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Some results on the Navier-Stokes equations with Navier boundary conditions

Abstract. I make an overview of some results concerning the Stokes and Navier-Stokes equations, supplemented with the Navier's type slip boundary conditions. I try to explain the interest for this problem, the main analytical results, and also the differences between the flat case and more general cases. Some recent results concerning the vanishing viscosity limits are also announced.

Keywords. Stokes equations, Navier-Stokes equations, existence, regularity, singular limits.

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1 - Introduction

In this paper I am collecting and enlarging the notes of a short course for PhD students and young researchers, given in September 2009 at the International School for Advanced Studies (SISSA/ISAS) of Trieste. The lectures have been given within the seventh meeting in hyperbolic conservation laws and fluid dynamics. In these lectures I took chance of presenting certain rather classical results, but also more recent and even current advances concerning a subject which is related with fluid dynamics, but also with the basic theory of elliptic systems of partial differential equations and with the behavior of solutions when singular perturbations are present. In particular -motivated by physical, numerical, and analytical insight- I am presenting some results and open problems concerning the *incompressible* Navier-Stokes system supplemented by certain slip boundary conditions. This paper cannot be considered as an exhaustive treatment of the subject, but just a limited collection of results along a research path I think most interesting, especially for young scientists, potentially oriented in doing a research in related fields. In the setting of the problem I am trying to emphasize the connections between modeling, numerical aspects, computational tools, and the mathematical analysis. Most of the results are strictly linked, but I am trying to look at them from different points of view, since taking a (limited) detour in related fields can give new insight, new inspiration, and also open new research avenues.

I am deliberately skipping many details in the various proofs, since I am trying to shed light on the ideas, with the hope of interesting the reader to this subject. In fact, in order that the reader can focus directly on the (hopefully) most relevant points, I am avoiding the most technical parts, but an extended and rather detailed bibliography is also added, where one can precisely find the missing details. In the references one can also find recent results and attempts to understand more about fluids and the fascinating research involving their mathematical analysis, modeling of turbulence, and also the numerical resolution of real-life flows.

The presentation is intended for a reader with at least some background of Sobolev spaces and of the basic variational results for elliptic and parabolic equations. The knowledge of the (nowadays) "elementary" results in mathematical fluid mechanics (existence of weak and strong solutions for the Navier-Stoke equations) is not necessary, even if for a better understanding at least a qualitative idea of the basic results is warmly welcome.

The paper is organized as follows: In Section 2 I introduce the problem I will consider and I give the main motivations for the study of viscous fluids with slip boundary conditions. In Section 3 I give some of the motivations that I believe relevant to study this problem. In Section 4 the main properties regarding the variational formulation of the linear stationary problem are recalled. In particular, the approach based on the introduction of the artificial compressibility is explained. In Section 5 I recall the formulation for the time-evolution nonlinear problem and some existence results. In Section 6 the connection with modeling of boundary conditions in LES is explained, together with some rigorous results. In Section 7 some results concerning the vanishing viscosity are recalled and some new results are announced.

2 - Setting of the problem

The incompressible (with constant density) Navier-Stokes equations read as

$$
\begin{aligned}
u_t - \nu \varDelta u + (u \cdot \nabla)\, u + \nabla p &= f \quad \text{in } \varOmega \times\,]0, T], \\
\nabla \cdot u &= 0 \quad \text{in } \varOmega \times\,]0, T],
\end{aligned}
\tag{1}
$$

where the open set $\varOmega \subset \mathbb{R}^n$, $n = 2, 3$, is the physical domain. The unknown u is the velocity vector field with n components (and to avoid a too heavy notation I do not use boldface symbols for vector fields). The scalar p is the pressure, while the positive number ν denotes the kinematic viscosity.

The differential operators ∇ and $\varDelta$ are the standard ones, while we observe that the convection term is properly defined in terms of coordinates as follows

$$
\big[(u \cdot \nabla)\, u\big]_i := u_k \partial_k u_i, \quad \text{for} \quad i = 1, \ldots, n,
$$

and in the paper I use (when needed) the Einstein convention of summation over repeated indices, while ∂_k denotes partial differentiation with respect to x_k. It is also worth noting that, due to the incompressibility, we can also write the convection term as

$$(u \cdot \nabla)\, u = \nabla \cdot (u \otimes u),$$

while in some cases the notation

$$(u \cdot \nabla)\, u = \big[\nabla u\big] u$$

is used, thinking of ∇u as a linear operator acting on the vector u.

For an axiomatic derivation of the equations and for the precise assumptions underlying the process, see Serrin [**125**]. Further details and other presentations can be found in Chorin and Marsden [**48**], Batchelor [**9**], Lamb [**93**], and Landau and Lifshitz [**94**].

In almost all the paper the space dimension will be $n = 3$, corresponding to the case with more interesting physical meaning. In the 2D case the theory is much more complete (regardless of the boundary conditions). In some cases I will restrict to the 2D case in order to point out some of the simplifications occurring in two dimensions. The initial-value-problem must be supplemented with a divergence-free initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega.$$

In presence of a domain Ω with boundary $\Gamma = \partial\Omega$, one has to add suitable boundary conditions. In literature, most of the results in domain with boundaries[1] are obtained by supplementing the system with Dirichlet (*no slip*) boundary conditions

$$u(x, t) = 0, \qquad \text{on } \Gamma \times\,]0, T]. \tag{2}$$

The Dirichlet boundary conditions have been proposed by Stokes [**131**] since the contrary assumption

> "*...implies an infinitely greater resistance to the sliding of one portion of fluid past another than to the sliding of fluid over a solid.*"

Condition (2) corresponds to consider that fluid particles adhere to the boundary, hence they have the same velocity (generally vanishing) of the solid boundary.

The basic results on existence of weak solutions, local existence of strong solutions, partial regularity and so on (in the periodic setting and with Dirichlet boundary

[1] In the whole space case one has just to assume suitable decay at infinity and of particular interest is also the problem in the space periodic setting.

conditions) can be found in many references, see for instance the books by Constantin and Foiaş [**54**], Doering and Gibbon [**57**], Galdi [**65**, **66**, **67**], Ladyžhenskaya [**92**], Sohr [**128**], and Temam [**137, 138**]. Moreover the reader is warmly encouraged to read the original sources, especially the masterpieces written by Leray [**97**] and Hopf [**75**].

Under the Dirichlet boundary conditions, one can (formally) use the velocity itself as test function, obtaining -with suitable integration by parts- the energy balance

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx + \nu\int_{\Omega}|\nabla u|^2\,dx = \int_{\Omega} f\cdot u\,dx.$$

This gives a control of the kinetic energy, which is critical to prove existence of weak solutions. Moreover, it is well-known that there are long-standing open questions related the 3D Navier-Stokes equations: Essentially we know global (in time) existence of solutions in a class in which we are not able to prove uniqueness, and uniqueness in a class in which we are able to prove just existence for small times. The "*Million dollar problem*" concerning the Navier-Stokes equations can be found in the web site http://www.claymath.org/millennium of the Clay Institute.

I do not want to focus on a so hard and seemingly elusive-to-any-attempt open problem, but I would like to take a slightly different path, a little bit more oriented towards applications, and where there are realistic chances to obtain some new (and non trivial) results.

2.1 - *The Navier boundary conditions*

It is well known that there are situations in which the boundary condition (2) may not be valid. From the historical point of view, the slip (with friction) boundary conditions proposed by Navier [**111**] (twenty years before the work of Stokes) were

$$\begin{aligned} &u\cdot\underline{\mathrm{n}}=0 && && \text{on } \Gamma\times\,]0,T],\\ &\beta\,u_\tau+\underline{\mathcal{T}}(u,p)=0, && \beta\geq 0, && \text{on } \Gamma\times\,]0,T], \end{aligned}\tag{3}$$

where $\underline{\mathrm{n}}$ denotes the exterior unit normal vector to Γ, while

$$u_\tau := u-(u\cdot\underline{\mathrm{n}})\,\underline{\mathrm{n}}\,,$$

denotes the tangential part[2] of the velocity. In addition

$$\underline{\mathcal{T}}(u,p) := \underline{t}(u,p)-(\underline{t}(u,p)\cdot\underline{\mathrm{n}})\,\underline{\mathrm{n}}$$

[2] I use this notation which is historical, but one can also write the same in terms of exterior product with the unit vector $\underline{\mathrm{n}}$, see also (46).

denotes the tangential part of the *Cauchy stress vector* $\underline{t}$ defined by

$$\underline{t}(u,p) := \underline{\mathrm{n}} \cdot \mathbb{T}(u,p) = \sum_{k=1}^{n} \mathbb{T}_{ik}(u,p)\, \underline{\mathrm{n}}_k,$$

and, if δ_{ij} denotes the Kronecker symbol,

$$\mathbb{T}_{ik}(u,p) := -\delta_{ik}\, p + \nu(\partial_k u_i + \partial_i u_k).$$

Since the term $\underline{\mathcal{T}}(u,p)$ in fact does not depend explicitly on the pressure, we use also the notation

$$\beta\, u_\tau + \underline{\mathcal{T}}(u) = 0, \qquad \beta \geq 0.$$

Probably Maxwell [**107**] first analyzed the two types of boundary conditions (conditions (3) proposed by Navier and condition (2) proposed by Stokes), observing that the same conditions may be derived also within the kinetic theory of gases. In fact, the Navier-Stokes equations can be obtained by taking suitable limits from the kinetic theory of gases and one obtains the slip conditions with

$$\beta \sim \frac{\text{mean free passes of molecules}}{\text{macroscopic length}}.$$

Thus, for certain range of the parameters, the no-slip condition

$$u_\tau = 0 \qquad \text{on } \Gamma \times]0,T]$$

can be recovered. In particular, the parameter β should depend on the viscosity ν and on the mean free-path λ, satisfying the pair of consistency conditions:

$$\begin{aligned} &\beta \to \infty && \text{as } \lambda \to 0 \text{ for } \nu \text{ fixed},\\ &\beta \to 0 && \text{as } \nu \to 0 \text{ for } \lambda \text{ fixed}. \end{aligned}$$

With the above asymptotics it is possible to recover in both cases the correct no-slip boundary conditions for viscous fluids and the no-penetration conditions for ideal fluids. Observe in fact that $u \cdot \underline{\mathrm{n}} = 0$ is the condition supplementing the Euler system, i.e., the Navier-Stokes equations with $\nu = 0$, see Section 7.

Remark 2.1. *In contrast to Stokes* [**131**] (*in* 1845) *who employed continuum mechanics, Navier* [**111**] (*in* 1823) *derived the equations by using some formal* (*and possibly out the range of applicability*) *analogy with the elasticity theory and the assumption that molecules are animated by attractive and repulsive forces. Overview on the historical connections can be found in Cannone and Friedlander* [**46**].

Under the boundary conditions (3) one can perform integrations by parts similar to those valid with Dirichlet boundary conditions, obtaining the energy balance

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |u|^2\,dx + \beta\int_{\Gamma} |u_{\tau}|^2\,dS + \nu\int_{\Omega} |\nabla u|^2\,dx = \int_{\Omega} f\cdot u\,dx,$$

but the most interesting features of the slip boundary conditions can be better explained when considering the vorticity field $\omega := \operatorname{curl} u = \nabla\times u$.

2.2 - *A couple of vector identities*

It is worth noting that, on flat portions of the boundary and if $\beta = 0$, the boundary conditions (3) and

$$\begin{aligned} u\cdot \underline{\mathrm{n}} &= 0 \qquad \text{on } \Gamma\times\,]0,T], \\ \omega\times \underline{\mathrm{n}} &= 0 \qquad \text{on } \Gamma\times\,]0,T], \end{aligned} \tag{4}$$

coincide. Observe that -in a certain sense- under the Dirichlet boundary conditions (see [**125**]) it holds that $\omega\cdot\underline{\mathrm{n}} = 0$, hence the slip conditions are really different from the no-slip ones in the light of behavior of the vorticity field.

This leads us to consider (4), even when the boundary is not flat. Note that the boundary conditions (4) are strongly related to the slip boundary conditions (3). In fact,

$$\underline{t}\cdot\underline{\tau} = \frac{\nu}{2}(\omega\times\underline{\mathrm{n}})\cdot\underline{\tau} - \nu u\cdot\frac{\partial\underline{\mathrm{n}}}{\partial\underline{\tau}} \qquad \text{on } \Gamma,$$

for each vector $\underline{\tau}$ tangential to the boundary. Note that the last term from the right-hand side is a lower order term, and that $\omega\times\underline{\mathrm{n}}$ and $\partial\underline{\mathrm{n}}/\partial\underline{\tau}$ are tangential to Γ, while $|\partial\underline{\mathrm{n}}/\partial\underline{\tau}|$ is the normal curvature in the $\underline{\tau}$ direction. The relevance of using boundary conditions involving the vorticity is that one can try to use the vorticity equation in order to obtain estimates on the vorticity field and consequently on the gradient of the velocity. Recall in fact that for divergence free vector fields it holds

$$-\Delta u = \operatorname{curl}\omega,$$

hence, one can invert the Laplace operator showing that (at least in terms of L^q estimates) ω and ∇u are equivalent. In the case of Dirichlet boundary conditions the partial differential equations for the vorticity

$$\omega_t - \nu\Delta\omega + (u\cdot\nabla)\,\omega - (\omega\cdot\nabla)\,u = \operatorname{curl} f,$$

seem difficult to be used since in the integration by parts[3] arise some boundary integrals which we are not able to control. In particular, most of the problems are created by the Laplacian of the vorticity (this explains why for the Euler equations this problem does not appear). Consequently new problems derive from the linear part of the equations, while the well-known limitations due to the nonlinear terms remains essentially the same. We set the problem in the flat case and for instance consider $\Omega = \mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : \ x_3 > 0\}$, with boundary $\Gamma := \{x \in \mathbb{R}^3 : \ x_3 = 0\}$. On the boundary we consider, for $\beta = 0$, the conditions (3) (now equivalent to (4)) and we obtain with direct computation

$$
\begin{cases}
\omega_1 = \partial_2 u_3 - \partial_3 u_2 = 0 + 0, \\
\omega_2 = \partial_3 u_1 - \partial_1 u_3 = 0 + 0, \\
\partial_3 \omega_3 = -\partial_1 \omega_1 - \partial_2 \omega_2 = 0 + 0,
\end{cases}
$$

where the last line follows since $\nabla \cdot \omega = 0$. With this at disposal we can show that

$$
-\int_\Omega \Delta\omega \cdot \omega \, dx = \int_\Omega |\nabla \omega|^2 \, dx,
$$

because the boundary term $\int_\Gamma \omega \cdot \partial\omega / \partial \underline{n} \, dS$ vanishes identically, and one can also compare this Gauss-Green formula with Lemma 5.2 in the general case.

The mathematics of the Navier-Stokes under these boundary conditions presents new problems: Many results are not straightforward and need some adaption, see also Málek and Rajagopal [**105**]. As a further example, the finite element numerical analysis requires some work, due to a) the choice of the basis functions and b) the storage of the information is not the same as for the Dirichlet conditions. A study of the numerical problems related to the implementation of (3) or (4) can be found in Girault [**71**] (in this paper they are called non-standard), John [**79, 78**], Liakos [**100**], and Verfürth [**142, 143**].

Remark 2.2. *In the case $n = 2$ the situation is considerably simpler, as we will see in the next sections. In fact, for $n = 2$, the second boundary condition in equation* (4) *is simply replaced by $\omega = 0$. Furthermore,*

$$
\underline{t} \cdot \underline{\tau} = \frac{\nu}{2}\omega - \nu u \cdot (k \, \underline{\tau}), \tag{5}
$$

[3] Clearly in the space-periodic case and for the Cauchy problem, the use of the vorticity represents a formidable tool. Problems in order to use the vorticity equation in the Dirichlet case turns out since we do not know the boundary values of ω, see also the results in Rautmann [**119**].

where k is the curvature of Γ. This vector identity is well-known and it can be used to construct weak solutions to the Euler equations in the 2D case by approximating the Euler equations with the Navier-Stokes equations supplemented by $u \cdot \underline{\mathrm{n}} = \omega = 0$ on Γ, see for instance J.L. Lions [**101**], *Bardos* [8], *Clopeau, Mikelić, and Robert* [**49**], *and in the stochastic context Bessaih and Flandoli* [**37**, **38**]. *See also Section* 7.1.

3 - Some problems in which the Navier conditions naturally arise

In this section I explain some of the motivations that make the Navier-slip boundary conditions interesting from different points of view and not only a mathematical game.

3.1 - *Certain physical situations linked with slip on the boundary*

In Serrin [**125**, § 64] and Truesdell [**140**] it is pointed out that when moderate pressure and low surface stresses are involved, the adherence condition is no longer true. In this respect several authors proposed various *slip* (generally nonlinear) conditions, modeling precise physical situations. Having in mind problems with high altitude aerodynamics and the interface of porous media, Serrin [**125**], Beavers and Joseph [**10**], and Kreĭn and Laptev [**90**] proposed various slip conditions. Recently, Fujita [**64**, **123**] performed the analysis with the "slip or leak with friction" boundary conditions. These conditions are of particular interest in the study of polymers, blood flow, and flow through filters. The boundary conditions studied in [**64**], are

$$\begin{aligned}
&u \cdot \underline{\mathrm{n}} = 0 \qquad \text{on } \Gamma \times]0,T],\\
&\text{if } |\underline{t}| < k|\underline{\mathrm{n}} \cdot \mathbb{T} \cdot \underline{\mathrm{n}}|, \quad \text{then } u_\tau = 0 \qquad \text{on } \Gamma \times]0,T],\\
&\text{if } |\underline{t}| = k|\underline{\mathrm{n}} \cdot \mathbb{T} \cdot \underline{\mathrm{n}}|, \quad \text{then } \exists\, \lambda \geq 0: \ u_\tau = -\lambda\, \underline{t} \qquad \text{on } \Gamma \times]0,T],
\end{aligned}$$

where $k > 0$ is a coefficient of friction. This problem is treated with the techniques of variational inequalities, and turns out to be a particular case of the nonlinear boundary conditions proposed in [**125**, p. 240]. These nonlinear (unilateral) conditions are very-strictly connected to both the Navier and the no-slip boundary conditions. See also Consiglieri [**50**] for related problems.

