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On the zero-viscosity limit for 3D Navier-Stokes equations under slip boundary conditions

Abstract. In this survey article we consider the initial-boundary value problem for the three-dimensional Navier-Stokes equations with Navier boundary conditions and study the problem of convergence of the solutions, as the viscosity goes to zero, to the solution of the Euler equations. We present some strong convergence results, obtained in collaboration with H. Beirão da Veiga (see [4] and [5]), in the case where the region of motion is a bounded domain with flat boundary.

Keywords. Navier-Stokes equations, slip boundary conditions, zero-viscosity limit.

Mathematics Subject Classification (2000): 35Q30, 76D05, 76D09.

1 - Introduction

The question whether solutions of the initial boundary value problem for the Navier-Stokes equations tend to a solution of the corresponding problem for the Euler equations, as the viscosity ν tends to zero, is a famous open question in Mathematical Fluid Mechanics. It is the so-called vanishing viscosity or zero viscosity problem or, even, inviscid limit problem. Unlike the case of the Cauchy problem, where vanishing viscosity limit results have been successfully studied by many authors (see, for instance, [**15**], [**16**], [**18**], [**23**], and the more recent papers [**3**], [**19**]), when the adherence condition (Dirichlet boundary condition) for the Navier-Stokes is prescribed, there arise difficulties mainly connected with the formation of boundary layers associated with the loss of boundary conditions. Indeed, for the

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Euler equations, the boundary condition $u = 0$ on $\partial\Omega$ is weakened to $u \cdot n = 0$, n being the outer unit normal to $\partial\Omega$. As clearly explained in the book [**7**] (see, also, [**8, 22**]) the presence of the small viscosity term and the difference in the boundary conditions have the following three main effects: the flow governed by the Navier-Stokes problem is drastically modified near the wall in a region whose thickness is proportional to the square root of the viscosity; this region may separate from the boundary; this separation produces vorticity. In 1905 Prandtl formally derived the equations describing the boundary layer, which are indeed known as Prandtl equations. Roughly speaking, one expects that, as the viscosity vanishes, a solution to the Navier-Stokes equations tends to a solution to the Euler equations, away from boundaries, and to a solution to the Prandtl equations within the boundary layer. However, only partial results are known in this regard, for particular domains or initial data. This kind of analysis is beyond our interest and, among the very wide bibliography, we refer to the first contributions given in [**11**] and [**20**] and to the more recent paper [**22**].

In the light of the previous considerations, it is quite natural to wonder if, imposing different and physically meaningful boundary conditions for the Navier-Stokes equations, a similar situation occurs. The idea of taking into account the vanishing viscosity issue with different boundary value problems is not new. Indeed, the special case of the so-called "free boundary conditions" for 2-D domains ($u \cdot n = 0$ and $\mathit{curl}\, u = 0$) goes back to Lions [**17**] in 1969. Here we take into account the following slip conditions on the boundary

$$u \cdot n = 0, \qquad (\mathit{curl}\, u) \times n = 0. \tag{1.1}$$

It is worth noting that in the presence of a flat boundary, which is the case considered in the sequel, they coincide with the classical Navier boundary conditions

$$u \cdot n = 0, \qquad t - (t \cdot n)n = 0, \tag{1.2}$$

where $t = \mathcal{T} \cdot n$ ($\mathcal{T}$ stress tensor) is the stress vector. These conditions, introduced by Navier in 1823 and derived by Maxwell in 1879 from the kinetic theory of gases, prescribe both that the flow cannot go out from the wall and that the shear stress vanishes on the boundary.

