{k=1}^{n} \ln \rho(x_{-k})$. Thus, by Birkhoff ergodic theorem, $\limsup \dfrac{1}{n} \ln \mathrm{Lip}\, f^{-n}_{\hat{x}} \leq -L + \dfrac{\varepsilon}{3}$ for $\hat{\mu}$-almost every $\hat{x} \in \widehat{X}$. Then, arguing as for V_ε, one finds a measurable function $S_\varepsilon : \widehat{X} \to [1, +\infty[$ such that: $\mathrm{Lip}\, f^{-n}_{\hat{x}} \leq S_\varepsilon(\hat{x})e^{-n(L-\varepsilon)}$ for $\hat{\mu}$ – a.e. $\hat{x} \in \widehat{X}$ and every $n \in \mathbb{N}$. To end the proof we set $\eta_\varepsilon := \dfrac{\alpha_\varepsilon}{S_\varepsilon}$. Taking into account the previous estimates, one checks by induction on $n \in \mathbb{N}$ that $f^{-n}_{\hat{x}}$ is defined on $D(x_0, \eta_\varepsilon(\hat{x}))$ for $\hat{\mu}$ – almost every $\hat{x} \in \widehat{X}$ and every $n \in \mathbb{N}$ (this uses the fact that $0 < \varepsilon_0 < \dfrac{L}{2}$ and $\alpha_\varepsilon(\tau(\hat{x})) \geq e^{-\varepsilon}\alpha_\varepsilon(\hat{x})$). □

• *Radon-Nikodym derivatives.* We aim here to reduce the problem to an estimate on some Radon-Nikodym derivatives. Let $0 < \varepsilon_0$ be given by Lemma 0.2.

For $0 < \varepsilon \leq \varepsilon_0$ and $n, N \in \mathbb{N}$ we set:

$$\widehat{X}_N^{\varepsilon} := \left\{\hat{x} \in \widehat{X} \ / \ \eta_\varepsilon(\hat{x}) \geq \frac{1}{N} \text{ and } S_\varepsilon(\hat{x}) \leq N\right\}; \qquad \hat{\nu}_N^{\varepsilon} := 1_{\hat{X}_N^{\varepsilon}}\hat{\mu}, \quad \nu_N^{\varepsilon} := \pi_* \hat{\nu}_N^{\varepsilon}.$$

For $0 < \varepsilon \leq L$ and $n, N \in \mathbb{N}$ we set:

$$R_n^{\varepsilon} := \{p \in \boldsymbol{C} \ / \ f^n(p) = p \text{ and } |(f^n)'(p)| \geq e^{n(L-\varepsilon)}\}; \qquad \mu_n^{\varepsilon} := d^{-n} \sum_{R_n^{\varepsilon}} \delta_p$$

$$R_n := R_n^L = \{p \in \boldsymbol{C} \ / \ f^n(p) = p \text{ and } |(f^n)'(p)| \geq 1\}; \quad \mu_n := \mu_n^L = d^{-n} \sum_{R_n} \delta_p.$$

Lemma 0.3. *If, for* $\varepsilon \in \left]0, \varepsilon_0\right]$, *any weak limit* σ *of* $\left(\mu_n^{\varepsilon}\right)_n$ *satisfies* $\frac{d\sigma}{d\nu_N^{\varepsilon'}} \geq 1$ *for some* $\varepsilon' > 0$ *and every* $N \in \mathbb{N}$ *then* $\mu_n^{\varepsilon} \to \mu$ *for every* $\varepsilon \in \left]0, L\right]$ *and* $L = \lim_n \frac{d^{-n}}{n} \sum_{R_n^*} \ln |(f^n)'(p)|$.

Proof. Assume first that $0 < \varepsilon \leq \varepsilon_0$. By assumption $\sigma \geq \nu_N^{\varepsilon'}$ for every $N \in \mathbb{N}$, letting $N \to +\infty$ one gets $\sigma \geq \mu$. This actually implies that $\sigma = \mu$ since $\sigma(J) \leq \limsup_n \mu_n^{\varepsilon}(J) \leq \lim_n \frac{d^n + 1}{d^n} = 1 = \mu(J)$. As this occurs for any weak limit of $\left(\mu_n^{\varepsilon}\right)_n$ we have shown that $\mu_n^{\varepsilon} \to \mu$. Similarly, as $\mu_n^{\varepsilon} \geq \mu_n^{\varepsilon_0}$ for $\varepsilon_0 \leq \varepsilon$, one sees that $\mu_n^{\varepsilon} \to \mu$ for $\varepsilon_0 \leq \varepsilon \leq L$ as well.