For laminar flows the Navier boundary conditions (3) also appears in the presence of rough boundaries, see Jäger and Mikelić [**76**, **77**] and Achdou, Pironneau, and Valentin [**1**]. Among other nonstandard boundary conditions I recall those studied by Begue et al. [**11**]

$$\begin{aligned} \underline{n} \times u &= 0 \qquad \text{on } \Gamma \times]0,T], \\ p &= 0 \qquad \text{on } \Gamma \times]0,T], \end{aligned}$$

and the "do-nothing" Neumann conditions, appealing for numerical studies in pipes, implemented in Heywood, Rannacher, and Turek [**74**]:

$$\frac{\partial u}{\partial \underline{n}} - p\,\underline{n} = K\,\underline{n} \qquad \text{on } \Gamma \times]0,T].$$

Some interest for slip boundary conditions has recently appeared also in problems of shape optimization (see Bucur *et al.* [**44, 45**]), especially in presence of rough boundaries.

My main interest about the Navier-type conditions comes from another theme and more precisely the modeling of turbulent flows. In the next section I will explain some of the results and connections with large scale approximation of fluids with very small viscosities.

3.2 - *Near wall models and turbulent flows*

My interest in non-standard boundary conditions started from the study of the numerical methods in turbulence. In order to briefly introduce the problem, I am summarizing the main points. It is not possible to compress any reasonable understanding of turbulent flows in a few pages, but I am trying to give at least motivations for the results that will follow. I also hope to interest the reader for a research field, which is still lacking of the needed mathematical rigor. For turbulent flows new insight and advances could come from the joint efforts of engineers, mathematician, and physicists. The aim is to try to attack what has been defined by Landau [**94**]

> ... *one of the great unsolved problems of classical physics.*

The reader interested in a better and deeper understanding of the field can see the books by Frisch [**63**], Pope [**117**], and Tennekes and Lumley [**139**]. As one can notice I will not define what a turbulent flow is, but one can stem on the fact that as the viscosity decreases, the flow becomes less and less stable and the motion becomes "*chaotic.*"[4] From the mathematical point of view one can say that uniqueness or stability are known only for large values of the viscosity (with respect to the velocity of the flow), while the most interesting effects appear when $\nu \sim 0$.

[4] This word is used here in a purely qualitative way.

It is well-known that Kolmogorov's [**89**] theory predicts that simulating turbulent flows by using the Navier-Stokes Equations requires $\mathcal{N} = O(Re^{9/4})$ degrees of freedom, where $Re = UL\,\nu^{-1}$ denotes the (non dimensional) Reynolds number and U and L are a typical velocity and length, respectively. This number $\mathcal{N}$ is too large, in comparison with memory and computational capacities of actual computers, to perform a Direct Numerical Simulation (DNS). Indeed, for realistic flows -such as for instance geophysical flows- the Reynolds number is order at least of 10^8, yielding $\mathcal{N}$ of order 10^{18}....

With some technical assumptions and certain physical guessing, Kolmogorov has been able to show that (at least for some classes of velocities considered as random variables with suitable properties) the coherent structures of the flow (nowadays called *eddies*) evolve into smaller and smaller ones, leaving the total energy unaltered, till the point where they are so small to be destroyed by the viscous mechanism[5] and this happens in a proper statistical sense. The Kolmogorov length η_K at which this occurs represents the smallest scale present in the flow and it is of the order of $Re^{-3/4}$. For scales below this length, the behavior is more or less the same of solutions of the (dissipative) heat equation. Even if this is not rigorous (because to obtain this behavior one has to postulate a mathematical knowledge of solutions that we do not have) it is one of the reasons why one aims at computing at least the "mean or large scales values" of the unknowns (u, p), those not involving scales smaller than η_K. This is not enough, since numerical simulations cannot reach a so small scale, but motivated also from the fact that some gross characteristics of the flow behave in a more orderly manner, Large Eddy Simulation (LES) is about approximating (spatial) averages of turbulent flows, see Foiaş *et al.* [**61**]. Thus, LES seeks to predict the dynamics (the motion) of the organized structures in the flow (the eddies) which are larger than some user-chosen length-scale α. The length $\alpha > 0$ is related to mesh-size of the grid used in the numerical simulation. It is clear that LES is a *computational tool*, for which one tries to give a sound mathematical justification. One of the great challenges of simulating turbulence is that equations describing averages of flow quantities cannot be obtained directly from the physics of fluids. On the other hand, the equations for the point-wise flow quantities are well-known, but intractable to direct solution and sensitive to small perturbations and uncertainties in problem data.

In the spirit of the ideas speculated probably the first time by Leonardo da Vinci [**56**], the LES approach corresponds in finding a suitable computational de-

[5] This mechanism has been postulated in the early thirties by the meteorologist Richardson [**120**].

composition

$$u = \overline{u} + u' \qquad \text{and} \qquad p = \overline{p} + p',$$

where the primed variables are turbulent fluctuations around the over-lined mean fields. In many cases fluctuations can be disregarded and this can be justified because in applications knowledge of the mean flow is enough to extract relevant information on the fluid motion. The "mean values" can be defined in several ways (time or space average, statistical averages, solution of elliptic problems ...). If one denotes the means fields by $(\overline{u}, \overline{p})$, and by assuming that the averaging operation (whatever it is) commutes with differential operators, one gets the *filtered Navier-Stokes equations*

$$\begin{aligned} \partial_t \overline{u} + \nabla \cdot (\overline{u \otimes u}) - \nu \Delta \overline{u} + \nabla \overline{p} &= \overline{f}, \\ \nabla \cdot \overline{u} &= 0. \end{aligned}$$

This immediately raises the question of the *interior closure problem*, that is the modeling of the second order tensor $R(u) = \overline{u \otimes u}$ in terms of the filtered variables $(\overline{u}, \overline{p})$. Classical Large Eddy Simulations (LES) models approximate $R(u)$ by $w \otimes w - \nu_T(k/k_c)\nabla^s w$ where $w \approx \overline{u}$, and $\nabla^s w := (\nabla w + \nabla w^T)$. Here $\nu_T \geq 0$ is an *eddy viscosity* based on a "cut-off frequency" k_c (for a general discussion see [**121**]).

Remark 3.1. *I am introducing the new variable w since when using any approximation for $R(u)$, one is not writing the differential equations satisfied by $\overline{u}$, but that satisfied by another field w, which is hopefully close enough to $\overline{u}$.*

In recent years the role of Large Eddy Simulation increased and attracted the attention of mathematicians. One can find extensive overview in the nowadays classic book by Sagaut [**121**]. See also Geurts [**70**], John [**79**], Lesieur, Métais, and Comte [**98**] and -for a more theoretical approach- the monograph I wrote with my co-workers [**32**].

Here, I do not want to discuss the interior closure modeling, or other specific issues of LES. I want to focus on the problem that, even if one has a disposal a set of partial differential equations describing in some sense the mean values of the flow, there is the need to describe the boundary conditions. The classical [**89**] theory of turbulence starts with the homogeneous and isotropic case and the flow is in a periodic box. In real life boundaries do exist and they are one of the biggest source of problems. As a folklore one can recall the sentence of Heisenberg

> *... the boundary is the invention of the devil ...*

Concerning this issue, it is also interesting to read the reprint of an old paper by von

Neumann [**145**] in one of the latest volume of the Bulletin of the American Mathematical Society.

In most cases in LES, the filtered velocity $\overline{u}$ is defined through a space-convolution

$$\overline{u}(x,t) = g_\alpha(x) * u(x,t) \tag{6}$$

with a rapidly decreasing smoothing kernel $g_\alpha(x)$ of width α. In several cases of practical interest g_α is a Gaussian, i.e.,

$$g_\alpha(x) := \left(\frac{6}{\pi}\right)^{3/2} \frac{1}{\alpha^3}\, \mathrm{e}^{-\frac{6|x|^2}{\alpha^2}}.$$

By definition, the value of $\overline{u}$ at a point x_0 on the boundary Γ will mainly depend on the behavior of u in a neighborhood of width α near that point: Even if u is extended to zero for each $x \notin \overline{\Omega}$, it is clear that in general $\overline{u}(x_0) \neq 0$.

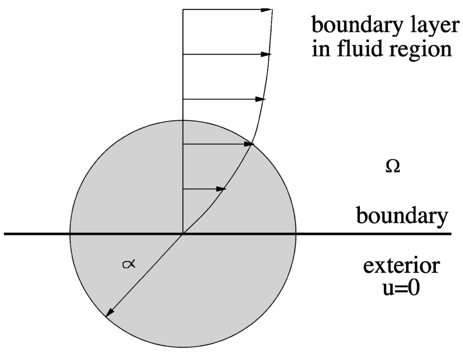

Fig. 1. Filtering the velocity does not yield homogeneous Dirichlet conditions at the boundary.

Following the approach of LES (again trying to understand the possible connections with the boundary conditions) I recall that another way of approximating the equations -avoiding eddy viscosities- consists in approaching $R(u)$ by a suitable quadratic term. It is curios that the first LES model has been introduced by Leray [**97**] with a different goal. In fact to construct weak solutions of the Navier-Stokes equations in the celebrated 1934 paper he solved (in $\mathbb{R}^3$) the differential problem

$$\begin{aligned} w_t - \nu\Delta w + \nabla\cdot(\overline{w}\otimes w) + \nabla q = f \qquad & \text{in } \mathbb{R}^3 \times]0,T], \\ \nabla\cdot w = 0 \qquad & \text{in } \mathbb{R}^3 \times]0,T]. \end{aligned} \tag{7}$$

In this model the transport is realized by $\overline{w}$, which is a field smoother than w itself. In particular in Leray's work $\overline{w}$ is defined by means of a smoothing by mollifiers, as in (6). This approximation has the same properties of the recent Leray-alpha LES model, where $\overline{w}$ satisfies[6] the elliptic equations

$$-\alpha^2 \Delta\overline{w} + \overline{w} = w.$$

One has to specify the boundary conditions for the above differential equation and this is again absolutely non-trivial. It is also relevant to observe that (in absence of boundaries) the solution w to (7) satisfies

$$w \to u, \qquad \text{as} \qquad \alpha \to 0,$$

where u is the velocity of the Navier-Stokes system. In this limit process the good properties of w (which is smooth and unique) are lost and this is Leray's proof for existence of possibly non unique weak-solutions.

Remark 3.2. *As one can understand this is not the real goal of LES, since one would like to approximate $\overline{u}$ and not a single trajectory u. A first rigorous result in this direction have been recently proved in a joint work with R. Lewandowski* [**33**] *for the so-called Stolz and Adams* [**132**, **2**] *Approximate Deconvolution Model* (*ADM*). *Even if we are forced to consider the periodic setting, to our knowledge such a "well posedness", i.e., proving that w converges to $\overline{u}$* (*which is the average of a weak solution*)*, was not previously known for any LES model.*

In order to describe the problems arising when studying the boundary conditions for a LES model, I first recall that in the boundary-layer theory several log-law and power-law asymptotics near the boundary are introduced, together with the fictitious boundaries, in order to model turbulent flows within a small region near to Γ. Roughly speaking, appropriate nonlinear boundary conditions are imposed on an artificial boundary that lies inside the computational domain. The boundary conditions may simulate (at least in a computational approach) the behavior of the *boundary-layer*, and they are modeled to take into (partial) account of the peculiar behavior of a fluid near the boundaries. In this respect we recall that Maxwell [**107**] observed

> *... it is almost certain that the stratum of gas next to a solid body is in a very different state from the rest of the gas.*

As we pointed out before, one basic problem in LES is turbulence driven by interaction of the flow with a solid wall. Mathematically, this is the problem of *specifying*

---

[6] This is why it is also called a differential filter and it is generally used in the periodic setting.

boundary conditions for flow averages. Flow averages are *inherently non-local*: they depend on the behavior of the unknown turbulent flow near the boundary. On the other hand, to be guided by the mathematical theory of the equations of fluid motion and seek boundary conditions that have hope of leading to a simple enough and well-posed problem, those boundary conditions should be *local*. From the mathematical point of view more complex conditions can be analyzed, but one has to keep in mind that LES is a computational tool, hence the equations need to be implemented in a efficient way. Introducing something too complex or numerically intractable will have the only effect of moving problems from one point to another one.

In LES the question of finding boundary conditions when using a constant averaging radius α is known as *Near Wall Modeling*. This is related to the extensive literature in Conventional Turbulence Modeling (CTM) on "wall-laws." CTM seeks to approximate long-time averages of flow-quantities and, conveniently for CTM, there is a lot of experimental and asymptotic information available about time-averaged turbulent boundary-layers. One common approach in CTM is to place an artificial boundary *inside* the flow domain and *outside* the boundary-layer, together with a Dirichlet condition for the stresses. The main difference between CTM and LES is that LES would like to describe inherently dynamic phenomena, so imposing a condition that $\overline{u}$ should match some equilibrium profile is probably not correct. One challenge in LES is how to use the extensive information on *time averaged turbulent* boundary-layers to generate NWM's that allow *time fluctuating* solution's behavior near the wall.

The classical approach (first introduced for the $k - \varepsilon$ model) consists in eliminating part of the boundary-layer, see Launder and Spalding [**95**]. The boundary that is considered is not the real boundary Γ, but it is an artificial one Γ_1, lying inside the volume of the flow, where one can impose

$$
\begin{aligned}
&\overline{u} \cdot \underline{\mathrm{n}} = 0 && \text{on } \Gamma_1 \times [0, T], \\
&\frac{u_{ws}^2}{|\overline{u}|}\overline{u}_\tau + \underline{\mathcal{T}}(\overline{u}, \overline{p}) = 0 && \text{on } \Gamma_1 \times [0, T].
\end{aligned}
\tag{8}
$$

In particular, when considering the Smagorinsky model[7] (studied with the above artificial boundary conditions by Parés [**115**]) the turbulent stress-tensor in (8) is

[7] This is the oldest LES model introduced by Smagorinsky [**127**] with the intent of studying geophysical flows. The mathematical properties have been studied starting with Ladyženskaya [**91**] since they fit with the theory of monotone operators. We do not treat here these equations, but we recall that they are still an intense research field. For the treatment of the Smagorinsky model with Navier boundary conditions, especially in the context of regularity, see Beirão da Veiga [**15**].

given by

$$\mathbb{T}_{ik}(\overline{u}, \overline{p}) = -\delta_{ik} p + (\nu + \nu_T)(\partial_k \overline{u}_i + \partial_i \overline{u}_k)$$

where ν is the usual kinematic viscosity, while $\nu_T = \nu_T(\alpha, \nabla^s \overline{u})$ is the turbulent viscosity. The quantity u_{ws} appearing in Eq. (8) is the so-called *wall shear velocity* (or skin friction velocity). It has the dimension of a length divided by a time and acts as a characteristic velocity for the turbulent flow; for more details, see Landau and Lifshitz [**94**, § 42-44] and Pope [**117**, § 7.1.3].

One recurring theme in these attempts is the use of *non-local* boundary conditions to incorporate solution's behavior in a strip near Γ, *via* an extra forcing function in the strip along the boundary. The problem remains however, difficult because the behavior of $\overline{u}$ on Γ depends on the behavior of u in a α-neighborhood of Γ.

As pointed out in Galdi and Layton [**68**] the physical intuition may suggest that

> ... *large coherent structures touching a wall do not penetrate, but instead slide along the wall and lose their energy.*

Consequently the boundary conditions of Navier may be revisited by linking the micro-scale λ of the kinetic theory of gases with the radius α of the averaging filter. Many NWM have been tested in the computational approach (Sagaut [**121**] and Piomelli and Balaras [**116**]), the results are not uniformly successful, and a positive outcome is very often based on a fine tuning of parameters. This is why new models require at least a positive background from the physical hypotheses and a coherent mathematical analysis. In particular, a direct application of the Navier slip-with-friction boundary conditions (3) is prevented by

1) The presence of recirculation regions;
2) The presence of fast time-fluctuating quantities.

The first problem is motivated by the fact that in recirculation regions the local Reynolds number is very different from the main stream, and it is natural to expect that β should depend (possibly in a nonlinear way) on a local Reynolds number related to the local slip speed, i.e., if u_τ is the local tangential velocity

$$\beta = \beta(\alpha, |u_\tau|).$$

Preliminary analysis has been performed by John, Layton, and Sahin [**80**] and Dunca et al. [**102**], and an appropriate power-law choice of β seems promising to improve the estimation of reattachment points. To emphasize the role of recirculation in real life flows I want to stress that in the simulation of the blood flow within the carotid-bifurcation recirculation has a prominent effect, see Quarteroni [**118**]. The limitation of the Navier law (3) in a boundary-layer theory is that it can well-describe time-averaged flow profiles, but also the information coming from fluctuating quantities in the wall-normal direction can play an important role in triggering separation and detachment. To try to overcome these limitations, Layton [**96**] recognized a particular class of boundary conditions, leading to conditions similar "in spirit" to the so-called *vorticity seeding methods*. We will see more details about this in Section 6. Observe in fact, that in the 2D case identity (5) implies the generation of vorticity at the boundary, proportional to the tangential velocity. In particular, in [**96**] the following boundary conditions are proposed to simulate the boundary effects

$$
\begin{aligned}
&u \cdot \underline{\mathrm{n}} = \alpha^2 \mathcal{G}(x,t) && \text{on } \Gamma \times]0,T], \\
&\frac{\nu}{\alpha U} u_\tau + \underline{\mathcal{T}}(u) = 0 && \text{on } \Gamma \times]0,T],
\end{aligned}
\tag{9}
$$

where $\mathcal{G}$ is a *highly oscillating* function in the time variable (hopefully a random variable in numerical tests), while it may be very smooth in the space variables and should satisfy the natural compatibility condition

$$
\int_\Gamma \mathcal{G}(x,t)\, dS = 0 \qquad \forall\, t \in (0,T),
\tag{10}
$$

which is required by the normal trace of a divergence-free vector field.