Our aim is to show that, once replacing the Dirichlet boundary conditions with the previous Navier boundary conditions, the flow is well described by the Euler equations in the limit as the viscosity tends to zero, at least in the case of flat boundaries. Note that we are interested in the strong convergence of smooth solutions in 3-D domains. More precisely, here we establish *a priori* estimates in $L^\infty(0, T; W^{3,p}(\Omega))$, independent of $\nu > 0$, with $p > \dfrac{3}{2}$, for the solution to the Navier-Stokes initial boundary value problem. Then, by suitable compactness arguments, we show the strong convergence of such a solution to the unique strong solution to the Euler equations in $W^{s,p}$-spaces, for any $s<3$. For similar results in the particular case where $p=2$, we refer to the paper [**27**]. This kind of results is part of the analysis on the vanishing viscosity problem made in collaboration with H. Beirão da Veiga in [**4**] and [**5**]. In particular, in paper [**4**] both the bi-dimensional and three-dimensional cases are considered. It is worth noting that in the bi-dimensional case, corresponding to the free boundary condition, the previous results are proved for any simply connected open set, with $\partial\Omega$ sufficiently regular. Actually, the results could be proved globally in time by following ideas already known in considering 2-D problems. Further, if the initial data is more regular and satisfies suitable compatibility conditions, it is possible to use an induction argument and extend the 3-D inviscid limit results to arbitrary $W^{k,p}$-spaces, $k>3$. This improvement is one of the results obtained in the recent paper [**5**].

2 - Some auxiliary results and statement of the main theorem

Let us consider the initial value problem for the 3-D incompressible Navier-Stokes equations:

$$
(2.1)\qquad \begin{cases} \partial_t u^\nu - \nu\Delta u^\nu + (u^\nu\cdot\nabla)u^\nu + \nabla\pi^\nu = 0, & \text{in } \Omega\times(0,T),\\ \nabla\cdot u^\nu = 0, & \text{in } \Omega\times(0,T),\\ u^\nu(0) = u_0, & \text{in } \Omega, \end{cases}
$$

where the velocity u^ν and the pressure π^ν are the unknowns, $\nu>0$ is the coefficient of kinematic viscosity, $(u^\nu\cdot\nabla)u^\nu = (\nabla u^\nu)u^\nu$ and u_0 is the initial data.

In order to work with flat boundaries and, at the same time, with a bounded domain, we consider here a cubic domain and impose the Navier boundary conditions on two opposite sides and periodicity in the other two directions, avoiding in this way singularities due to the corner points. Therefore, throughout the paper, Ω denotes a three dimensional cube of unitary length $\Omega = (]0,1[)^3$ and Γ the two opposite faces of Ω in the x_3 direction, i.e.

$$
\Gamma = \{x : |x_1|<1, |x_2|<1, x_3=0\} \cup \{x : |x_1|<1, |x_2|<1, x_3=1\}.
$$

We impose the Navier boundary conditions (1.1) on Γ. In the sequel we often use the term boundary referring to Γ. Setting $x'=(x_1,x_2)$, we say that a function is x'-periodic if it is periodic in both the two directions x_1 and x_2. Hence, denoting the vorticity field $curl\, u^\nu$ by ω^ν, we can write the boundary conditions (1.1) as follows

$$\begin{cases} u_3^\nu = \omega_1^\nu = \omega_2^\nu = 0 \ \text{ on } \Gamma, \\ u^\nu \text{ is } x' - \ \text{periodic}. \end{cases} \tag{2.2}$$

Let us also introduce the initial value problem for the 3-D incompressible Euler equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \pi = 0, & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{2.3}$$

with the boundary condition

$$u \cdot n = 0 \text{ on } \Gamma. \tag{2.4}$$

We mainly work with the solution u^ν of the Navier-Stokes equations (2.1). Therefore, in order to ease the notation, from now on we denote this solution by u, except when both solutions u^ν and u appear at the same time. The same simplification will be used for the vorticity.

We introduce some auxiliary results. We will resort to the use of the following well known estimates (see [**2**] and [**21**]), whose proof can be obtained by straightforward calculations. Note that in the sequel we use Einstein's summation convention.