Setting now $\varphi_n(p) := \frac{1}{n} \ln |(f^n)'(p)|$ one has for $M > 0$

$$\begin{aligned}\mu_n^{\varepsilon}(J)(L-\varepsilon) &\leq d^{-n} \sum_{R_n^{\varepsilon}} \varphi_n(p) \leq d^{-n} \sum_{R_n} \varphi_n(p) \\ &= \int_J \ln |f'| \mu_n \leq \int_J Max\left(\ln |f'|, -M\right) \mu_n\end{aligned}$$

as we just saw that $\mu_n^{\varepsilon} \to \mu$ and $\mu_n = \mu_n^L \to \mu$, making $n \to +\infty$ leads to

$$\begin{aligned}(L-\varepsilon) \leq \liminf d^{-n} \sum_{R_n} \varphi_n(p) &\leq \limsup d^{-n} \sum_{R_n} \varphi_n(p) \\ &\leq \int_J Max(\ln |f'|, -M) \mu\end{aligned}$$

to obtain $\lim d^{-n} \sum_{R_n} \varphi_n(p) = L$ it suffices to make first $M \to +\infty$ and then $\varepsilon \to 0$.

Since there are less than $2nd^{\frac{n}{2}}$ periodic points whose period strictly divides n, one may replace R_n by $R_n^* := \{p \in \boldsymbol{P}^1 / \ p \text{ has exact period } n \text{ and } |(f^n)'(p)| \geq 1\}$. □

• *The heart of the proof.* We assume here that $0<\varepsilon<\frac{\varepsilon_0}{2}$. Let $\hat{a}\in\widehat{X}_N^{\varepsilon}$ and $a:=\pi(\hat{a})$. For every $r>0$ we denote by D_r the closed disc centered at a of radius r. According to the Lemma 0.3, it suffices to show that *any weak limit* σ of $\left(\mu_n^{2\varepsilon}\right)_n$ satisfies

$$\sigma(D_{r'})\geq \nu_N^{\varepsilon}(D_{r'}),\quad \forall N\in\mathbb{N},\ \forall\, 0<r'<\frac{1}{N}. \tag{0.1}$$

Let us pick $r'<r<\frac{1}{N}$. We set $\widehat{D}_r:=\pi^{-1}(D_r)$ and:

$$\widehat{C}_n:=\{\hat{x}\in\widehat{D}_r\cap\widehat{X}_N^{\varepsilon}\ /\ f_{\hat{x}}^{-n}(D_r)\cap D_{r'}\neq\emptyset\}.$$

Let also consider the collection S_n of sets of the form $f_{\hat{x}}^{-n}(D_r)$ where $\hat{x}$ runs in $\widehat{C}_n$. As $f_{\hat{x}}^{-n}$ is an inverse branch on D_r of the ramified cover f^n, one sees that the sets of the collection S_n are mutually disjoint.

Let us see how (0.1) may be deduced from two further estimates. Using Brouwer fixed point theorem and the estimate on $Lip\, f_{\hat{x}}^{-n}$ (Lemma 0.2) we will get

$$d^{-n}\left(\text{Card } S_n\right)\leq \mu_n^{2\varepsilon}(D_r)\ \text{ for } n \text{ big enough} \tag{0.2}$$

on the other hand, the constant Jacobian property $f^{n*}\mu=d^n\mu$ will lead to

$$d^{-n}\left(\text{Card } S_n\right)\mu(D_r)\geq\hat{\mu}\left(\hat{f}^{-n}(\widehat{D}_r\cap\widehat{X}_N^{\varepsilon})\cap\widehat{D}_{r'}\right). \tag{0.3}$$

Combining (0.2) and (0.3) yields:

$$\hat{\mu}\left(\hat{f}^{-n}(\widehat{D}_r\cap\widehat{X}_N^{\varepsilon})\cap\widehat{D}_{r'}\right)\leq\mu(D_r)\mu_n^{2\varepsilon}(D_r)$$

which, by the mixing property of $\hat{\mu}$, implies

$$\nu_N^{\varepsilon}(D_r)\mu(D_{r'})=\hat{\mu}(\widehat{D}_r\cap\widehat{X}_N^{\varepsilon})\hat{\mu}(\widehat{D}_{r'})\leq\mu(D_r)\sigma(D_r)$$

since $\mu(D_{r'})>0$, one gets (0.1) by making $r\to r'$.