This way of reasoning is also similar to the introduction of stochastic fluctuations to simulate the micro-scale effects. A comprehensive introduction to stochastic partial differential equations in fluid mechanics can be found in Monin and Yaglom [**109**], Bensoussan and Temam [**27**], Višik and Fursikov [**144**], and Flandoli [**60**].

3.3 - *Vanishing viscosity limits*

To conclude the introduction I observe that another motivation for the study of slip boundary conditions are the recent advances on the vanishing viscosity limits obtained by Xiao and Xin [**148**] and by Beirão da Veiga and Crispo [**25**, **26**]. The main idea is that under Dirichlet boundary conditions one cannot expect to have convergence (in strong norms) of the solutions to the Navier-Stokes equations, towards those of the Euler equations with the same data, as $\nu \to 0$: There is the *boundary-layer*, characterized by large gradients and this prevents from proving results of convergence, see Constantin [**52**]. It is also well-known that in presence of boundaries one cannot expect convergence (or one can expect convergence only in special settings, see the review in Mazzucato [**108**]) if the boundary-layer has some effect, see also Asano [**6**] and Sammartino and Caflisch [**122**]. In 3D the situation is complicated, also for the fact that we do not know the existence of reasonably weak solutions to the Euler equations. Anyway, in the case of weak solutions u^ν (corresponding to the positive viscosity ν) to the Navier-Stokes equations Kato [**83**] proved that

$$
\begin{gathered}
u^\nu \to u^E \quad \text{in} \quad L^2(\Omega), \text{ uniformly in } t \in [0,T], \quad \text{as} \quad \nu \to 0, \\
\text{if and only if} \\
\nu \int_0^T \|\nabla u^\nu(\tau)\|_{L^2(\Omega^\nu)}\, d\tau \to 0, \quad \text{as} \quad \nu \to 0,
\end{gathered} \tag{11}
$$

where u^E is the solution to the Euler equation and Ω^ν is a boundary strip of width $1/\nu$. Recent results have been also proved by Kelliher [**86, 87**] and Wang, Wang, and Xin [**147**].

On the other hand in the whole space or in the periodic case it is well-known that convergence takes place, see Section 7. Probably the most difficult part is showing strong convergence with respect to the norm of the initial datum, part of the so-called *sharp* convergence result.

The Navier conditions represent an intermediate path between having no boundaries and having a solid boundary (in fact they have been used also for the free-boundary problem). To understand why the Navier conditions allow to simplify the problem, one can observe that if $\beta = 0$ and in the half-space case, one can extend the solution (u, p) to the whole space by extending in a even (with respect to $\{x_3 = 0\}$) way u_1, u_2, and p, while extending in a odd way u_3. With this procedure one obtains a new couple $(\widetilde{u}, \widetilde{p})$ which is a solution the Navier-Stokes equations in $\mathbb{R}^3$. With this trick the problem is reduced to the Cauchy one and this is tractable with the standard tools, see for instance [**81**]. On the other hand, the generic non-flat case is much more difficult, because the reflection technique does not seem to work and one has to introduce new tools also to study the existence of strong solutions to the Navier-Stokes problem. Moreover, the 2D theory is much more complete also in the non-flat case, but some sharp results were still lacking, see Section 7, where recent developments are explained. On the other hand, the approach to the 3D case is particularly new. There have been important advances in the last three years [**148, 25, 26**] and I review some of the new results. In addition, I announce some recent results concerning the well-posedness of the boundary value problem in the non-flat case and I recall that in the general 3D case some questions remain still open, and others are subject of the current research.

4 - The linear stationary problem

In this section I present the basic results concerning the linear and stationary Stokes problem

$$
\begin{aligned}
-\nu\Delta u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{aligned}
\tag{12}
$$

with the slip boundary conditions (3). I am starting with this simplified problem since its understanding represents one of the main building blocks to deal with the Navier-Stokes equations under the slip boundary conditions.

In particular, we will see that existence of weak solutions derives from a fairly standard application of the Lax-Milgram lemma, once the precise functional setting has been introduced. We observe that the property of uniqueness is not straightforward: It depends on the geometry of the domain and on the values of the parameters entering in the boundary conditions. Moreover, the regularity of weak solutions (if the data of the problem are smooth enough) represents a non-trivial result. The $H^2(\Omega)$ regularity of solutions, which represents the counterpart of the Cattabriga [**47**] results for the Dirichlet problem, has been first proved by Solonnikov and Ščadilov [**130**]. Here, I will present a summary of the self-contained proof provided by Beirão da Veiga [**14, 16**], which is particularly important for the estimates on the pressure (I wish also to mention the results in [**13**] for the Dirichlet problem). Based on a subtle treatment of the pressure, the regularity is obtained by means of the usual Nirenberg translations method [**114**]. I also observe that the proof I am presenting does not make any use of the vorticity equation, which is one of the main tools at disposal when dealing with problems without boundary or with the Navier boundary conditions. Next, I will briefly present also some simplifications and extension obtained in the Master Thesis of Borselli [**39**] and by myself [**30**] in the context of very-weak solutions, and which make substantial use of the vorticity equation.

4.1 - *Notation*

Here and in the sequel Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{1,1}$ (Lipschitz-continuous first derivatives). We use the classical Sobolev spaces $W^{k,q}(\Omega)$ with norm $\|\,.\,\|_{k,q}$ and we also write $H^k(\Omega) = W^{k,2}(\Omega)$ (we use standard notation and symbols, see also Adams [**3**] and Brezis [**42**]). If $k = 0$, we write simply $\|\,.\,\|_q := \|\,.\,\|_{0,q}$ and, since the Hilbert case $q = 2$ represent the cornerstone, to simplify the notation we set

$$\|\,.\,\| := \|\,.\,\| = \|\,.\,\|_{L^2}.$$

The number k can also be a real one and we will not distinguish between scalar, vector, or tensor valued function spaces (the reader can extrapolate from the context the correct meaning of the symbols). We also use fractional trace spaces on Γ and we denote by $\|\,.\,\|_{k,p,\Gamma}$ its norm (with $\|\,.\,\|_{s,\Gamma} := \|\,.\,\|_{s,2,\Gamma}$). As usual $W^{k,p}_0(\Omega)$ denotes the closure, with respect to the norm of $W^{k,p}(\Omega)$, of smooth and with compact support functions. Since in the sequel we will deal many times with tangential vector fields, we use the following symbol

$$H^1_\tau := \big\{v \in H^1(\Omega) : (v \cdot \underline{\mathrm{n}})_{|\Gamma} = 0\big\},$$

and we also remark that, as for the standard Poincaré inequality, $\|\nabla v\|$ is a norm in H^1_τ equivalent to the canonical norm $\|v\|_{1,2}$, see e.g. Galdi [**65**] (the same holds also in the non Hilbertian case).

We use also the symbol $(X)'$ to denote the topological dual of the linear space X and $\langle\,.\,,\,.\,\rangle$ denotes the duality paring. Generally we will denote the duality pairing of spaces of functions defined on the boundary Γ by $\langle\,.\,,\,.\,\rangle_\Gamma$.

The pressure in the Stokes system enters only with first derivatives and hence it is determined up to an additive constant. To uniquely determine the pressure generally one imposes a vanishing mean value. In the sequel we denote by $X_\#$ the subspace of functions of X with vanishing mean value.

4.2 - *On a generalized Stokes system*

Contrary to the Dirichlet case, the geometry of the domain is very important when studying boundary values problems with Navier slip boundary conditions. In the next sections we will also understand some of the substantial differences between the flat and non-flat case. Here, we recall some of the restrictions which are needed in order to study the linear case. In particular, the presence of symmetries can give rise problems of non uniqueness. These issues are treated in great detail in [**14**] and we report here the main conclusions. We say that the domain Ω is *axially symmetric* if it can be generated by a revolution around a given axis l_1 (or even around by two orthogonal axes l_1 and l_2). We also assume that the origin of the coordinates belongs to both axes and denote by $\underline{l}_i$ the unit vector with the same direction of l_i. Define the linear space (zero, one, or two dimensional)

$$Z := \{z : \ z = k_i\,\underline{l}_i \times x, \quad k_i \in \mathbb{R}\},$$

with summation over repeated indices. The summation is taken over zero, one or two indices, depending on the symmetries of the domain (obviously if there are no symmetries $Z = \{0\}$).

We say that we are in the special case if Ω is symmetric and $\beta = 0$ and otherwise in the generic case. We also use the following notation

$$H^1_z := \{v \in H^1 : \langle v, z \rangle = 0, \quad \forall z \in Z\}$$

and

$$H^1_{z,\tau} := H^1_z \cap H^1_\tau.$$

Clearly it follows that $H^1_\tau = H^1_{z,\tau} \oplus Z$, while in the generic case $H^1(\Omega) = H^1_z$.

I start considering the following "generalized Stokes" system

$$\begin{aligned} -\nu \Delta u - \mu \nabla(\nabla \cdot u) + \nabla p = f(x) \quad &\text{in } \Omega, \\ \lambda p + \nabla \cdot u = g(x) \quad &\text{in } \Omega, \end{aligned} \tag{13}$$

under the general non homogeneous boundary conditions

$$\begin{aligned} u \cdot \underline{\mathrm{n}} = a(x) \qquad &\text{on } \Gamma, \\ \beta u_\tau + \underline{T}(u) = b(x) \qquad &\text{on } \Gamma, \end{aligned} \tag{14}$$

where $a(x)$ and $b(x)$ are a given scalar field and a given tangential vector field on Γ and the constants μ, ν, and λ satisfy the assumptions

$$\nu > 0, \qquad \mu + \nu > 0 \qquad \text{and} \qquad \lambda \geq 0.$$

The (only apparent) complication coming from the study of a more general system is motivated by the fact that it allows to give a better understanding of the role of the pressure and of boundary data. Moreover, it can be used also to study problems involving compressible fluids and clearly when $\mu = \lambda = g(x) = 0$ one re-obtains the classical Stokes system.

If $\lambda > 0$ then in the "generic case" there is a unique solution to (13)-(14). On the other hand, if $\lambda = 0$ the necessary and sufficient condition for existence is the compatibility condition

$$\int_\Omega g \, dx = \int_\Gamma a \, dS, \tag{15}$$

which derives from the Gauss-Green formula applied to $(13)_2$ with $(14)_1$. The velocity u is uniquely determined, while p is determined up to an additive constant, which is generally set in such a way[8] that

$$\overline{p} := p - \frac{1}{|\Omega|} \int_\Omega p \, dx = 0.$$

[8] In this section the over-lined variable has nothing to do with the LES variables of the previous sections.

On the contrary, in the "special case" there are non-zero solutions to the homogeneous problem, that are rigid motions. In fact, the kernel of the linear problem coincides with the linear space Z. If we take the couple $(z, 0)$ with $z \in Z$, this is a solution of the problem (13)-(14) with $a = f = 0$ and $b = g = 0$, since $\Delta z = 0$, $\nabla \cdot z = 0$, $(z \cdot \underline{\mathrm{n}})_{|\Gamma} = 0$ and $\underline{\mathcal{T}}(z)_{|\Gamma} = 0$. The converse is also true but the proof (see [**14**, App. 1]) requires some care: If (u, p) is a weak solution of the homogeneous problem, then necessarily $u \in Z$ and $p = 0$ (or a constant if $\lambda = 0$). In the special case the solution can be decomposed as $u_0 = u + z$, where u is the particular solution to the non-homogeneous problem such that $\int_\Omega u\, z\, dx = 0$ for all $z \in Z$. Moreover, the function u (which is unique) exists if and only if the compatibility condition

$$\int_\Omega f \cdot (\underline{l}_i \times x)\, dx = -\nu \int_\Gamma b \cdot (\underline{l}_i \times x)\, dS \tag{16}$$

is satisfied.

Remark 4.1. *The proof of the existence of weak solutions I am presenting is based on some kind of* *<u>artificial compressibility</u>* *method. In fact, the term λp has been added to the divergence equation. This tool will greatly simplify the problem for two reasons:*

1. *The resulting variational problem is coercive over the set of all tangential vector fields, and not only on the subspace of divergence-free functions;*
2. *The use of functions without prescribed divergence allows us to use the same as test functions. This will be of particular interest for proving higher regularity, since the classical Nirenberg translation method* [**114**] *can be used directly.*

Obviously the price to be paid is that the estimates (*cf.* (18)) *we obtain directly are not independent of λ. One has to use in a clever way some well-known inequalities concerning a function and its derivatives* (*as those proved by Duvaut and Lions* [**58**]) *in order to recover results independent of $\lambda > 0$.*

Remark 4.2. *The use of function spaces without a constraint on the divergence is particularly important also in view of the numerical analysis and implementation of numerical schemes for the Stokes problem. The reader can find details on the numerical implementation and additional* (*with respect to the Laplacian, which seems apparently similar*) *problems involving the Stokes system, e.g., in Girault and Raviart* [**72**] *and Brezzi and Fortin* [**43**].

The first result I am recalling is the following.

Theorem 4.1. *Assume that*

$$f \in (H^1_\tau)', \quad g \in L^2(\Omega), \quad a \in H^{1/2}(\Gamma) \quad \text{and} \quad b \in H^{-1/2}(\Gamma),$$

where b is tangential to Γ. In the special case (*i.e., if Ω is symmetric and if $\beta = 0$*) *assume also the* (*necessary*) *compatibility condition* (16). *One has the following results:*

(a) *If $\lambda > 0$ the problem* (13)-(14) *has a unique weak solution $(u, p) \in H^1_z \times L^2(\Omega)$. Moreover, the following estimate in terms of the data holds true:*

$$\|u\|_1^2 + \lambda\|p\|^2 + \|\overline{p}\| \leq c([f]_{-1}^2 + \|a\|^2_{1/2,\Gamma} + \|b\|^2_{-1/2,\Gamma}) + \frac{c}{\lambda}(\|g\|^2 + \|a\|^2_{1/2,\Gamma}).$$

(b) *If $\lambda \geq 0$ and* (15) *holds, the problem* (13)-(14) *has a unique weak solution $(u, p) \in H^1_z \times L^2_\#(\Omega)$, where $L^2_\#(\Omega) := \{f \in L^2(\Omega) : \int_\Omega f\, dx = 0\}$. If $\lambda = 0$ the pressure p is unique up to a constant. Moreover, it holds*

$$\|u\|_1^2 + (1 + \lambda)\|p\|^2 \leq c([f]_{-1}^2 + \|g\|^2 + \|a\|^2_{1/2,\Gamma} + \|b\|^2_{-1/2,\Gamma}).$$

(c) *In the special case* (*hence $Z \neq \{0\}$*) *the general solution is given by $(u + z, p)$, where (u, p) is the particular solution described in points* (a) *or* (b) (*hence u is orthogonal to Z*) *and z is an arbitrary element of Z.*

Here, the symbol $[f]_{-1}$ denotes the norm of f as an element of $(H^1_\tau)'$.

Proof. I give a sketch of the proof of the most important steps, while complete details can be found in [**14**, Sec. 2]. To write the weak formulation we formally multiply (13) by a smooth vector field ϕ tangential to the boundary and we observe that (u, p) is a solution if and only if

$$B(u, \phi) - \int_\Omega p\, \nabla \cdot \phi\, dx = \int_\Omega f\, \phi\, dx + \int_\Gamma (b - \beta\, u)\, \phi\, dS,$$

where

$$\begin{aligned} B(u, \phi) &:= \int_\Omega \Big[\nu\, \nabla^s u \nabla^s \phi + (\mu - \nu)(\nabla \cdot u)(\nabla \cdot \phi)\Big]\, dx \\ &= -\int_\Omega \big[\nu\, \Delta u + \mu\, \nabla(\nabla \cdot u)\big]\phi + \int_\Gamma \underline{T}(u)\, \phi\, dS \end{aligned}$$

and we recall that $\nabla^s f := (1/2)(\nabla f + \nabla f^T)$. In order to impose the value of the normal component of the velocity, we reduce the problem to an homogeneous one by taking a vector w such that

$$\begin{aligned}
\nabla \cdot w &= g && \text{in } \Omega,\\
w \cdot \underline{\mathrm{n}} &= 0 && \text{on } \Gamma,\\
\|w\|_1 &\leq (\|a\|_{1/2} + \|g\|).
\end{aligned}$$

This can be done in several (rather standard) ways[9], for instance by solving the divergence equation by the Bogovskiĭ formula or as in [**14**, § 8]. Hence, by setting

$$u = w + v,$$

we obtain that $V = (v, p) \in H^1_{\tau,z} \times L^2$ must satisfy the following equality

$$a_\lambda(V, \Phi) = L(\Phi), \tag{17}$$

for all $\Phi = (\phi, \psi) \in H^1_{\tau,z} \times L^2$, where

$$\begin{aligned}
a_\lambda(V, \Phi) &:= B(v, \phi) - \langle p, \nabla \cdot \phi \rangle + \beta \langle v, \phi \rangle_\Gamma + \lambda \langle p, \psi \rangle + \langle \nabla \cdot v, \psi \rangle,\\
L(\Phi) &:= -B(w, \phi) + \langle f, \phi \rangle - \beta \langle w, \phi \rangle_\Gamma + \langle b, \phi \rangle_\Gamma + \langle g, \psi \rangle - \langle \nabla \cdot w, \psi \rangle.
\end{aligned}$$

To study the variational problem (17) we need the following result.