Proposition 2.1. *For each $p > 1$ and sufficiently regular vector field v, one has*

$$\left|\nabla |v|^{\frac{p}{2}}\right|^2 \le \left(\frac{p}{2}\right)^2 |v|^{p-2} |\nabla v|^2, \tag{2.5}$$

and

$$\begin{gathered} -\int \Delta v \cdot (|v|^{p-2} v) dx = \\ \frac{1}{2}\int |v|^{p-2} |\nabla v|^2 dx + 4\frac{p-2}{p^2} \int \left|\nabla |v|^{\frac{p}{2}}\right|^2 dx - \int_\Gamma |v|^{p-2} (\partial_i v_j) n_i v_j d\Gamma. \end{gathered} \tag{2.6}$$

We set

$$\zeta = curl\, \omega, \qquad \chi = curl\, \zeta.$$

Lemma 2.1. *Assume that $\omega_1 = \omega_2 = 0$ on Γ and $\nabla \cdot \omega = 0$ in Ω. Then, on Γ,*

$$\partial_3 \omega_3 = 0, \tag{2.7}$$

$$(\partial_i \omega_j) n_i \omega_j = 0, \tag{2.8}$$

and

$$\zeta_3 = 0. \tag{2.9}$$

Proof. The first identity is trivial since, as the boundary is flat, $\partial_1 \omega_1 = \partial_2 \omega_2 = 0$. The second follows by appealing to (2.7) and to $n_1 = n_2 = 0$. Finally the definition of ζ_3 and $\partial_1 \omega_2 = \partial_2 \omega_1 = 0$ imply the last identity. □

Lemma 2.2. *Let u be a vector field in Ω, and $\omega = curl\, u$. Assume that $u_3 = \omega_1 = \omega_2 = 0$ on Γ. Then the vector fields $(u \cdot \nabla)\omega$ and $(\omega \cdot \nabla)u$ are normal to Γ.*

Proof. The proof is straightforward, recalling that $\partial_3 u_1 = \omega_2 + \partial_1 u_3 = 0$ on Γ, and similarly for $\partial_3 u_2$. □

Lemma 2.3. *Assume, in addition to the hypothesis of Lemma* 2.2, that ω satisfies the equation

$$\partial_t \omega - \nu \Delta \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0, \tag{2.10}$$

where $\nu > 0$. Then

$$(curl\, \zeta) \times n = 0 \quad on \quad \Gamma.$$

Proof. Since ω is normal to the boundary Γ, so is $\partial_t \omega$. By appealing to Lemma 2.2, it follows that $-\Delta \omega = curl\, \zeta$ is normal to the boundary. □

Lemma 2.4. *Under the assumptions of Lemma* 2.3 *one has*

$$(\partial_i \zeta_j) n_i \zeta_j = 0 \quad on \quad \Gamma. \tag{2.11}$$

Proof. The thesis follows from identity (2.9) and Lemma 2.3. Note that $\partial_3 \zeta_1 = \chi_2 + \partial_1 \zeta_3$, and similarly for $\partial_3 \zeta_2$. □

Lemma 2.5. *Under the assumptions of Lemma* 2.3 *one has*

$$(\partial_i \chi_j) n_i \chi_j = 0 \quad on \quad \Gamma. \tag{2.12}$$

Proof. By Lemma 2.3, one has $\chi \times n = 0$. Hence $\chi_1 = \chi_2 = 0$ on Γ. Further, by appealing to $\nabla \cdot \chi = 0$, it follows that $(\partial_3 \chi_3)\chi_3 = 0$. □

The previous Proposition and lemmas are the main tools in order to prove the following theorem.

Theorem 2.1. *Let be $p > \frac{3}{2}$, $\nu_0 > 0$ and assume that u_0 belongs to $W^{3,p}(\Omega)$, is divergence free in Ω and satisfies the boundary conditions* (2.2). *For each fixed*

$\nu \in (0, \nu_0]$ *let* u^ν *be the solution to the initial boundary value problem* (2.1), (2.2). *Then*

$$\begin{cases} u^\nu \rightharpoonup u & in \quad L^\infty(0, T_0; W^{3,p}(\Omega)) \quad weak - *, \\ u^\nu \to u & in \quad C([0, T_0]; W^{s,p}(\Omega)), \quad for\ each\ s < 3, \end{cases} \tag{2.13}$$

where u is the unique solution to the Euler equations (2.3), (2.4). *Further,*

$$\partial_t u^\nu \to \partial_t u \quad in \quad L^\infty(0, T_0; W^{1,p}(\Omega)) \tag{2.14}$$

and, if $p > 2$,

$$\partial_t u^\nu \to \partial_t u \quad in \quad L^p(0, T_0; W^{1,3p}(\Omega)). \tag{2.15}$$