Let us now prove the estimate (0.2). We have to show that D_r contains at least $\left(\text{Card } S_n\right)$ elements of $R_n^{2\varepsilon}$ when n is big enough. For every $\hat{x}\in\widehat{C}_n\subset\widehat{X}_N^{\varepsilon}$ one has $\eta_\varepsilon(\hat{x})\geq\frac{1}{N}$ and $S_\varepsilon(\hat{x})\leq N$ and thus the map $f_{\hat{x}}^{-n}$ is defined on D_r $\left(r<\frac{1}{N}\right)$ and $\text{Diam}\, f_{\hat{x}}^{-n}(D_r)\leq 2r\ \text{Lip}\, f_{\hat{x}}^{-n}\leq 2rS_\varepsilon(\hat{x})e^{-n(L-\varepsilon)}\leq 2rNe^{-n(L-\varepsilon)}$.

As moreover $f_{\hat{x}}^{-n}(D_r)$ meets $D_{r'}$, there exists n_0, which depends only on ε, r and r', such that $f_{\hat{x}}^{-n}(D_r)\subset D_r$ for every $\hat{x}\in\widehat{C}_n$ and $n\geq n_0$. Thus, by Brouwer theorem, $f_{\hat{x}}^{-n}$ has a fixed point $p_n\in f_{\hat{x}}^{-n}(D_r)$ for every $\hat{x}\in\widehat{C}_n$ and $n\geq n_0$. Since the elements of S_n are mutually disjoint sets, we have produced $\left(\text{Card } S_n\right)$ fixed points of f^n in D_r

for $n \geq n_0$. It remains to check that these fixed points belong to $R_n^{2\varepsilon}$. This follows immediately from the estimates on $\mathrm{Lip}\, f_{\hat{x}}^{-n}$. Indeed: $|(f^n)'(p_n)| = |(f_{\hat{x}}^{-n})'(p_n)|^{-1} \geq \left(\mathrm{Lip}\, f_{\hat{x}}^{-n}\right)^{-1} \geq N^{-1} e^{n(L-\varepsilon)} \geq e^{n(L-2\varepsilon)}$ for n big enough.

Finally we prove the estimate (0.3). Let us first observe that

$$\pi\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}_N^{\varepsilon}) \cap \widehat{D}_{r'}\big) \subset \bigcup_{\hat{x} \in \hat{C}_n} f_{\hat{x}}^{-n}(D_r) \tag{0.4}$$

this can be easily seen : if $\hat{u} \in \hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}_N^{\varepsilon}) \cap \widehat{D}_{r'}$ then $u_0 = \pi(\hat{u}) \in D_{r'} \cap f_{\hat{x}}^{-n}(D_r)$ where $\hat{x} := \hat{f}^n(\hat{u}) \in \widehat{D}_r \cap \widehat{X}_N^{\varepsilon}$.

By the constant Jacobian property we have $\mu\big(f_{\hat{x}}^{-n}(D_r)\big) = d^{-n}\mu(D_r)$ and, since the sets $f_{\hat{x}}^{-n}(D_r)$ of the collection S_n are mutually disjoint, we obtain

$$\mu\Bigg(\bigcup_{\hat{x} \in \hat{C}_n} f_{\hat{x}}^{-n}(D_r)\Bigg) = \big(\mathrm{Card}\, S_n\big)\, d^{-n}\mu(D_r). \tag{0.5}$$

Combining (0.4) with (0.5) yields (0.3):

$$\big(\mathrm{Card}\, S_n\big)\, d^{-n}\mu(D_r) \geq \mu\big[\pi\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}_N^{\varepsilon}) \cap \widehat{D}_{r'}\big)\big]$$

$$= \hat{\mu}\big[\pi^{-1} \circ \pi\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}_N^{\varepsilon}) \cap \widehat{D}_{r'}\big)\big] \geq \hat{\mu}\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}_N^{\varepsilon}) \cap \widehat{D}_{r'}\big).$$

□

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