Lemma 4.1. *For all $\lambda > 0$ the bilinear form $a_\lambda(\,.\,,\,.\,)$ is continuous and coercive in $(H^1_{\tau,z} \times L^2(\Omega))^2$ and the linear operator L is continuous on $H^1_{\tau,z} \times L^2(\Omega)$.*

Proof. The continuity of L is proved by a direct computation, since by using the explicit expression of $B(\,.\,,\,.\,)$ one obtains directly

$$|L(\Phi)| \leq c(\|a\|_{1/2,\Gamma} + [f]_{-1} + \|b\|_{-1/2,\Gamma})\|\nabla \phi\| + c\|a\|_{1/2,\Gamma}\|\psi\| + \|g\|\|\psi\|,$$

and in the case that (15) holds, one can also prove that

$$|L(\Phi)| \leq c([f]_{-1} + \|g\| + \|a\|_{1/2,\Gamma} + \|b\|_{-1/2,\Gamma})\|\nabla \phi\|.$$

The verification of the other property concerning the bilinear form is slightly more complicated. We first observe that the following vector identity holds true: Assume that $v \in H^1_\tau$, then

$$B(v, v) = \nu\, \|\nabla v\|^2 + \mu\, \|\nabla \cdot v\|^2 - \nu \int_\Gamma \partial_i n_k v_i v_k \, dS.$$

[9] Observe that when we do not assume (15) it is enough to take any lifting from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$. Condition (15) is needed in the case $\lambda = 0$, or if we want to pass to the limit as λ vanishes. In the special case one has also to construct w such that it belongs to H^1_z, but this can be done simply projecting the function w previously constructed.

The proof is obtained with an integration by parts. First one gets the same equality, but with the boundary term $\nu \int_{\Gamma} \partial_i v_k v_i n_k \, dS$. Next, one can smoothly extend the normal unit vector field $\underline{n}$ to a small neighborhood of Γ (see for instance Nečas [**113**] for Lipschitz prolongation concerning $C^{0,1}$-boundaries) and, by observing that v is tangential and that $v \cdot \underline{n}$ vanishes identically[10] on Γ, we get

$$0 = (v \cdot \nabla)(v \cdot \underline{n}) = v_i \partial_i (v_k n_k) = v_i (\partial_i v_k) n_k + v_i v_k (\partial_i n_k) \quad \text{on } \Gamma.$$

With this result we can reduce the order of the boundary term. Hence it follows that

$$\exists\, c_0 = c_0(\Omega): \quad B(v,v) \geq \overline{\nu}\, \|\nabla v\|^2 - c_0 \nu\, \|v\|_{\Gamma}^2,$$

with

$$\overline{\nu} = \begin{cases} \nu, & \text{if } \mu \geq 0, \\ \nu + \mu, & \text{if } -\nu < \mu < 0. \end{cases}$$

Next, we observe that by a rather standard compactness argument (see [**130**]) the following inequality holds true

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N}: \quad \|v\|_{\Gamma} \leq \varepsilon \|\nabla v\| + N B(v,v) \qquad \forall\, v \in H^1_{z,\tau}.$$

Consequently, it follows that

$$B(v,v) + c_0 \nu\, \|v\|_{\Gamma}^2 \geq \overline{\nu}\, \|\nabla v\|^2,$$

and the left-hand side is a norm on $H^1_{\tau,z}$. Having at disposal these tools one can show (some work has to be done, cf. [**14**, p. 1090] for further details) that *for each fixed* $\lambda > 0$

$$a_\lambda(V,V) \geq c_1 \|\nabla v\|^2 + \lambda \|p\|^2,$$

and this proves the coercivity of the bilinear form a_λ, for any given λ. □

By using the Lax-Milgram lemma one obtains that, for each fixed $\lambda > 0$ there exists a unique $V_\lambda = (v_\lambda, p_\lambda) \in H^1_{\tau,z} \times L^2(\Omega)$ which satisfies (17). By using (v_λ, p_λ) as test function we also get

$$\|\nabla v_\lambda\|^2 + \lambda \|p_\lambda\|^2 \leq c\Big[[f]_{-1}^2 + \|a\|_{1/2,\Gamma}^2 + \|b\|_{-1/2,\Gamma}^2 \Big] + \frac{c}{\lambda}\Big[\|a\|_{1/2,\Gamma}^2 + \|g\|^2 \Big], \tag{18}$$

and the estimate is depending of λ. It is clear that if $a = g = 0$, i.e., the classical incompressible Stokes system, one can obtain directly an information on the velocity

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[10] We will use several times in the sequel such identities.

v_λ, since the norm of this therm is not depending on λ. In particular since

$$a_\lambda(V_\lambda, \Phi) = L(\Phi)$$

and the problem is linear it is easy to see that there exists a sequence $\lambda_n \to 0$ and a function $v \in H^1_{\tau,z}$ such that $v_{\lambda_n} \rightharpoonup v$, which is solution of the problem with $\lambda = 0$. In this process, which is nevertheless standard, one is loosing all information on the pressure, which can be recovered by De Rham theorem in some weak (distributional) sense.

It is now clear that passing to the limit keeping also the information on the pressure will greatly improve the result. Clearly, in the non-homogeneous case the situation is even more difficult since this argument does not work even for the velocity, this is why we study the non-homogeneous case in order to focus on a possible variational treatment of the constraint on the divergence.

In order to prove the estimate in the statement (a) of Theorem 4.1, we observe that

$$\int_\Omega p_\lambda \nabla \cdot \phi \, dx = B(v_\lambda, \phi) - \langle f, \phi \rangle,$$

hence

$$\nabla p_\lambda = f + \nu \Delta v_\lambda + \mu \nabla(\nabla \cdot v_\lambda), \quad \text{in the sense of } H^{-1} := (H^1_0(\Omega))'.$$

This implies that

$$\|\nabla p_\lambda\|_{-1} \leq [f]_{-1} + (\nu + |\mu|)(\|\nabla v_\lambda\| + \|w\|_1),$$

and now we use in a crucial way the following result.

Proposition 4.1. *Let be given $p \in L^2(\Omega)$. There exists a constant c depending only on Ω such that*

$$\|\overline{p}\| \leq c\|\nabla p\|_{-1}, \qquad \forall\, p \in L^2(\Omega). \tag{19}$$

The classical proof of this result can be found in Duvaut and Lions [**58**] and Tartar [**134**]. A simplified and self-contained proof can be found also in Beirão da Veiga [**14, 16**]. See also Bourgain and Brezis [**40**]. Observe also that this is the "easy" version of this inequality, since we need it for a function we know a-priori to be in $L^2(\Omega)$. Stronger results with the same estimate for distributions with first derivatives in $L^2(\Omega)$ are known, see Nečas [**112**].

By using the estimate we know on v_λ we can add (19) to both sides of (18) to get the desired estimate. Moreover, if the compatibility condition (15) holds true (note that this is not needed if $\lambda \neq 0$) we can prove in the same way that

$$\|\nabla v_\lambda\|^2 + \lambda\|p_\lambda\|^2 \leq c\Big[\,[f]^2_{-1} + \|g\|^2 + \|a\|^2_{1/2,\Gamma} + \|b\|^2_{-1/2,\Gamma}\Big]$$

and the estimate of statement (b) follows by adding again to both sides the estimate on p in $L^2(\Omega)$ coming from (19). Note also, that $\overline{p} = p$ since, by using as test function the couple $(\phi, \psi) = (0, 1)$, one gets

$$\lambda \int_\Omega p\,dx = \int_\Omega g\,dx - \int_\Gamma a\,dS = 0.$$

Once one has proved those estimates, one can also handle the case $\lambda = 0$. Being the norm of V_λ independent of λ one can take sub-sequences $\lambda_k \to 0^+$ such that $V_{\lambda_k} = (v_{\lambda_k}, p_{\lambda_k})$ weakly converges to $V = (v, p) \in H^1_{\tau,z} \times L^2(\Omega)$. By passing to the limit it follows that

$$a_0(V, \Phi) = L(\Phi).$$

The limit $V = (u, p)$ is unique because the solution of the Stokes problem with $\lambda = 0$ is unique. In fact, the energy estimate gives $B(u, u) = 0$, hence $u = 0$. Consequently, $\int_\Omega p \nabla \cdot \phi\,dx = 0$, hence $\nabla p = 0$ in H^{-1}, but being the mean value of p zero, it follows that $p = 0$. □

In the case of more regular data one can prove the following result.

Theorem 4.2. *Assume that Γ is of class $C^{2,1}$. Let λ and the data f, g, a, and b satisfy the conditions assumed in one of the cases considered in Theorem* 4.1, *and let (u, p) be the corresponding weak solution. Assume moreover that*

$$f \in L^2(\Omega), \quad g \in H^1(\Omega), \quad a \in H^{3/2}(\Gamma) \quad \textit{and} \quad b \in H^{1/2}(\Gamma).$$

Then the couple (u, p) belongs to $H^2(\Omega) \times H^1(\Omega)$. Moreover, in case (b), *it holds*

$$\|u\|_2^2 + (1 + \lambda)\|p\|_1^2 \le c(\|f\|^2 + \|g\|_1^2 + \|a\|_{3/2,\Gamma}^2 + \|b\|_{1/2,\Gamma}^2),$$

where c is independent of λ. In case (a), *the above estimate is satisfied by replacing c by $c(\lambda)$, where $c(\lambda)$ tends to infinity as λ goes to zero. In the special case, the above estimates are satisfied by the particular solution $u \in H^1_{\tau,z}$, i.e., by the solution u for which $\langle u, z\rangle = 0$. Clearly, the solutions $u_0 = u + z$ are regular, as well.*

Proof. I do not give the proof of this result which is rather technical even if it is elementary in the approach. I would like just to give the main idea in the simplified case of the domain with a flat boundary, since it is illuminating on the techniques one has to use also in general cases. Nevertheless, the proof in a general domain requires several tricks and a very complete and detailed proof is given in [**14**, § 3-6]. Observe also that obtaining a bound in $L^2(\Omega)$ for the second order derivatives of the velocity and for the first order derivatives of the pressure is the hardest part in the study of the Stokes problem. With this at disposal, and if the data are smoother, one can prove further regularity. In particular, the estimates obtained concern the so-called regularity *up-to-the boundary*, which is a result technically much harder than local or *interior* results.

The first step is rather standard and consists in the construction of a weak solution with an arbitrary support. This is clearly needed in order to localize and obtain the local estimates in each open set of a finite covering of $\overline{\Omega}$. One needs to derive the variational formulation satisfied by $(\theta v, \theta p)$, where θ is a function of class $C_0^{1,1}(\mathbb{R}^3)$. In fact $(\theta v, \theta p)$ will turn out to be a weak solution in Ω of a problem with modified data, but the modified data keep the same regularity of the data of the problem. It is clear that in the interior of the support of θ, if $(\theta v, \theta p)$ is regular, then also (v, p) is regular. This is accomplished if we are able, for each $x_0 \in \overline{\Omega}$, to prove regularity of $(\theta v, \theta p)$ for some θ with support containing x_0. If x_0 belongs to the interior of Ω and by choosing θ with support small enough (not touching Γ) one can reduce the problem to the interior one.

The case in which $x_0 \in \Gamma$ is more delicate. In this case one can write, in a small neighborhood of x_0, the boundary as the graph of a smooth function; next one can flatten the boundary in the usual way by means of an invertible smooth transformation. Let us write $(x_1, x_2, x_3) = (x', x_3)$. Then, there is a positive real a and a real function $x_3 = h(x')$, of class C^3 defined on the ball $\{x \in \mathbb{R}^2 : |x'| < a\}$, such that the points x for which $x_3 = h(x')$ belong to Γ, the points such that $h(x') < x_3 < a + h(x')$ belong to Ω, and the points x such that $-a + h(x') < x_3 < h(x')$ belong to the complementary of Ω. The change of variables which flattens the boundary is defined by $y = Tx$

$$(y_1, y_2, y_3) := (x_1, x_2, x_3 - h(x'))$$

and we set $\widetilde{f}(y) := f(T^{-1}(y))$. What is different from the usual Dirichlet problem is that we do not have to preserve the zero value of the velocity on the boundary, but one need to send tangential vector fields on Γ into tangential vector fields on $\{x_3 = 0\}$. If we do not take care of this property we cannot have consistent results: We need to use a co-variant transformation which sends *tangential* vector fields on Γ into vector fields on $x_3 = 0$ such that the third component vanishes. Such a change of coordinates is given by

$$\begin{aligned} &\widetilde{v}_j = v_j, \qquad \text{for } j = 1, 2, \\ &\widetilde{v}_3(y) = v_3 - (\partial_1 h) v_1 - (\partial_2 h) v_2. \end{aligned}$$

Then one can use the translations in the horizontal directions, while some care is needed to obtain information in the directions which are orthogonal to the boundary.

In order to show one main computation, which is crucial for the proof, I consider a much simpler problem, but which nevertheless furnishes the main ideas in order to treat the previous case. I consider the flat case, where one has not to make the change of coordinates[11] and in particular I consider the problem in $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : \ x_3 > 0\}$. For full details see [**16**].

We apply the translation method and we can do this in the horizontal directions. Let $j = 1, 2$ be fixed and let $h \neq 0$. Define the translation operator by

$$\tau_h z(x) = z(x + h\, e_j),$$

where e_j is the j-element of the canonical basis of $\mathbb{R}^3$ defined by $[e_j]_i = \delta_{ij}$. In the variational formulation set $\phi = \tau_{-h}\phi$ and $\psi = \tau_{-h}\psi$. Take the difference with the variational formulation with test function $\Phi = (\phi, \psi)$ and use the change-of-variables identity

$$\int_\Omega \tau_h v\, z\, dx = \int_\Omega v\, \tau_{-h} z\, dx.$$

It follows that

$$a\Big(\frac{\tau_h V - V}{h}, \Phi\Big) = \Big\langle \frac{\tau_h L - L}{h}, \Phi \Big\rangle,$$

and by using the properties of the bilinear form $a(\,.\,,\,.\,)$ and of L we obtain

$$\begin{aligned}\overline{\nu}^2 \Big\| \frac{\nabla(\tau_h v - v)}{h} \Big\|^2 &+ \frac{\overline{\nu}^2}{(\nu + |\mu|)^2} \Big\| \frac{\tau_h p - p}{h} \Big\|^2 \\ &\leq c\Big[(\nu + |\mu|)\|\partial_j g\| + \|\partial_j a\|_{1/2,\Gamma} + \Big(1 + \frac{\overline{\nu}}{\nu + |\mu|}\Big)\|f\| + [b]_{1/2,\Gamma}\Big]^2.\end{aligned}$$

In particular this follows since $\|\tau_h g - g\| \leq h\|\partial_j g\|$ and also using similar estimates for a. The regularity of the data and classical (cf. [**42**]) results on Sobolev spaces concerning the equivalence between Sobolev norms and L^2-norms of differential quotients, imply that

$$\begin{aligned}\overline{\nu}\|\nabla_h^2 v\|^2 &+ \frac{\overline{\nu}^2}{(\nu + |\mu|)^2}\|\nabla_h p\|^2 \\ &\leq c\Big[(\nu + |\mu|)\|\nabla g\| + \|a\|_{3/2,\Gamma} + \Big(1 + \frac{\overline{\nu}}{\nu + |\mu|}\Big)\|f\| + \|b\|_{1/2,\Gamma}\Big]^2.\end{aligned}$$

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[11] In the case $\Omega = \mathbb{R}^3_+$ there is also a small technical complications due to the fact that the problem does not have a variational formulation in H^1_τ, but only in the space $\widehat{H}^1_\tau$, which is the closure with respect to the semi-norm $\|\nabla v\|$. Functions in $\widehat{H}^1_\tau$ do not belong to $L^2(\mathbb{R}^3_+)$, but just lo $L^6(\mathbb{R}^3_+)$. At this point this is inessential, since we want to study *second order* derivatives of the velocity.

Here ∇_h^2 denotes the second order derivatives, except for ∂_3^2, while ∇_h denotes the horizontal gradient, i.e., first order derivatives except ∂_3. To recover the regularity of the remaining derivatives, we cannot use the elementary tool that works for the Poisson equation $-\Delta u = f$: In that case one simply writes by comparison

$$\partial_3^2 u = \partial_1^2 u + \partial_2^2 u - f,$$

and it is possible to estimate $\partial_3^2 u$ in terms of quantities form the right-hand side which are in $L^2(\Omega)$ by the previous step.

If we use exactly the same tool for the Stokes problem, we will have in the right-hand side also the term $\partial_3 p$ for which we do not have any estimate, yet. To this end we consider 2×2 linear system obtained by taking the 3rd scalar equation together with the equation obtained by differentiation of the 4th equation with respect to x_3

$$\begin{aligned} -(\nu + \mu)\, \partial_3^2 u_3 + \partial_3 p &= f_3 + \nu \Delta_h u_3 + \mu\, \partial_3(\nabla_h \cdot u_h) := \mathcal{F}, \\ \partial_3^2 u_3 + \lambda\, \partial_3 p &= \partial_3 g - \partial_3(\nabla_h \cdot u_h) := \mathcal{G}, \end{aligned}$$

where Δ_h is the Laplacean with respect to the first two variables, and obviously $u_h := (u_1, u_2)$. This algebraic linear system in the unknown $(\partial_3^2 u_3, \partial_3 p)$ has always a unique solutions since its determinant is

$$-1 - \lambda(\nu + \mu) \leq -1.$$

The previous result, obtained with the translation method, shows that both $\mathcal{F}$ and $\mathcal{G}$ belong to $L^2(\Omega)$. We can now solve the system with respect to the unknowns $\partial_3^2 u_3$ and $\partial_3 p$ obtaining them as a linear (continuous) combination of $\mathcal{F}$, $\mathcal{G} \in L^2(\Omega)$, hence showing the desired result.