In the next Section 3, for fixed $\nu > 0$, we prove the *a priori* estimates independent on ν that lead us to the strong inviscid limit result, developed in Section 4. We do not study the effective construction of the solution. However, as in the case of more classical boundary conditions, this may be obtained, for instance, via the classical Faedo-Galerkin approximation procedure (see [**9**], [**17**], [**25**]). To this end, due to the periodicity condition, it is standard to assume that the initial data has null mean value in the periodicity directions: $\overline{u_1^0} \equiv \int_\Omega u_1^0 dx = 0$, $\overline{u_2^0} \equiv \int_\Omega u_2^0 dx = 0$ (see [**14**], [**26**]). Then, it is easy to see that this property is preserved by the solution for all $t \geq 0$. However the lack of this assumption does not really affect the solution, since one could easily bring back this case by replacing the initial data $u^0(x)$ with $u^0(x) - \overline{u^0}$ and adding a linear term of the kind $\overline{u^0} \cdot \nabla u$ in the equations. This is the reason way in [**4**] and [**5**] we neglect the null mean value assumption. Obviously, in this case, in estimates like those of Lemma 3.1, one should add an L^p norm of u on the left-hand sides, which, however, is bounded thanks to the energy estimate.

Moreover it is worth noting that in references [**12**] and [**13**] the authors prove very general and complete strong solvability results for solutions to the Navier-Stokes equations under different, even nonhomogeneous, boundary conditions for a bounded domain with smooth boundary, where the condition (1.1) is assumed on the whole $\partial\Omega$. The uniqueness of strong solutions is also standard.

Concerning the existence and uniqueness of a local in time smooth solution to the Euler equation, we refer to the classical papers [**6, 10, 24**].

Finally, note that the estimates obtained in this section, except for Proposition 2.1, are strictly connected with the flatness of the boundary. Loosing the previous geometrical information we are not able to extend Theorem 2.1 to general boundaries. In order to a better understanding of the real obstacles due to the passage from a flat to a non-flat boundary, that prevent us to get similar results for more general boundaries, we refer the interested reader to the paper [**5**], Section 3.

3 - A priori estimates for the Navier-Stokes initial-boundary value problem

We use standard notations for Lebesgue spaces, Sobolev spaces and corresponding norms. Further, the symbol c denotes a numerical positive constant whose value is unessential to our aims. Hence, its value may change even in the same equation. In particular it never depends on the viscosity ν but it can depend on an arbitrarily fixed upper bound for ν.

In what follows, we assume u, ω, ζ and χ as smooth as necessary for the subsequent calculations to make sense.

Let us consider the vorticity equation (2.10), obtained by taking the curl of equation $(2.1)_1$ and using the vector identity $(u\cdot\nabla)u=\frac{1}{2}\nabla|u|^2-u\times curl\, u$. By multiplying both sides of equation (2.10) by $|\omega|^{p-2}\omega$, by integrating in Ω, and by taking into account Proposition 2.1, one gets the general relation

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p+\frac{\nu}{2}\int|\omega|^{p-2}|\nabla\omega|^2dx+4\nu\frac{p-2}{p^2}\int\left|\nabla|\omega|^{\frac{p}{2}}\right|^2dx$$
$$+\frac{1}{p}\int(u\cdot\nabla)|\omega|^pdx-\int|\omega|^{p-2}\big((\omega\cdot\nabla)u\big)\cdot\omega dx$$
$$=\nu\int_\Gamma|\omega|^{p-2}(\partial_i\omega_j)n_i\omega_jd\Gamma.$$

Due to $\nabla\cdot u=0$ in Ω and $u\cdot n=0$ on Γ, the third integral on the left hand side vanishes. Hence,

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p+\nu\frac{2(2p-3)}{p^2}\int\left|\nabla|\omega|^{\frac{p}{2}}\right|^2dx\\&\le\int|\nabla u||\omega|^pdx+\nu\int_\Gamma|\omega|^{p-2}(\partial_i\omega_j)n_i\omega_jd\Gamma,\end{aligned}\tag{3.1}$$

where we have used (2.5).

Next we follow the above argument with ω replaced by ζ. By applying the operator *curl* to both sides of equation (2.10) one gets, with obvious notation,

$$\partial_t\zeta-\nu\Delta\zeta+(u\cdot\nabla)\zeta+\sum c(Du)(D\omega)=0.\tag{3.2}$$

Next, multiply both sides of the above equation by $|\zeta|^{p-2}\zeta$, integrate in Ω, and take into account Proposition 2.1. An obvious extension of the above argument gives

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p+\nu\frac{2(2p-3)}{p^2}\int\left|\nabla|\zeta|^{\frac{p}{2}}\right|^2dx\\&\le\int|\nabla u||\nabla\omega||\zeta|^{p-1}dx+\nu\int_\Gamma|\zeta|^{p-2}(\partial_i\zeta_j)n_i\zeta_jd\Gamma.\end{aligned}\tag{3.3}$$

Finally, by applying the operator *curl* to the equation (3.2), and by following devices similar to that used in obtaining (3.3), we get the estimate

$$\begin{aligned}(3.4)\quad &\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + \nu\frac{2(2p-3)}{p^2}\int \big|\nabla|\chi|^{\frac{p}{2}}\big|^2 dx \\ &\leq \int (\,|Du||D^2\omega| + |D^2u||D\omega|)|\chi|^{p-1}dx + \nu\int_{\Gamma} |\chi|^{p-2}(\partial_i\chi_j)n_i\chi_j d\Gamma.\end{aligned}$$

By appealing to equations (2.8), (2.11) and (2.12) one obtains the following crucial result.

Proposition 3.1. *If the boundary is flat, the boundary integrals in equations* (3.1), (3.3) *and* (3.4) *vanish.*

In the sequel we shall highly appeal to the following equivalences for the norms.

Lemma 3.1. *Let u, ω, ζ, χ be as above. Then, for each non-negative integer k, for any $p > 1$, one has the following norm-equivalence results:*

$$\|\omega\|_{k,p} \simeq \|u\|_{k+1,p}; \quad \|\zeta\|_{k,p} \simeq \|u\|_{k+2,p}; \quad \|\chi\|_{k,p} \simeq \|u\|_{k+3,p}.$$

The result follows from standard regularity results for which we refer to [**1**] and to the more recent paper [**28**]. Actually, the first claim follows from $curl\, u = \omega$ and $\nabla \cdot u = 0$ in Ω, together with the boundary condition $u \cdot n = 0$. The second claim follows from $-\Delta u = \zeta$ in Ω, together with the boundary conditions $\partial_3 u_1 = \partial_3 u_2 = u_3 = 0$. Finally, the third claim follows from the second claim, by taking into account that $curl\, \zeta = \chi$ and $\nabla \cdot \zeta = 0$ in Ω, and that $\zeta \cdot n = 0$ on Γ.

From the continuous immersion of $W^{1,2}$ in L^6 it follows that

$$\|g\|_{3p}^p \leq c(\|\nabla|g|^{\frac{p}{2}}\|_2^2 + \|g\|_p^p). \tag{3.5}$$

We may use this estimate in equations (3.1), (3.3) and (3.4). From (3.3) and (3.5), one gets

$$\frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p + c\nu\|\zeta\|_{3p}^p \leq \int |\nabla u||\nabla\omega||\zeta|^{p-1}dx + c\nu\|\zeta\|_p^p, \tag{3.6}$$

where we assume that $p > \dfrac{3}{2}$. Further, the integral on the right hand side of (3.6) is bounded by $\|\nabla u\|_\infty \|\nabla\omega\|_p \|\zeta\|_p^{p-1}$. Since $W^{1,p} \subset L^\infty$, for $p > 3$, the following result holds.