This does not end yet the proof, since we are still missing the regularity of the terms

$$\partial_3^2 u_1 \quad \text{and} \quad \partial_3^2 u_2.$$

To obtain such terms we take the first two equations and we observe that, by comparison,

$$\nu \partial_3^2 u_j = -\nu \sum_{k=1}^{2} \partial_k^2 u_j + \mu\, \partial_j \left(\sum_{k=1}^{2} \partial_k u_k \right) + \partial_j p - f_j, \qquad j = 1, 2.$$

Since previous results proved that the right-hand side belongs to $L^2(\Omega)$, this finally shows that all second order derivatives of the velocity and all first derivatives of the pressure are square summable. The precise estimate for these quantities in terms of the data follows from the proof, and the details are left to the reader. □

4.3 - *Further remarks on strong solutions*

We give now an alternative proof of the regularity result, at least in the flat case. The proof here is given in the case $\beta = \lambda = g = 0$ and it is taken from results of Borselli in her thesis [**39**] and by myself [**30**]. The proof is based on the application of the curl operator and on standard regularity results for the Poisson problem.[12] The main advantage of the technique we introduce is that one can study also the non Hilbertian case. In particular, estimates in $L^q(\Omega)$, $q \neq 2$ for second derivatives of u and first derivatives of p can be obtained, by assuming as starting point just classical the $W^{2,q}$-regularity for the Poisson equation, under different boundary conditions, see Bers, John, and Schechter [**28**]. In particular, in this case one can avoid the complicated technique of singular integrals needed to deal with the Dirichlet problems as in [**47**]. See also Simader and Sohr [**126**] for another approach to the Dirichlet problem for the Laplacean and the Stokes operators.

Here to avoid technicalities at infinity, we assume that the domain instead of $\Omega = \mathbb{R}^3_+$ is the "cube" $\Omega =]-1,1[^2 \times]0,1[$, with the flat boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where

$$\Gamma_i = \Big\{ x \in \mathbb{R}^3 : |x_1|, |x_2| < 1,\ x_3 = i \Big\}, \qquad \text{for} \quad i = 0, 1,$$

while the problem is assumed periodic (with period 2) in the other two directions. We also define $x' := (x_1, x_2)$ and we call "x'-periodic" any function that is periodic with period 2 in both x_1 and x_2. We impose the following Navier's (slip without friction) boundary conditions on Γ:

$$\begin{cases} -\partial_3 u^1 = a^1 & \text{on } \Gamma_0 \qquad \partial_3 u^1 = 0 \quad \text{on } \Gamma_1, \\ -\partial_3 u^2 = a^2 & \text{on } \Gamma_0 \qquad \partial_3 u^2 = 0 \quad \text{on } \Gamma_1, \\ -u^3 = b & \text{on } \Gamma_0 \qquad u^3 = 0 \quad \text{on } \Gamma_1, \end{cases} \tag{20}$$

where a^i and b are given functions. The Navier's boundary conditions become the above ones (20) since the outer unit vector is $n = (0, 0, (-1)^{i+1})$ on Γ_i, the domain is flat, and $\beta = 0$. For simplicity we set homogeneous boundary conditions on Γ_1.

Remark 4.3. *In this case there are problems of possible non-uniqueness, even if there is no axial symmetry. In fact, if (u, p) is a solution also $(u + u_0, p + p_0)$ with $u_0 := (c_1, c_2, 0,)$ and $p_0 := c_3$, for $c_i \in \mathbb{R}$ is a solution. Hence, to have uniqueness, we fix the mean value of u^1, u^2, and p equal to zero. In this functional setting the usual Poincaré-Sobolev inequalities still hold true.*

[12] Related results can be found also in Bae and Jin [**7**], by a different method, based on suitable reflections of u and p.

Here the subscript $''_{(\#)}''$ denotes vector fields with the first two components with vanishing mean value, while $''_{\#}''$ denotes scalar fields with vanishing mean value.

Theorem 4.3. *Let (u,p) be a weak solution of the Stokes problem with the boundary conditions* (4). *Let be given, for $1<q<+\infty$, $f\in L^q(\Omega)$, $a^i\in W^{1-1/q,q}(\Gamma_0)$, and $b\in W^{2-1/q,q}(\Gamma_0)$, such that $\int_\Omega f^i+\int_{\Gamma_0} a^i=0$, for $i=1,2$, and $\int_{\Gamma_0} b\,dS=0$. Then, there exists a unique solution of the Stokes problem* (12)-(20) *such that $(u,p)\in W^{2,q}_{(\#)}(\Omega)\times W^{1,q}_{\#}(\Omega)$ and*

$$\|u\|_{2,q}+\|p\|_{1,q}\leq C\|(f,a,b)\|_{0,q},$$

for some $C=C(q,\Omega,\nu)>0$, where we set

$$\|(f,a,b)\|_{0,q}:=\|f\|_{L^q(\Omega)}+\sum_{i=1}^{2}\|a^i\|_{W^{1-1/q,q}(\Gamma_0)}+\|b\|_{W^{2-1/q,q}(\Gamma_0)}.$$

We recall that, while the $L^2(\Omega)$ results can be obtained by the translation method, the L^q-estimates need probably in a substantial way the use of potential theory and of Green functions. In the case of slip boundary conditions one can prove in an elementary way the L^q-estimates by using as starting block the same results for the Laplace equation. The interest for solutions with derivatives in $L^q(\Omega)$ is motivated by the fact that in exterior domains the $L^2(\Omega)$ setting is not satisfactory in many situations, see Galdi [**65, 66**]. Another situation in which one needs to deal naturally with L^q-solutions is the problem of existence of *very-weak solutions*. These solutions satisfy the equation with all derivatives transferred on the test function, by suitable integration by parts. Hence, one can speak of L^q-very-weak solutions of the Stokes and Navier-Stokes equations. For reasons due to the nonlinear term and restrictions deriving from Sobolev embeddings, for the (non-linear) Navier-Stokes 3D equations one is forced to consider very-weak solutions in $L^q(\Omega)$ with $q\geq 3$. Recent results concerning very-weak solutions with different boundary conditions in the stationary (linear and non-linear case) are those by Galdi, Simader, and Sohr [**69**], Kim [88], and myself [**30**].

Proof. [Proof of Theorem 4.3] We know in advance that a unique weak solution of the Stokes problem exists, and we need just to prove that it has stronger regularity properties. If one wish can justify rigorously all calculations by using the translation method, or by approximating the solution with Galerkin approximate functions (u_n,p_n) and then showing estimates uniform in n.

By taking the curl of (12) we write the equation satisfied by ω and, since $\operatorname{curl}(\nabla p)=0$, then

$$-\nu\Delta\omega=\operatorname{curl} f$$

that is a (vector) Poisson problem. By using the same calculations seen in Section 2.2 we get

$$\begin{cases} \omega_1 = \partial_2 u_3 - \partial_3 u_2 = \alpha_2 & \text{on } \Gamma_0, \\ \omega_2 = \partial_3 u_1 - \partial_1 u_3 = -\alpha_1 & \text{on } \Gamma_0, \\ \partial_3 \omega_3 = -\partial_1 \omega_1 - \partial_2 \omega_2 = \partial_2 \alpha_1 - \partial_1 \alpha_2 & \text{on } \Gamma_0, \end{cases}$$

where $\partial_2 \alpha_1 - \partial_1 \alpha_2 \in W^{-1/q}(\Gamma_0)$. The three components of the vorticity can be treated separately and hence we have three *uncoupled* Poisson problems. The first two with Dirichlet boundary conditions and the third one with Neumann conditions.[13] It is easy to see that the compatibility conditions for the Neumann problem are automatically satisfied and with the standard regularity results for the Poisson problem [**28**] we obtain

$$\omega \in W^{1,q}(\Omega).$$

Next, we use the identity,

$$\operatorname{curl} \operatorname{curl} u = -\Delta u + \nabla(\nabla \cdot u),$$

and since $\nabla \cdot u = 0$ we get the equality that has a very relevant role in the theory of incompressible fluids.

$$-\Delta u = \operatorname{curl} \omega. \tag{21}$$

In the whole space this is the main tool to recover regularity on the gradient of u by that of the vorticity. In fact, in $\mathbb{R}^3$ the Biot-Savart law implies

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\nabla \frac{1}{|y|} \right) \times \omega(x+y)\, dy.$$

One has an explicit expression of u in terms of ω via the integral representation for the solution of the Poisson equation.

In our setting the Poisson equation for u is supplemented by the boundary conditions (20) and we have other three uncoupled Poisson equations: We have two Neumann problems (for u_1 and u_2) and a Dirichlet problem for u_3. Again the standard regularity for the Laplace equation gives

$$u \in W^{2,q}(\Omega).$$

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[13] This observation have bee used in [**17**] to solve the (vector) equation (21) by means of the Green functions in the half-space, under different boundary conditions. Apart technical complications due to the flattening of the boundary, this is the technique used to study problem (32) as we will see later on.

Finally, the regularity of the pressure is obtained by comparison, since

$$\nabla p = \nu \Delta u + f \in L^q(\Omega),$$

ending the proof. □

Note added in proof. I have been recently informed that similar results, for the Stokes problem with Navier boundary conditions have been proved by C. Amrouche and his coworkers. Their proof is based on a different and more general approach which uses the theory of vector potentials.

5 - The time-evolution problem

In this section I recall some results concerning the time-evolution problem. Both the linear and the non-linear cases present more or less the same difficulties (in terms of existence and uniqueness) of the corresponding problem with Dirichlet boundary conditions. In the previous section we considered solution with prescribed divergence $g \in L^2(\Omega)$. This was functional to a better understanding of the proofs, but here we come back to the setting we have in mind of *incompressible* fluids. Hence, from now on all solutions will be divergence-free and we introduce a couple of spaces to better study the problem. The space H is the usual one in the treatment of the Navier-Stokes equations:

$$H := \{ v \in L^2(\Omega) : \ \nabla \cdot v = 0 \text{ and } (v \cdot \underline{\mathrm{n}})_{|\Gamma} = 0 \},$$

where the divergence is in the sense of distribution and the normal trace is intended in the sense of $H^{-1/2}(\Gamma)$. In addition, if we set

$$G := \{ \nabla p : p \in H^1_{\#}(\Omega) \},$$

then $L^2(\Omega) = H \oplus G$.

The second space we need is slightly different from the usual one, since in the no-slip case it is a subspace of $H^1_0(\Omega)$. Here we define

$$V := \{ v \in H^1_\tau(\Omega) : \ \nabla \cdot v = 0 \},$$

with norm $\|v\|_V = \|\nabla v\|$.

In a standard way we can define the linear operator $A : \ V \to V'$ by the identity

$$\langle Au, v \rangle = a(u, v) := \nu \int_\Omega \nabla^s u \nabla^s v + \beta < u, v >_\Gamma, \qquad \forall\, u,\, v \in V.$$

The results of the previous section show that A is an homeomorphism from V onto V'

and we can consider the restriction (as an unbounded operator) to H, with domain

$$D(A) = \{v \in V : Av \in H\}.$$

Moreover, we have also the following characterization of the domain of A

$$D(A) = \{v \in H^2(\Omega) : \nabla \cdot v = 0, \ (v \cdot \underline{\mathrm{n}})_{|\Gamma} = 0, \ \text{and } \beta\, v_\tau + \underline{\mathcal{T}}(v)_{|\Gamma} = 0\},$$

and that the operator A satisfies

$$Au = -P\Delta u,$$

where P is the projection operator $L^2(\Omega) \to H$. The regularity results for the stationary problem can be reformulated as follows: if $f \in H$ (and $g = 0$, $a = 0$), then the unique weak solution (u, p) of (13)-(14) belongs to $D(A) \times H^1_{\#}(\Omega)$. Moreover, it also turns out that A is a self-adjoint maximal monotone operator. See also [**42**] for a review of maximal monotone operators and the application to evolution problems. Hence, we have the following result, see [**18**].

Proposition 5.1. *The operator A is maximal monotone and self-adjoint in H. In particular, it is also generator of an analytical* (*and compact*) *semigroup of contractions in H.*

5.1 - *The linear problem*

With the results proved in the previous section one can successfully treat the time-dependent problem

$$\begin{aligned} u_t - \nu\Delta u + \nabla p &= f && \text{in }]0,T] \times \Omega, \\ \nabla \cdot u &= 0 && \text{in }]0,T] \times \Omega, \\ u(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned} \tag{22}$$

under the slip boundary conditions

$$\begin{aligned} u \cdot \underline{\mathrm{n}} &= 0 \quad \text{on } \Gamma \times]0,T], \\ \beta\, u_\tau + \underline{\mathcal{T}}(u) &= 0 \quad \text{on } \Gamma \times]0,T]. \end{aligned}$$

Definition 5.1. *We say that a couple* (u, p) *is a weak solution of* (22) *with the slip conditions* (3) *if $u(t) \in V$ for a.e. $t \in (0,T)$, if $u_0 = u(0)$, and if*

$$\frac{d}{dt}(u(t), v) + \nu \int_\Omega \nabla^s u \cdot \nabla^s v \, dx + \beta \langle u, v \rangle_\Gamma = \langle f, v \rangle,$$

for all $v \in V$, in the sense of $\mathcal{D}'(]0,T[)$.

The time-evolution boundary value problem of (22) with slip boundary conditions can be also written as an abstract (functional) equation as follows

$$\begin{aligned} \frac{du}{dt} + Au = f \qquad & \text{in } H, \\ u(0) = u_0. & \end{aligned}$$

The properties of A imply in a standard way that if $u_0 \in D(A)$ and $f \in C^1(0,T;H)$, then there is a unique solution $u \in C(0,T;D(A)) \cap C^1(0,T;H)$ of the initial boundary-value problem. The usual techniques allow us to study also the nonlinear problem

$$\begin{aligned} \frac{du}{dt} + Au + P[(u\cdot\nabla)\,u] = f \qquad & \text{in } H, \\ u(0) = u_0, & \end{aligned}$$

and to prove existence of strong solutions for small data.

Theorem 5.1. *Let $f \in L^2(0,+\infty;L^2(\Omega))$ and $u_0 \in V$. There exists a positive T such that the problem* (1)-(3) *has a unique solution* (u,p) *satisfying*

$$\begin{aligned} & u \in L^2(0,T;H^2(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)), \\ & p \in L^2(0,T;H^1_{\#}(\Omega)). \end{aligned}$$

In addition, if the data are small enough (*in terms of the viscosity*) *the solution is global in time.*

It is clear that for our interest concerning flows with extremely small viscosity, such a result is not satisfactory, so there is need to study larger classes of (weak) solutions, which are global in time. Consequently, it is also worth studying some notions classical for the Navier-Stokes equation. In particular, one would like to have the existence of weak solutions, the energy inequality, and the enstrophy balance.

5.2 - *The nonlinear problem: Weak and strong solutions*

In this section it will be studied the time-evolution problem by means of (generalized) energy estimates. Here, we obtain a couple of differential inequalities satisfied by smooth solutions, which can be used to obtain a-priori estimates on Galerkin approximate functions and to prove existence results by standard techniques. We first derive some integrations by parts formulas that will be used in the sequel. In particular in this section we use as boundary conditions (4).

We start with an identity involved in the energy budget, see Beirão da Veiga and Berselli [**24**].

Lemma 5.1. *Let u and ϕ be two smooth enough vector fields, tangential to the boundary Γ. Then it follows*

$$-\int_{\Omega} \Delta u_i\, \phi_i\, dx = \int_{\Omega} \nabla u_i \nabla \phi_i\, dx - \int_{\Gamma} (\omega \times n)_i\, \phi_i\, dS + \int_{\Gamma} \phi_i\, u_k\, \partial_i n_k\, dS, \tag{23}$$

where $\omega = \operatorname{curl} u$.

Proof. We use index notation and ε_{ijk} is the totally anti-symmetric Ricci tensor. We observe that, for $i = 1, 2, 3$,

$$[\omega \times n]_i = \varepsilon_{ijk}\, \omega_j\, n_k = \varepsilon_{ijk}\, (\varepsilon_{jlm}\, \partial_l u_m)\, n_k = (\delta_{kl}\, \delta_{im} - \delta_{km}\, \delta_{il})\, n_k\, \partial_l u_m, \qquad \text{on } \Gamma.$$

Hence

$$n_k\, \partial_k u_i - n_k\, \partial_i u_k = \big[\omega \times \underline{\mathrm{n}}\big]_i \qquad \text{on } \Gamma. \tag{24}$$

Since u is tangential to the boundary, it follows that $(\tau \cdot \nabla)\,(u \cdot \underline{\mathrm{n}})_{|\Gamma} \equiv 0$, for each vector field τ tangential to the boundary. A straightforward argument (ϕ is tangential, as well) shows that $(\phi \cdot \nabla)\,(u \cdot \underline{\mathrm{n}})_{|\Gamma}$ vanishes, i.e.,

$$n_k\, \phi_i\, \partial_i u_k = -u_k\, \phi_i\, \partial_i n_k \qquad \text{on } \Gamma,$$

(the reader can compare it also with the results in the proof of Lemma 4.1). Finally, by using the classic Gauss-Green formula, and the above identities on Γ we deduce formula (23). □

The second identity is concerned with the vorticity field. In particular, we have seen in Section 2.2 that in the flat case one can freely integrate by parts the Laplacian of ω, since no boundary terms arise. Here we show the counterpart of this result in a general domain, which is still tractable, but gives rise to some lower order terms.