Theorem 3.1. *Assume that* $p > 3$. *Then*

$$\frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p + c\nu\|\zeta\|_{3p}^p \le c\|\zeta\|_p^{p+1} + c\nu\|\zeta\|_p^p. \tag{3.7}$$

Similarly, from (3.4) we obtain, if $p > \frac{3}{2}$,

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + c\nu\|\chi\|_{3p}^p \le \int\Big(|\,D\,u\,||D^2\omega| + |D^2u|^2\Big)|\chi|^{p-1}dx + c\nu\|\chi\|_p^p.$$

Note that, for $p > \frac{3}{2}$, one has $\|Du\|_\infty \le c\|Du\|_{2,p}$ and

$$\||D^2u|^2\|_p = \|D^2u\|_{2p}^2 \le c\|D^3u\|_p^2,$$

since $W^{1,p} \subset L^{2p}$. So, the following result holds.

Theorem 3.2. *Assume that* $p > \frac{3}{2}$. *Then*

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + c\nu\|\chi\|_{3p}^p \le c\|\chi\|_p^{p+1} + c\nu\|\chi\|_p^p. \tag{3.8}$$

We are in position to get the *a-priori* estimates, which could be used to prove the existence of the local smooth solution. We fix a positive constant ν_0 and assume that $\nu \le \nu_0$. From comparison theorems for ordinary differential equations applied to (3.8), it follows that $\|\chi(t)\|_p \le y(t)$, where $y(t)$ satisfies

$$\begin{cases} \dfrac{dy}{dt} = cy^2 + cy, \\ y(0) = y_0 =: \|\chi(0)\|_p. \end{cases} \tag{3.9}$$

The solution to equation (3.9) is given by

$$\frac{y}{1+y} = \frac{y_0}{1+y_0}e^{ct}.$$

Note that $y(t)$ is no-negative, increasing, and goes to ∞ as t goes to T^*, where the blow-up time T^* is defined by

$$e^{cT^*} = \frac{1+y_0}{y_0}.$$

We fix a value $T_0 \in (0, T^*)$. Then $y(t) \le y(T_0)$ for each $t \le T_0$. For instance, define T_0 by

$$e^{cT_0} = \frac{1}{2}\left(1 + \frac{1+y_0}{y_0}\right).$$

It follows that $y(T_0) = 1 + 2y_0$. Hence

$$\|\chi\|_{L^\infty(0,T_0;L^p)} \leq 1 + 2\|\chi(0)\|_p. \tag{3.10}$$

Next, we turn back to the equation (3.8). By integrating it over $(0,t)$, for $t \in (0,T_0)$, and by using (3.10), with a straightforward manipulation it is easy to verify that

$$\|\chi\|_{L^\infty(0,T_0;L^p)} + \nu^{\frac{1}{p}}\|\chi\|_{L^p(0,T_0;L^{3p})} \leq M_0,$$

where

$$M_0^p \leq c(1 + \|\chi(0)\|_p^p + \|\chi(0)\|_p^{p+1}).$$

Denote by N_0 positive constants that depend on M_0. One has

$$\|\omega\|_{L^\infty(0,T_0;W^{2,p})} + \nu^{\frac{1}{p}}\|\omega\|_{L^p(0,T_0;W^{2,3p})} \leq N_0. \tag{3.11}$$

Consequently,

$$\|u\|_{L^\infty(0,T_0;W^{3,p})} + \nu^{\frac{1}{p}}\|u\|_{L^p(0,T_0;W^{3,3p})} \leq N_0. \tag{3.12}$$

From (3.11), it follows that

$$\|\nu\Delta\omega\|_{L^\infty(0,T_0;L^p)} \leq \nu N_0. \tag{3.13}$$

Further,

$$\|(u\cdot\nabla)\omega\|_{L^\infty(0,T_0;W^{1,p})} + \|(\omega\cdot\nabla)u\|_{L^\infty(0,T_0;W^{2,p})} \leq N_0.$$

From (2.10), together with the above estimates it follows, in particular, that

$$\|\partial_t\omega\|_{L^\infty(0,T_0;L^p)} + \|\partial_t u\|_{L^\infty(0,T_0;W^{1,p})} \leq N_0. \tag{3.14}$$