Lemma 5.2. *Assume that u is divergence-free and that on Γ the slip conditions* (4) *hold true. Then*

$$-\frac{\partial \omega}{\partial \underline{\mathrm{n}}} \cdot \omega = (\varepsilon_{1jk}\, \varepsilon_{1\beta\gamma} + \varepsilon_{2jk}\, \varepsilon_{2\beta\gamma} + \varepsilon_{3jk}\, \varepsilon_{3\beta\gamma})\, \omega_j\, \omega_\beta\, \partial_k n_\gamma\,. \tag{25}$$

In particular, exists $c = c(\Omega)$ such that

$$-\int_{\Omega} \Delta\omega \cdot \omega\, dx \geq \int_{\Omega} |\nabla \omega|^2\, dx - c \int_{\Gamma} |\omega|^2\, dS. \tag{26}$$

Proof. The vorticity ω is parallel to the normal unit vector on Γ. Hence $(\tau \cdot \nabla)\,(\omega \times \underline{\mathrm{n}})\big|_{\Gamma} \equiv 0$ for each vector field τ tangential to the boundary. Since on the boundary ω is orthogonal to tangent vectors, it follows that $\omega \times \nabla[\,(\omega \times \underline{\mathrm{n}})_i\,] \equiv 0$ for $i = 1, 2, 3$, on Γ. In more explicit coordinates we can write, for $i,\ \alpha = 1, 2, 3$,

$$\varepsilon_{ijk}\,\varepsilon_{\alpha\beta\gamma}\,\omega_j\,\partial_k(\omega_\beta\,n_\gamma) = 0, \qquad \text{on } \Gamma. \tag{27}$$

Hence, by considering Eq. (27) for (i, α) equal to $(1, 1)$, $(2, 2)$, and $(3, 3)$ we get, respectively:

$$\begin{aligned}
n_3\,\omega_2\,\partial_3\omega_2 + n_2\,\omega_3\,\partial_2\omega_3 - n_2\,\omega_2\,\partial_3\omega_3 - n_3\,\omega_3\,\partial_2\omega_2 + \varepsilon_{1jk}\,\varepsilon_{1\beta\gamma}\,\omega_j\,\omega_\beta\,\partial_k n_\gamma &= 0,\\
n_1\,\omega_3\,\partial_1\omega_3 + n_3\,\omega_1\,\partial_3\omega_1 - n_3\,\omega_3\,\partial_1\omega_1 - n_1\,\omega_1\,\partial_3\omega_3 + \varepsilon_{2jk}\,\varepsilon_{2\beta\gamma}\,\omega_j\,\omega_\beta\,\partial_k n_\gamma &= 0,\\
n_2\,\omega_1\,\partial_2\omega_1 + n_1\,\omega_2\,\partial_1\omega_2 - n_1\,\omega_1\,\partial_2\omega_2 - n_2\,\omega_2\,\partial_1\omega_1 + \varepsilon_{3jk}\,\varepsilon_{3\beta\gamma}\,\omega_j\,\omega_\beta\,\partial_k n_\gamma &= 0.
\end{aligned}$$

Next, by adding term-by-term, the above equations together with

$$\begin{aligned}
(n_2\,\omega_2\,\partial_2\omega_2 - n_2\,\omega_2\,\partial_2\omega_2) + (n_3\,\omega_3\,\partial_3\omega_3 - n_3\,\omega_3\,\partial_3\omega_3)&\\
+ (n_1\,\omega_1\,\partial_1\omega_1 - n_1\,\omega_1\,\partial_1\omega_1) &= 0,
\end{aligned}$$

we show that on Γ

$$n_i\,\omega_k\,\partial_i\omega_k - (\omega_i\,n_i)(\partial_k\omega_k) + (\varepsilon_{1jk}\,\varepsilon_{1\beta\gamma} + \varepsilon_{2jk}\,\varepsilon_{2\beta\gamma} + \varepsilon_{3jk}\,\varepsilon_{3\beta\gamma})\,\omega_j\,\omega_\beta\,\partial_k n_\gamma = 0.$$

Finally, since $\nabla\cdot\omega = 0$ we get (25). Equation (26) follows by appealing to the well known Green's formula

$$-\int_\Omega \Delta\omega\cdot\omega\,dx = \int_\Omega |\nabla\omega|^2\,dx - \int_\Gamma \frac{\partial\omega}{\partial\underline{\mathrm{n}}}\cdot\omega\,dS\,,$$

since (25) shows that

$$\exists\, c = c(\Omega) > 0: \qquad \left|\frac{\partial\omega(x)}{\partial\underline{\mathrm{n}}}\cdot\omega(x)\right| \le c\,|\omega(x)|^2, \qquad \forall\, x \in \Gamma.$$

$\square$

We give now the following precise definition of the classes of solutions we consider

Definition 5.2 (Weak solution *à la* Leray-Hopf). *We say that $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ is a weak solution to* (1)-(4) *if the two following conditions hold:*

i) *For each $\phi \in C^\infty([0, T] \times \overline{\Omega})$ satisfying $\nabla\cdot\phi = 0$ in $\Omega \times [0, T]$, $\phi(T) = 0$ in Ω, and $\phi\cdot n = 0$ on $\Gamma \times [0, T]$.*

$$\begin{aligned}
\int_0^T\!\!\int_\Omega \big(-u\,\phi_t + \nu\nabla u\nabla\phi + (u\cdot\nabla)\,u\,\phi\big)\,dx\,dt + \nu\int_0^T\!\!\int_\Gamma \phi\cdot\nabla\underline{\mathrm{n}}\cdot u\,dS\,dt&\\
= \int_\Omega u_0(x)\,\phi(x, 0)\,dx;&
\end{aligned}$$

ii) *There exists* $c = c(\Omega) \geq 0$ *such that the energy estimate*

$$\|u(t)\|^2 + \nu \int_0^t \|\nabla u(s)\|^2 \, ds \leq \|u_0\|^2 \, e^{2ct},$$

is satisfied for all $t \in [0, T]$.

In order to understand a little bit more about the meaning of this definition let us derive the energy balance.

Lemma 5.3. *Let* u *be a smooth solution of* (1)-(4) *in* $[0, T]$. *Then, there exists a positive constant* $c = c(\Omega)$ *such that*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |u|^2 dx + \nu \int_{\Omega} |\nabla u|^2 \, dx - \nu c \int_{\Gamma} |u|^2 \, dS \leq 0. \tag{28}$$

Proof. The proof follows immediately by taking the scalar product of (1) with u, by integrating over Ω, and by using results of Lemma 5.1. Note again that the first order derivatives of the (extended) normal unit vector $\underline{n}$ are uniformly bounded, since the domain is smooth. □

We also observe that the inequality can be proved to be true also for weak solutions, by means of standard arguments of approximation and of semi-continuity of the norm, see [**54, 67**].

Next, we give the definition of strong solution.

Definition 5.3 (Strong solution). *We say that a weak solution* u *is strong in* $[0, T]$ *if*

$$\nabla u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \, .$$

We say that a weak solution u *is strong in* $[0, T_1[$ *if* u *is strong in* $[0, T]$ *for each* $T < T_1$.

Trace theorems imply that for strong solutions the condition $\omega \times \underline{n} = 0$ holds in $H^{-1/2}(\Gamma)$. In addition, standard tools (following the same lines of the proof in [**54**]) show uniqueness of strong solutions in the much wider class of weak solutions.

To show existence of strong solutions, a very interesting tool in the setting we are exploring is that of using the vorticity equation: By applying the curl operator to (1) we get

$$\begin{aligned} \omega_t + (u \cdot \nabla)\, \omega - \nu \Delta \omega = (\omega \cdot \nabla)\, u \qquad & \text{in } \Omega \times]0, T], \\ \nabla \cdot \omega = 0 \qquad & \text{in } \Omega \times]0, T], \end{aligned} \tag{29}$$

supplemented with the boundary conditions $u \cdot \underline{n} = \omega \times \underline{n} = 0$ on Γ. In order to deduce the enstrophy balance, we take the scalar product of $(29)_1$ with ω, and we integrate over Ω. By appealing to (26) we obtain the following result.

Lemma 5.4. *Let u be a strong solution of* (1)-(4) *in* $[0,T]$. *Then, there exists a positive constant $c = c(\Omega)$ such that*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 dx + \nu \int_\Omega |\nabla\omega|^2\, dx - c\nu \int_\Gamma |\omega|^2\, dS \leq \left| \int_\Omega (\omega\cdot\nabla)\, u \cdot \omega\, dx \right|. \tag{30}$$

Inequality (30) allows us to bound (at least for small times/small data) the vorticity in natural function spaces. It is also well known, that the presence in the right-hand side of the *vortex stretching term* (that, at least at first glance, behaves in the same way as $\|\omega\|_3^3$) is the main obstacle in proving global existence results for strong solutions, even for the Cauchy problem in $\mathbb{R}^3$.

To employ inequality (30) we must observe that it concerns the $L^2(\Omega)$-norm of the vorticity and of its first order derivatives, while the definition of strong solutions involves the full gradient of u. In order to deduce suitable estimates we shall show that it is possible to bound the gradient of velocity, by the curl (at least in the $L^2(\Omega)$-setting). More precisely, we have the following result.

Lemma 5.5. *Let $u \in V$ be a function satisfying* (4). *Then, there exists a positive constant $c = c(\Omega)$ such that*

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \leq c(\Omega)\int_\Omega |u|^2\, dx + \int_\Omega |\omega|^2 dx. \tag{31}$$

In addition, if $\omega \in H^1(\Omega)$, then $u \in H^2(\Omega)$ and its $H^2(\Omega)$-norm can be bounded by $\|\omega\|_{H^1}$.

Proof. Since $\nabla \cdot u = 0$ in Ω, we write again (21) and in particular we have that u satisfies the system

$$\begin{aligned} -\Delta u &= \operatorname{curl}\omega && \text{in } \Omega,\\ u\cdot n &= 0 && \text{on } \Gamma,\\ \omega\times n &= 0 && \text{on } \Gamma. \end{aligned} \tag{32}$$

Next, we multiply both sides of the first equation (32) by u, and integrate over Ω. By appealing to Lemma 5.1 it follows that

$$\int_\Omega |\nabla u|^2\, dx + \int_\Gamma u_i\, u_k\, \partial_i n_k\, dS = \int_\Omega \operatorname{curl}\omega \cdot\, u\, dx\,.$$

This last equation can be written in the equivalent form

$$\int_{\Omega} |\nabla u|^2 \, dx + \int_{\Gamma} u_i \, u_k \, \partial_i n_k \, dS = \int_{\Gamma} (\omega \times n) \cdot u \, dS + \int_{\Omega} |\omega|^2 \, dx \,. \tag{33}$$

The boundary integral from the right-hand side of (33) vanishes and the smoothness of Γ implies that the absolute value of the second integral from the left-hand side of (33) is bounded by a multiple of $\int_{\Gamma} |u|^2 \, dS$. Hence, the standard trace inequality implies (31).

The L^2-regularity of second order derivatives follows by standard arguments. Similar estimates in $L^q(\Omega)$ can be obtained, by multiplying by $|u|^{q-2}u$ and performing the same integration by parts. Related calculations are also sketched in Section 7. □

Remark 5.1. *The arguments of this section are the same as in* [**24**] *and we observe that, in order to use inequality* (31), *we need a bound for the* $L^2(\Omega)$*-norm of* u *to ensure the* H^1*-a-priori estimate for the solution. Since we are considering the time-evolution problem, the above bound follows from the energy estimate. However, we observe that if* Ω *is convex, then this last device is superfluous since the integrand that appears in the surface integral in the left-hand-side of* (33) *is* (*almost*) *everywhere non-negative.*

A relevant role in this approach is played by the estimates of u and ∇u in terms of ω, also by explicit formulas with Green matrices. These are obtained by a precise analysis of the system (32), or more generally of the system

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u \cdot n &= 0 && \text{on } \Gamma, \\ \omega \times n &= 0 && \text{on } \Gamma \end{aligned}$$

which falls in the class of Petrovskiĭ elliptic systems. We recall that in Petrovskiĭ's systems different equations and unknowns have the same "differentiability order," see in [**129**, p. 126]. This fact allows us to obtain a representation formula with a single Green's matrix. The importance of these systems is also due to the fact that they are a relevant subclass of Agmon-Douglis-Nirenberg [4] (ADN) elliptic systems, and they share the same properties of self-adjoint ADN systems. In addition, for these systems the H^2-regularity can be used to prove the full regularity of solutions, provided that the data are smooth.

The explicit verification that this system satisfies the requirements is done in [**24**]. I do not reproduce it here, but for the reader's convenience I observe that –on the contrary– the Stokes problem

$$
\begin{aligned}
-\Delta u + \nabla p &= f \qquad \text{in } \Omega,\\
\nabla \cdot u &= 0 \qquad \text{in } \Omega,\\
u \cdot n &= 0 \qquad \text{on } \Gamma,\\
\omega \times n &= 0 \qquad \text{on } \Gamma,
\end{aligned}
$$

is not of Petrovskiĭ type. There is a difference between proving existence of smooth (u, p) and proving suitable relations between velocity and its curl. For an introduction to the above subject we recommend the reader to look up in the proof of Proposition 2.2 in [**138**], where the Stokes system is considered under the Dirichlet boundary condition: The Stokes system is still not of Petrovskiĭ type also in this case.

5.3 - *Existence of solutions*

I conclude this section by giving a sketch of the proof of the existence results for weak and strong solutions.

By using standard techniques, one can approximate the problem (for instance via a Galerkin method) and then one can prove a-priori estimates independent of the regularization. In fact, by taking into account the trace inequality and of (28)-(30) we prove the following differential inequalities:

$$
\begin{cases}
\dfrac{1}{2}\dfrac{d}{dt}\|u\|^2 dx + \dfrac{\nu}{2}\|\nabla u\|^2 \leq c(\Omega)\|u\|^2,\\[2mm]
\dfrac{1}{2}\dfrac{d}{dt}\|\omega\|^2 + \dfrac{\nu}{2}\|\nabla \omega\|^2 \leq c(\Omega)\|\omega\|^2 + \left|\displaystyle\int_\Omega (\omega \cdot \nabla)\, u \cdot \omega\, dx\right|.
\end{cases}
$$

Next, one can estimate the last term from the right-hand side of the second differential inequality as follows:

$$
\left|\int_\Omega (\omega \cdot \nabla)\, u \cdot \omega\, dx\right| \leq \|\omega\|_6 \|\nabla u\|_2 \|\omega\|_3,
$$

and with the standard tools of interpolation and Sobolev inequalities one obtains

$$
\left|\int_\Omega (\omega \cdot \nabla)\, u \cdot \omega\, dx\right| \leq C\Big[\|\omega\|^2_{H^1}(\|\omega\| + \|u\|)\|\omega\|^{1/2}\|\omega\|^{1/2}_{H^1}\Big].
$$

Next, by using also the energy inequality we get the following estimates:

$$
\begin{cases}
\dfrac{1}{2}\dfrac{d}{dt}\|u\|^2 + \dfrac{\nu}{2}\|\nabla u\|^2 \leq c(\Omega)\|u\|^2,\\[2mm]
\dfrac{1}{2}\dfrac{d}{dt}\|\omega\|^2 + \dfrac{\nu}{2}\|\nabla \omega\|^2 \leq c(\Omega)\left(\dfrac{\|\omega\|^6}{\nu^3} + \dfrac{\|\omega\|^2}{\nu} + \nu\|\omega\|^2 + \|\omega\|^3\right),
\end{cases}
\tag{34}
$$

and by using Poincaré-type estimates (see also Lemma 7.1) especially the second one can be improved to

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \frac{\nu}{2}\|\nabla\omega\|^2 \le c(\Omega)\left(\frac{\|\omega\|^6}{\nu^3} + \nu\|\omega\|^2\right).$$

This is enough to show that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and that $\omega \in L^\infty(0,T^*;L^2(\Omega)) \cap L^2(0,T^*;H^1(\Omega))$, for a positive T^* depending on the data. With the techniques introduced by Hopf [**75**], (see, e.g. ref. [**54, 138**]) one can easily show the following result.

Proposition 5.2. *Let $u_0 \in H$ be given. Then, for each $T > 0$ there exists at least one weak solution of the 3D Navier-Stokes equations* (1) *with the boundary conditions* (4) *in* $[0,T]$. *In addition, if $u_0 \in V$ then there is a $T^* = T^*(\|\nabla u_0\|,\nu) > 0$ such that a unique strong solution exists in* $[0,T^*[$.

Related question are treated by Girault [**71**] with an approach using vector-valued potentials. In addition, with a different variational formulation, existence and uniqueness of weak solutions to the stationary (Navier-)Stokes equations with the "non-standard" boundary conditions (4) can be given in simply connected domains. For related question of non-uniqueness, see Foiaş and Temam [**62**] with the characterization of curl/div-free vector fields in non-simply connected domains.

From Lemma 5.5 it follows that if we are able to bound the $L^2(\Omega)$-norm of the curl of a weak solution u, we are also able to bound the full gradient of this solution. Beside the physical meaning of the vorticity, this is the mathematical reason for the use of the vorticity equation. The precise analytical study of the vorticity equation and of the relation between u and ω has been motivated by the technicalities needed to prove the following regularity criterion in [**24**].

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with boundary Γ of class $C^{3,\alpha}$, for some $\alpha > 0$. Suppose that $u_0 \in V$ and u is a weak solution to* (1)-(4) *in* $[0,T]$. *Let "$\angle$" denote the angle between two unit vectors, identified with the length of a geodesic connecting them on a spherical unit surface. Define*

$$\theta(x,y,t) := \angle\left(\frac{\omega(x,t)}{|\omega(x,t)|}, \frac{\omega(y,t)}{|\omega(y,t)|}\right)$$

and suppose either that there exist $\beta \in [1/2,1]$ and $g \in L^a(0,T;L^b(\Omega))$, where

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2} \quad \textit{with} \quad a \in \left[\frac{4}{2\beta-1},\infty\right],$$

such that

$$\sin\theta(x, y, t) \leq g(t, x)|x - y|^{\beta}, \qquad a.e.\ x, y \in \Omega, \{\text{a.e. t} \in]0, \text{T}[,$$

or that there exists $\beta \in]0, 1/2]$ *such that*

$$\sin\theta(x, y, t) \leq c|x - y|^{\beta}, \qquad a.e.\ x, y \in \Omega,\ a.e.\ t \in]0, T[,$$

and that

$$\omega \in L^2(0, T; L^s(\Omega)), \qquad with\ s = \frac{3}{\beta + 1}\,.$$

Then, the solution u *is a strong solution in* $[0, T]$, *hence it is smooth.*

This result is the extension of the geometrical criterion introduced by Constantin and Fefferman [**53**], and subsequently improved in [**23**], to the bounded domain case. A simplified argument in the flat case can be found in [**17**], while remarks concerning the open problem with the Dirichlet boundary conditions can be found in [**19**]. Roughly speaking, the results of Theorem 5.2 show that if the vorticity does not change too much its direction (regardless its size), then the solution remains smooth. For instance this happens in the two dimensional case, where we know that the vorticity is a scalar, always orthogonal to the plane of motion. Related results linking regularity under geometrical assumptions on direction of vorticity and velocity, can be found in [**29, 31**] and Vasseur [**141**].

6 - On a possible modeling of turbulent phenomena

As we described in the introduction, certain interest for the Navier-type boundary conditions comes also for LES modeling. In this section I recall the results of Berselli and Romito [**34**], presenting the main ideas of the proof, and I also announce a new result.

6.1 - *Setting of the problem*

To better explain the problem, let us consider two dimensional case. We recall that the 2D Navier-Stokes equations and the limit as the viscosity vanishes in the case of *homogeneous* slip boundary conditions has been also recently studied by many authors, see especially the work of Clopeau, Mikelić, and Robert [**49**], Lopes Filho, Nussenzveig Lopes, and Planas [**103**], Mucha [**110**], and Kelliher [**85**]. We shall study the following boundary-initial value problem (we set $\nu = 1$ for simplicity and the equations are accordingly normalized):

$$
\begin{cases}
\partial_t u - \Delta u + (u \cdot \nabla)\, u + \nabla p = f & \text{in } \Omega \times]0,T],\\
\nabla \cdot u = 0 & \text{in } \Omega \times]0,T],\\
u \cdot \underline{\mathrm{n}} = \alpha^{\gamma} \mathcal{G}(x,t) & \text{on } \Gamma \times]0,T],\\
u \cdot \underline{\tau} + \alpha\, \underline{\mathrm{n}} \cdot \nabla^s u \cdot \underline{\tau} = 0 & \text{on } \Gamma \times]0,T],\\
u(x,0) = u_0(x) & \text{in } \Omega.
\end{cases}
\tag{35}
$$

The above problem with $\gamma = 2$ describes the experiment in [**10**], where $\alpha > 0$ represents the characteristic pore size and the system is laminar.

Note that a similar problem, but involving the Smagorinsky–Ladyžhenskaya turbulence model, together with a nonlinear dependence on u of the friction coefficient, has been studied in [**115**] but there the normal datum $\mathcal{G}$ is not allowed to depend on the time variable (cf. also Eq. (8)). As explained in Section 3.2, following ideas developed by Layton [**96**] our main interest was to describe time-dependent phenomena. Hence, we looked for weak hypotheses on $\mathcal{G}(x,t)$ with respect to the time variable (without any essential restriction on the space regularity) nevertheless allowing us to prove existence of weak solutions to the Navier–Stokes equations, see Theorem 6.1. This is motivated by the attempt of describing detachment of the boundary-layer and other phenomena which are inherently not stationary. In particular, in [**34**] our analysis focused mainly on two main points:

1) To show the existence of weak solutions in the sense of Leray and Hopf (since we do not want to deal with any weaker concept of solution);
2) To use the most elementary tools of functional analysis.

These two goals were motivated by the fact that we wanted to consider solutions in a very standard sense (the same we explained in the previous section) and results were also oriented to an audience of applied mathematicians.

Remark 6.1. *In the case of non-homogeneous no-slip conditions, several results of existence and uniqueness of other "less-standard" classes of solutions can be found in Amann* [**5**].

Our assumption of a non vanishing normal datum can be justified with the following argument: Suppose that we have a fictitious boundary Γ_1 and we want to impose a condition on it in order to resolve numerically the equation in a smaller domain $\Omega_1 \subset \Omega$ that rules out the boundary-layer (see Figure 2 below).

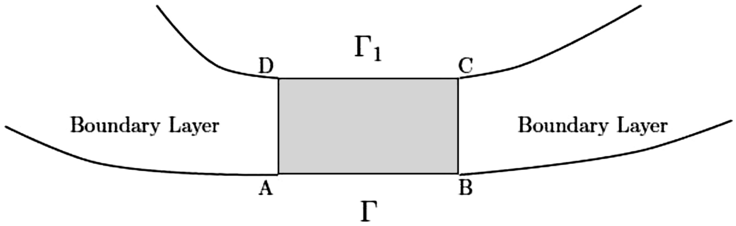

Fig. 2. The fictitious boundary.

We have to require, by the incompressibility of the flow, that

$$\int_{\{ABCD\}} \nabla \cdot u \, dx = \int_{\partial\{ABCD\}} u \cdot \underline{\mathrm{n}} \, dS = 0$$

for each (also curvilinear or infinitesimal) "rectangle" $\{ABCD\}$ touching the boundary Γ as in the figure. Since the behavior of the flow is not known, in general we have

$$\int_{\{CD\}} u \cdot \underline{\mathrm{n}} \, ds = -\left[\int_{\{BC\}} u \cdot \underline{\mathrm{n}} \, ds + \int_{\{DA\}} u \cdot \underline{\mathrm{n}} \, ds \right] \neq 0,$$

while the line integral over the segment $\{AB\}$ vanishes, because on the "physical boundary" Γ both the Navier and no-slip conditions impose that $u \cdot \underline{\mathrm{n}} = 0$. This may justify the introduction of a non vanishing normal flux, also with very low regularity properties, namely, the same shared by the trace of a turbulent flow in the boundary-layer region.

The main result is an existence and uniqueness theorem for weak solutions of the 2D Navier–Stokes system (35), with boundary conditions (9).

Theorem 6.1. *Assume that $\Omega \subset \mathbb{R}^2$ is smooth and bounded, that $\mathcal{G}$ belongs to $H^{\frac{1}{2}+\varepsilon}(0, T; H^{\frac{1}{2}}(\Gamma))$, for some $\varepsilon > 0$, and that the compatibility condition* (10) *is satisfied. Assume that $f \in L^2((0, T) \times \Omega)$ and $u_0 \in H$. Then, there exists a unique weak solution*

$$u \in L^\infty(0, T; H) \cap L^2(0, T; V),$$

of problem (35), *with $\gamma = 1$.*

6.2 - *The linear time-evolution problem*

As used in many similar problems, the first step is a very precise analysis of a linearized problem:

$$
(36) \qquad \begin{cases}
\partial_t z - \Delta z + \nabla q = 0 & \text{in } \Omega \times]0,T], \\
\nabla \cdot z = 0 & \text{in } \Omega \times]0,T], \\
z \cdot \underline{\mathrm{n}} = \alpha\, \mathcal{G} & \text{on } \Gamma \times]0,T], \\
\alpha\, \underline{\mathrm{n}} \cdot \nabla^s z \cdot \underline{\tau} + z \cdot \underline{\tau} = 0 & \text{on } \Gamma \times]0,T], \\
z(x,0) = G(x,0) & \text{in } \Omega,
\end{cases}
$$

where G is a suitable lifting of $\mathcal{G}$. In fact, we define G and Π to solve

$$
(37) \qquad \begin{cases}
-\Delta G + \nabla \Pi = 0 & \text{in } \Omega \times]0,T], \\
\nabla \cdot G = 0 & \text{in } \Omega \times]0,T], \\
G \cdot \underline{\mathrm{n}} = \alpha\, \mathcal{G}(x,t) & \text{on } \Gamma \times]0,T], \\
\alpha\, \underline{\mathrm{n}} \cdot \nabla^s G \cdot \underline{\tau} + G \cdot \underline{\tau} = 0 & \text{on } \Gamma \times]0,T],
\end{cases}
$$

where the time-variable in system (37) is just a parameter. The same regularity results of the previous section show the following result.

Proposition 6.1. *Let be given $\mathcal{G} \in H^{1/2+\varepsilon}(0,T;H^{1/2}(\Gamma))$, satisfying the compatibility condition* (10)*. Then, there exists a unique G solution of* (37) *such that*

$$
G(x,t) \in H^{1/2+\varepsilon}(0,T;H^1(\Omega)).
$$

Moreover, there is a constant C_0, depending only on Ω, such that

$$
\|\nabla G\| + \|\Pi\| \leq C_0(\alpha + \alpha^{\frac{1}{2}}) \|\mathcal{G}\|_{\frac{1}{2},\Gamma}.
$$

Notice also that, in the sequel it will be enough to have

$$
(38) \qquad G(x,t) \in H^{1/2+\varepsilon}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).
$$

In a classical way we treat the nonlinear problem as a perturbation of the linear system (36) and, by introducing the new unknowns,

$$
Z(x,t) = z(x,t) - G(x,t) \qquad \text{and} \qquad Q(x,t) = q(x,t) - \Pi(x,t),
$$

we are reduced to a homogeneous problem for (Z,Q):

$$
(39) \qquad \begin{cases}
\partial_t Z - \Delta Z + \nabla Q = -\partial_t G & \text{in } \Omega \times]0,T], \\
\nabla \cdot Z = 0 & \text{in } \Omega \times]0,T], \\
Z \cdot \underline{\mathrm{n}} = 0 & \text{on } \Gamma \times]0,T], \\
\alpha\, \underline{\mathrm{n}} \cdot \nabla^s Z \cdot \underline{\tau} + Z \cdot \underline{\tau} = 0 & \text{on } \Gamma \times]0,T], \\
Z(x,0) = 0 & \text{in } \Omega.
\end{cases}
$$

The above problem is not completely standard: The right-hand side is not enough smooth, but in addition it does not have the properties which make possible to treat it in an "usual" way. For instance, one can note that $\partial_t G$ does not belong to the domain of the Stokes operator, since $\partial_t G \cdot \underline{n} \neq 0$ and this is the main difficulty. In fact, if this term belongs to the domain of A, then one can use some integration by parts in the Duhamel formula

$$Z(t) = \int_0^t -\partial_t G(s)\, e^{-A(t-s)}\, ds$$

and shift the problems in the time variable into a requirement of more space regularity of G. This is for example one of the main techniques used in stochastic Pde's. The hypotheses we have on $\mathcal{G}$ (and consequently on G) are then responsible for a different approach. The main result is summarized here.

Theorem 6.2. *Assume that* (G, Π) *is a solution to system* (37), *with* G *satisfying the regularity property* (38). *Then, there exists a unique solution* (z, q) *to system* (36) *such that*

$$z \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega)).$$

Moreover,

$$\sup_{0\leq t\leq T} \|z(t)\|^2 + \int_0^T \left(\|\nabla^s z(s)\|^2 + \frac{1}{\alpha}\|z(s)\cdot \underline{\tau}\|_\Gamma^2 \right) ds + \|z\|^2_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))}$$
$$\leq C\left(\|G\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} + \int_0^T \|\nabla^s G(s)\|^2\, ds \right).$$

Proof. We show essentially how to derive the a-priori estimates, stressing the point where the regularity of G plays a significant role.

Since we just know that $\partial_t G \in H^{-\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))$, we introduce a sequence $\{G^N\}_{N\in\mathbb{N}} \subset H^1(\mathbb{R};L^2(\Omega))$ of approximate functions such that

(a) $G^N{}_{|_{[0,T]}} \to G$ in $H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))$, as $N \to \infty$,

(b) $\|\partial_t G^N\|_{L^2(0,T;L^2(\Omega))} = N$.

By the results of the previous section, we know that there exists a Hilbert basis $\{\phi_n\}_{n\in\mathbb{N}}$ of the space V, made of smooth functions, such that

$$\alpha\, \underline{n}\cdot \nabla^s \phi_n \cdot \underline{\tau} + \phi_n \cdot \underline{\tau} = 0.$$

Now, let $Z_n^N(t,x) = \sum_{k=1}^n \zeta_{n,k}^N(t)\phi_k(x)$ be the solution of the following (finite-dimensional) linear system of ordinary differential equations for $\zeta_{n,k}^N$ [14]:

$$\frac{d}{dt}\int_\Omega Z_n^N \cdot \phi_k + \int_\Omega \nabla^s Z_n^N \cdot \nabla^s \phi_k + \frac{1}{\alpha}\int_\Gamma (Z_n^N \cdot \underline{\tau})\,(\phi_k \cdot \underline{\tau}) = -\frac{d}{dt}\int_\Omega G^N \cdot \phi_k,$$

for $t \in (0,T)$ and $k = 1,\dots,n$ and with $\int_\Omega Z_n^N(x,0)\cdot\phi_k(x)\,dx = 0$. By using a standard argument this system of ODEs has a unique solution $Z_n^N \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and by using Z_n^N itself[15] as test function one easily obtains the following estimate:

$$\sup_{0\le t\le T}\|Z_n^N(t)\|^2 + \int_0^T\left(\|\nabla^s Z_n^N(s)\|^2 + \frac{1}{\alpha}\|Z_n^N(s)\|_\Gamma^2\right)ds \le C\|G^N\|^2_{H^1(0,T;L^2(\Omega))},$$

with a constant C, depending only on Ω.

The difficulties come from the fact that this estimate, is uniform in n, but not uniform in N due to property (b) of the approximate sequence $\{G^N\}_{N\in\mathbb{N}}$. Hence, we need other a priori estimates on the solutions Z_n^N of the finite-dimensional problem. We again multiply the equations by Z_n^N, but now we estimate the right-hand side in the following way:

$$\begin{aligned}
&\sup_{0\le t\le T}\|Z_n^N(t)\|^2 + \int_0^T\left(\|\nabla^s Z_n^N(s)\|^2 + \frac{1}{\alpha}\|Z_n^N(s)\|_\Gamma^2\right)ds \\
&\le \left|\int_0^T\int_\Omega \partial_t G^n \cdot Z_n^N\,dx\,ds\right| \\
&\le \|\partial_t G^N\|_{H^{-\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}\|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))} \\
&\le \|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}\|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))}.
\end{aligned} \tag{40}$$

We need now an uniform estimate (with respect to both n and N) of Z_n^N in the space $H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))$. We shall use the Fourier transform characterization of the norm of fractional Sobolev spaces and if

$$\widetilde{Z}_n^N = \begin{cases} Z_n^N & \text{for } t \in [0,T], \\ 0 & \text{elsewhere,}\end{cases}$$

[14] The index N concerns the smoothing of G, while the index n is the dimension of the subspace of V in which we look for approximate functions.

[15] This is obtained by multiplying each equation by the term $\zeta_{n,k}^N$ and summing over k.

we can write the following equality:

$$\frac{d}{dt}\int_\Omega \widetilde{Z}_n^N \cdot \phi_k + \int_\Omega \nabla^s \widetilde{Z}_n^N \cdot \nabla^s \phi_k + \frac{1}{\alpha}\int_\Gamma \widetilde{Z}_n^N \cdot \phi_k \, dS$$

$$= -\frac{d}{dt}\int_\Omega \widetilde{G}^N \cdot \phi_k + \delta(t)\int_\Omega G^N(0)\cdot \phi_k - \delta(t-T)\int_\Omega (Z_n^N(T) + G^N(T))\cdot \phi_k,$$

for each $k = 1, \dots, n$, where $\delta(\,\cdot\,)$ is the usual Dirac's *delta function*. By passing to the frequency variable ξ, ($\widehat{Z}_n^N$ denotes the Fourier transform of $\widetilde{Z}_n^N$) the above equation reads as follows:

$$-\mathtt{i}\,\xi\int_\Omega \widehat{Z}_n^N \cdot \phi_k + \int_\Omega \nabla^s \widehat{Z}_n^N \cdot \nabla^s \phi_k + \frac{1}{\alpha}\int_\Gamma \widehat{Z}_n^N \cdot \phi_k \, dS$$

$$= \mathtt{i}\,\xi\int_\Omega \widehat{G}^N \cdot \phi_k + \int_\Omega G^N(0)\cdot \phi_k - \mathtt{e}^{-\mathtt{i}\xi T}\int_\Omega (Z_n^N(T) + G^N(T))\cdot \phi_k.$$

Consequently, by multiplying by $\overline{\widehat{Z}}_n^N$ (the complex conjugate of $\widehat{Z}_n^N$) we get -with some integration by parts-

$$-\mathtt{i}\,\xi\|\widehat{Z}_n^N(\xi)\|^2 + \|\nabla^s \widehat{Z}_n^N(\xi)\|^2 + \frac{1}{\alpha}\|\widehat{Z}_n^N(\xi)\|_\Gamma^2$$

$$= \mathtt{i}\,\xi\int_\Omega \widehat{G}^N \cdot \overline{\widehat{Z}_n^N} + \int_\Omega G^N(0)\cdot \overline{\widehat{Z}_n^N} - \mathtt{e}^{-\mathtt{i}\xi T}\int_\Omega (Z_n^N(T) + G^N(T))\cdot \overline{\widehat{Z}_n^N}.$$

We take the imaginary part and multiply both sides of the previous formula by $|\xi|^{2\lambda-1}$, with $\lambda < \dfrac{1}{2}$ so that, by using Young's inequality, one gets

$$|\xi|^{2\lambda}\|\widehat{Z}_n^N(\xi)\|^2 \le C|\xi|^{2\lambda}\|\widehat{G}^N\|^2 + C|\xi|^{2\lambda-2}(\|G^N(T)\| + \|Z_n^N(T)\| + \|G^N(0)\|)^2.$$

In order to estimate the integral $\int_{\mathbb{R}} |\xi|^{2\lambda}\|\widehat{Z}_n^N(\xi)\|^2\,d\xi$, we split it into two parts: the "inner" integral and the "outer" one. By the above estimate, we prove that

$$\int_{|\xi|>1} |\xi|^{2\lambda}\|\widehat{Z}_n^N(\xi)\|^2 d\xi \le C\int_{\mathbb{R}} |\xi|^{2\lambda}\|\widehat{G}^N\|^2$$

$$+ C(\|G^N(T)\| + \|Z_n^N(T)\| + \|G^N(0)\|)^2 \int_{|\xi|>1} |\xi|^{2\lambda-2} d\xi.$$

The first term on the right-hand side is controlled by $C\|G^N\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}$, while (40) implies that

$$\|Z_n^N(T)\|^2 \leq C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}\|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))}.$$

Next, $\|G^N(0)\|$ is bounded by $\|G(0)\|$, and finally, by using the Morrey inequality $H^{1/2+\varepsilon}(0,T) \subset C([0,T])$, we get

$$\|G^N(T)\| \leq \|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}.$$

Observe that for the validity of the Morrey inequality it is essential that $\varepsilon > 0$ and observe also that the last integral is finite due to $\lambda < 1/2$.