Denoting by π the pressure field associated to the solution u, from the previous estimates one has

$$\|\nabla\pi\|_{L^\infty(0,T_0;W^{1,p})} \leq N_0 + \nu N_0.$$

Since, for $t > \tau$,

$$|\omega(t) - \omega(\tau)|^p = \left|\int_\tau^t \partial_s\omega(s)ds\right|^p \leq (t-\tau)^{p-1}\int_\tau^t |\partial_s\omega(s)|^p ds, \quad \text{a.e. in } \Omega,$$

one has

$$\|\omega(t) - \omega(\tau)\|_p^p \leq (t-\tau)^{p-1}\int_\Omega\int_\tau^t |\partial_s\omega(s)|^p ds\, dx \leq (t-\tau)^p\|\partial_t\omega\|^p_{L^\infty(0,T_0;\,L^p)}.$$

Hence, since $\|\cdot\|_{s,p} \leq c\|\cdot\|_{2,p}^{\frac{s}{2}}\|\cdot\|_{0,p}^{1-\frac{s}{2}}$, for $0<s<2$, from (3.11) and (3.14) it follows that

$$\|\omega\|_{C^{0,1-\frac{s}{2}}([0,T_0];W^{s,p})} \leq c\|\omega\|_{L^\infty(0,T_0;\,W^{2,p})}^{\frac{s}{2}}\|\partial_t\omega\|_{L^\infty(0,T_0;\,L^p)}^{1-\frac{s}{2}} \leq N_0. \tag{3.15}$$

4 - Proof of the vanishing viscosity limit result

The a-priori estimates obtained in the previous section are sufficient for passing to the limit and proving Theorem 2.1. Recall that in the above arguments the vector fields u and ω stand for u^ν and ω^ν respectively. From the above uniform estimates and well known compactness arguments, there is a suitable subsequence of u^ν and a vector field u such that one has $(2.13)_1$ and also the following property

$$\partial_t u^\nu \rightharpoonup \partial_t u \quad \text{in} \quad L^\infty(0,T_0;\,W^{1,p}) \quad \text{weak} - *.$$

The limit property $(2.13)_2$ follows from (3.15) by appealing to the Ascoli-Arzela compact embedding theorem. Note that we may pass to the limit directly in the equation (2.1), as $\nu \to 0$, and obtain that u is a solution to the Euler system (2.3), (2.4). The uniqueness of the strong solution to the Euler equations ensures the weak$-*$ convergence of the whole sequence u^ν in $L^\infty(0,T_0;\,W^{3,p}(\Omega))$ and the strong convergence in $C([0,T_0];\,W^{s,p}(\Omega))$.

Next, we take the difference between the vorticity equations for the Euler and the Navier-Stokes equations, respectively,

$$\begin{aligned}&\partial_t\omega - \partial_t\omega^\nu + (u-u^\nu)\cdot\nabla\omega + u^\nu\cdot\nabla(\omega-\omega^\nu)\\ &\quad + \omega^\nu\cdot\nabla(u^\nu-u) + (\omega^\nu-\omega)\cdot\nabla u = -\nu\Delta\omega^\nu.\end{aligned}$$

From the convergence results already obtained, together with estimates (3.11), (3.12) and the regularity of the solution to the Euler equations, it follows that the non-linear terms on the left hand side of the above equation go to zero, as ν goes to zero, in $C([0,T_0];W^{s,p})$, for any $s<1$. In view of (3.13), equation (2.14) follows. In particular, the above non-linear terms go to zero in $C([0,T_0];L^{3p})$, if $p>2$. On the other hand, from (3.11), it follows that

$$\|\nu\Delta\omega\|_{L^p(0,T_0;L^{3p})} \leq N_0\nu^{\frac{1}{p'}}. \tag{4.1}$$

Hence the convergence (2.15) holds.

If in the above argument we appeal to (3.7) instead of (3.8), we obtain similar results, where $W^{3,p}$ is replaced by $W^{2,p}$, and $s<2$. In this case it must be $p>3$ everywhere.

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