The inner part is estimated as follows, by using Parseval's theorem, Poincaré inequality, and estimate (40):

$$\begin{aligned}
\int_{|\xi|\leq 1} |\xi|^{2\lambda}\|\widehat{Z}_n^N\|^2\,d\xi &\leq \int_{\mathbb{R}} \|\widehat{Z}_n^N\|^2\,d\xi = \int_0^T \|Z_n^N(t)\|^2\,dt \\
&\leq C\int_0^T \|\nabla^s Z_n^N\|^2 dt \\
&\leq C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}\|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))}.
\end{aligned}$$

In conclusion, by collecting all of the above estimates we finally get that, for each $\varepsilon \in \big(0,\dfrac{1}{2}\big)$, there exists a constant C, depending only on Ω and ε, such that

$$\|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))} \leq C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))},$$

which, together with (40), shows that Z_n^N is bounded, uniformly in n and N, in the spaces $H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))$, $L^\infty(0,T;L^2(\Omega))$, and $L^2(0,T;H^1(\Omega))$.

As usual, it is possible to extract a (diagonal) sub-sequence converging weakly in $L^2(0,T;V)$, weakly$*$ in $L^\infty(0,T;H)$, and strongly in $L^2((0,T)\times\Omega)$ to the unique solution Z of problem (39) with the required estimates in terms of the data. $\square$

Remark 6.2. *This approach with the Fourier transform with respect to the time variable is used, for instance, in Lions* [**101**] *to prove estimates on the fractional derivative of the solution. One main difference between* [**101**] (*and to our knowledge all previous works involving fractional derivatives for the Navier–Stokes equations*) *and our result is that the starting point is the existence of a weak solution, on which it is possible to prove additional estimates. On the contrary, in our case the existence of a weak solution derives from the fractional derivative estimates itself and at present it does not seem possible to prove the usual existence results without this trick.*

From the proof it is clear that the result holds also for $\Omega \subset \mathbb{R}^n$. The restriction to the two dimensional case is present when we consider the full non-linear problem, due to the usual limitations in estimating the convective term; see also Remark 6.3.

6.3 - *The nonlinear problem*

We finally consider the nonlinear problem and we end the proof of Theorem 6.1. Again, we make use of an auxiliary problem: We introduce the new variables

$$U = u - z \qquad \text{and} \qquad P = p - q,$$

where (z, q) is the solution to the linear time-evolution problem (36), and the pair (U, P) solves the following problem:

$$\begin{cases} \partial_t U - \Delta U + [(U+z)\cdot\nabla](U+z) + \nabla P = f & \text{in } \Omega \times]0,T], \\ \nabla \cdot U = 0 & \text{in } \Omega \times]0,T], \\ U \cdot \underline{\mathrm{n}} = 0 & \text{on } \Gamma \times]0,T], \\ \alpha\, \underline{\mathrm{n}} \cdot \nabla^s U \cdot \underline{\tau} + U \cdot \underline{\tau} = 0 & \text{on } \Gamma \times]0,T], \\ U(x,0) = u_0(x) - G(x,0) & \text{in } \Omega. \end{cases} \tag{41}$$

By virtue of Theorem 6.2, the existence Theorem 6.1 for the nonlinear problem is a straightforward consequence of the following proposition.

Proposition 6.2. *Assume that* (G, Π) *is a solution to system* (37), *with* $G \in H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. *Then, there exists a unique solution*

$$U \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

to problem (41). *Moreover, the following estimate holds true: For all* $0 \leq t \leq T$

$$\begin{aligned} &\sup_{0\leq s\leq t} \|U(s)\|^2 + \int_0^t \Big(\|\nabla^s U(\sigma)\|^2 + \frac{1}{\alpha} \|U(\sigma)\|_\Gamma^2 \Big)\, d\sigma \\ &\leq \|u_0 - G(\cdot,0)\|^2 \mathrm{e}^{A(t)} + C \int_0^t (\|f(\sigma)\|^2 + \|\nabla z(\sigma)\|^2 \|z(\sigma)\|) \mathrm{e}^{A(t)-A(\sigma)}\, d\sigma, \end{aligned} \tag{42}$$

where

$$A(t) = Ct + C\Big(1 + \|z\|^2_{L^\infty(0,T;L^2(\Omega))}\Big) \int_0^t \|\nabla z(s)\|^2\, ds$$

and C *is a constant depending only on* Ω.

Proof. To prove the necessary a-priori estimate, we multiply (41) by U and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2+\|\nabla^s U\|^2+\frac{1}{\alpha}\|U\|_\Gamma^2=\int_\Omega U\cdot[(U+z)\cdot\nabla](U+z)\,dx+\int_\Omega f\cdot U\,dx.$$

The estimation of the integral involving f is straightforward, since it is bounded by $\|f\|^2+\|U\|^2$. We estimate the nonlinear term in from the right-hand side by using the Gagliardo-Nirenberg inequality

$$\|u\|_4\leq C\|u\|^{1/2}\|\nabla u\|^{1/2},\qquad \forall\, u\in H^1_\tau(\Omega). \tag{43}$$

Note that such an inequality is a little bit more general than the so-called *Ladyžhenskaya inequality* (cf. [**92**, § 1]), because here the functions are not vanishing on the boundary of Ω and the constant C depends on Ω.

Next, we observe that since $\nabla\cdot U=0$ and $U\cdot\underline{\mathrm{n}}=0$, then

$$\int_\Omega U\cdot(U\cdot\nabla)\,U\,dx=0\quad\text{and}\quad\int_\Omega(U\cdot\nabla)\,z\cdot U\,dx=-\int_\Omega(U\cdot\nabla)\,U\cdot z\,dx,$$

and, by using repeatedly (43), we get

$$\begin{aligned}\left|\int_\Omega U\cdot[(U+z)\cdot\nabla](U+z)\,dx\right|&\leq 2\|U\|_4\|z\|_4\|\nabla U\|+\|U\|_4\|z\|_4\|\nabla z\|\\&\leq\frac{1}{2}\|\nabla U\|^2+C\|z\|_4^4\|U\|^2+C\|z\|_4^{\frac{4}{3}}\|\nabla z\|^{\frac{4}{3}}\|U\|^{\frac{2}{3}}\\&\leq\frac{1}{2}\|\nabla U\|^2+C\|z\|^2\|\nabla z\|^2\|U\|^2+C\|z\|^{\frac{2}{3}}\|\nabla z\|^2\|U\|^{\frac{2}{3}}\\&\leq\frac{1}{2}\|\nabla U\|^2+C(1+\|z\|^2)\|\nabla z\|^2\|U\|^2+C\|\nabla z\|^2\|z\|.\end{aligned}$$

By Theorem 6.2, $z\in L^\infty(0,T;H)\cap L^2(0,T;V)$, both terms $(1+\|z\|^2)\|\nabla z\|^2$ and $\|\nabla z\|^2\|z\|$ are integrable in time. Consequently by Gronwall's lemma, we can deduce that U is bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$. Moreover, formula (42) also follows. Finally, uniqueness of the solution can be proved by using similar arguments. Indeed, if $\mathcal{U}$ is the difference between two solutions U_1 and U_2, one easily gets

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|^2\leq(\|U_2\|_4^4+\|z\|_4^4+\|\nabla z\|^2)\|\mathcal{U}\|^2$$

and, since $\mathcal{U}(0)=0$, from Gronwall's lemma it follows that $\mathcal{U}\equiv 0$. □

Remark 6.3. *In the proof of the result of this section we used in a fundamental way estimate* (43). *In the three-dimensional case this inequality is no* *longer true. Instead, it holds that*

$$\|u\|_{L^4} \leq C\|u\|^{1/4}\|\nabla u\|^{3/4} \qquad \forall\, u \in H^1_\tau(\Omega),$$

which in this case can be used to prove just local existence of weak solutions. The global result proved in the two-dimensional case depends in an essential manner on the stronger estimate, and this is the critical difference between the two cases.

In ref. [**34**] we also studied the behavior of the solution as the parameter $\alpha \to 0$ (other convergence results, under similar assumptions, have been also proved in [**85**]). In view of the considerations of the previous section, one can expect that, as the boundary-layer becomes thinner and thinner, the solutions will look closer and closer to the classical solutions corresponding to the *no-slip* boundary condition. Indeed, this is the case, as shown by Theorem 6.3.

Let u_α be the solution of (35) (we emphasize the dependence on α in this framework) and let v be the solution to the Navier–Stokes equations with the same initial value and no-slip boundary conditions. We have

$$u_\alpha = v + \mathcal{O}(\alpha^{\frac{1}{3}}),$$

so that the "no-slip solution" represents the average behavior, once one neglects the effect at the boundary. The term $u_\alpha - v$ can be seen as the "fluctuation term", which takes into account the nontrivial dynamics at the boundary. The error estimate we derive is consistent with the first step in the homogenization procedure employed by Jäger and Mikelić [**77**] to obtain the law of Beavers and Joseph.

Theorem 6.3. *Assume* $u_0 \in V$, $\mathcal{G} \in H^{\frac{1}{2}+\varepsilon}(0, T; H^{\frac{1}{2}}(\Gamma))$ *satisfying the compatibility condition* (10), *and* $f \in L^2((0, T) \times \Omega)$. *Then*

$$\sup_{0\leq t\leq T} \|u_\alpha - v\|^2 + \int_0^T \left(\|\nabla^s(u_\alpha - v)\|^2 + \frac{1}{\alpha}\|(u_\alpha - v)\cdot \underline{\tau}\|^2_\Gamma \right) dt = \mathcal{O}(\alpha^{\frac{2}{3}}).$$

In particular, u_α *converges to* v *in* $L^\infty(0, T; L^2(\Omega))$ *and* $L^2(0, T; H^1(\Omega))$.

We do not give the proof of this result, but we refer to [**34**] for further details, which nevertheless use the same functional tools employed before.

I take now the opportunity of presenting a new result. As we will see in the next section with more details, in the two dimensional case one can treat the problem of existence of weak solutions for the Euler equations, by means of a suitable vanishing viscosity limit: One has to approximate the Euler equations, by the Navier-Stokes ones, with the slip boundary conditions $u \cdot \underline{n} = \omega = 0$ on Γ (recall that in two dimensions the vorticity is a vector only with the third component different from zero), see also Section 7.1. This tool (without appealing to the stream function) has been used for the time-evolution case, first by Bardos [8] treating also non-homogeneous data, see also Secchi [**124**] for further extensions in the flat case. The results of this section (by using the same approach) can be adapted to prove also the following result.

Theorem 6.4. *Assume that $\Omega \subset \mathbb{R}^2$ is smooth and bounded, that $\mathcal{G} \in H^{\frac{1}{2}+\varepsilon}(0,T;H^{\frac{1}{2}}(\Gamma))$, for some $\varepsilon > 0$, and that the compatibility condition* (10) *is satisfied. Assume that $f \in L^2((0,T)\times\Omega)$ and $u_0 \in H$. Then there exists a weak solution*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V),$$

of the Euler system

$$\begin{cases} \partial_t u + (u\cdot\nabla)\,u + \nabla p = f & \text{in } \Omega \times]0,T], \\ \nabla \cdot u = 0 & \text{in } \Omega \times]0,T], \\ u \cdot \underline{\mathrm{n}} = \alpha\,\mathcal{G}(x,t) & \text{on } \Gamma \times]0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The technical improvement concerns relaxing to $\mathcal{G} \in H^{1/2+\varepsilon}(0,T;H^{1/2}(\Gamma))$, the condition $\mathcal{G} \in H^1(0,T;H^{1/2}(\Gamma))$ required in the previous references [**8**, **124**]. The details of the proof, which is nevertheless a straightforward application of the technique explained in this section, will appear in [**36**].

7 - Vanishing viscosity limits

In this section I consider some of the problems related with vanishing viscosity limits for the Navier-Stokes equations. To this end, in this section I will write explicitly the dependence on the viscosity

$$\begin{aligned} u_t^\nu - \nu\Delta u^\nu + (u^\nu\cdot\nabla)\,u^\nu + \nabla p^\nu &= f \qquad \text{in } \Omega \times]0,T], \\ \nabla\cdot u^\nu &= 0 \qquad \text{in } \Omega \times]0,T]. \end{aligned} \tag{44}$$

In particular, when $\nu \downarrow 0$ the Navier-Stokes equations converge "formally" to the Euler equations:

$$\begin{aligned} u_t^E + (u^E\cdot\nabla)\,u^E + \nabla p^E &= f \qquad \text{in } \Omega \times]0,T], \\ \nabla\cdot u^E &= 0 \qquad \text{in } \Omega \times]0,T], \end{aligned}$$

and the boundary condition associated to the Euler's boundary value problem is simply

$$u^E \cdot \underline{\mathrm{n}} = 0 \qquad \text{on } \Gamma \times]0, T].$$

It is clear that, in the case of the Dirichlet boundary conditions, the problem of the convergence of u^ν (solution to the Navier-Stokes equations with viscosity $\nu > 0$) to u^E (solution to the Euler equations) involves a singular limit for a change of order of the boundary conditions and the tangential part of the velocity may develop large gradients. On the other hand, also in the whole space (hence no-boundaries) the problem presents some difficulties. The limit as $\nu \to 0$ has been probably studied for the first time in this simplified setting by Swann [**133**], even if he points out that other better results are known in the two dimensional case. We will turn later to the 2D case. In the same years Kato [**81**] proved related results. Roughly speaking, the main idea is that if one takes enough derivatives of the equation (D^α for a multi-index α such that $|\alpha| \geq 3$), and multiplies the Euler (or Navier-Stokes) equations by $D^\alpha u$ suitable integration by parts of the convective term show that

$$\left| \int_{\mathbb{R}^3} D^\alpha (u \cdot \nabla)\, u\, D^\alpha u\, dx \right| \leq c \int_{\mathbb{R}^3} |D^{\alpha-\beta} u| |D^\beta u| |D^\alpha u|\, dx, \qquad \text{for } 0 \leq |\beta| < |\alpha|.$$

This takes place because the term

$$\int_{\mathbb{R}^3} (u \cdot \nabla)\, D^\alpha u\, D^\alpha u\, dx$$

vanishes, and the term with $|\alpha| + 1$ derivatives is canceled out. By using the Sobolev embedding $H^3(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, one can show that, if $|\alpha| = 3$, it follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|u^\nu\|^2_{H^3} + \nu \|u^\nu\|^2_{H^4} \leq C \|u^\nu\|^3_{H^3}, \\
&\frac{1}{2}\frac{d}{dt} \|u^E\|^2_{H^3} \leq C \|u^E\|^3_{H^3},
\end{aligned}$$

where the constant C does *not*[16] depend on ν. These two differential inequalities can be used to show (for instance by means of the Galerkin method) local existence of a solution of both the Euler and Navier-Stokes equations belonging to

[16] The constant does not depend on ν, since we do not use the Laplacian term in the left-hand side to "absorb" the highest derivatives of u. This can be done if $|\alpha| \geq 3$. Compare this estimate with (34).

$L^\infty(0,T^*;H^3(\mathbb{R}^3))$, for some $T^*>0$ depending only on the H^3-norm of the initial datum. Observe also that solutions in this class are classical and unique. See also the details in Temam [**135, 136**] with similar calculations in a bounded domain for the Euler equations, also in the non-hilbertian setting with $u_0 \in W^{2,q}(\Omega)$, for $q>3$.

With this approach, since u^ν is bounded uniformly, we can extract a sequence $\nu_n \downarrow 0$ such that u^{ν_n} converges weakly* in $L^\infty(0,T^*;H^3(\mathbb{R}^3))$ to some u. Then, uniqueness shows that the whole family u^ν converges to the solution of the problem with $\nu=0$, i.e., that $u=u^E$, and that the convergence takes place also in $C(0,T;H^{3-\varepsilon}(\mathbb{R}^3)) \cap L^p(0,T;H^3(\mathbb{R}^3))$, for all finite p.

These results of *weak* convergence are very interesting and generally rather difficult to be obtained. Nevertheless, they are not completely satisfactory, since the Hadamard well-posedness requires (among other results) to find a space X such that the initial datum $u_0 \in X$ is the same for all viscosities (also $\nu=0$) and to prove

$$u^\nu \to u^E, \text{ as } \nu \downarrow 0 \quad \text{in } C(0,T^*;X),$$

with strong convergence. As pointed out by Kato [**84**] this is generally the most difficult part in the theory of evolution equations. A sharp result have been proved by Ebin and Marsden [**59**], but they needed to work in the much more regular spaces $H^s(\mathbb{R}^3)$, with $s>13/2$.

The *sharp* convergence result, with the initial datum in the space $H^3(\mathbb{R}^3)$, has been later obtained by Kato [**82**] as by-product of a more general result on his "perturbation theory" for abstract equations. Simpler proofs, based on a delicate smoothing of the initial data, have been also given by Beirão da Veiga [**12**] and Masmoudi [**106**], who studied also the fractional case $H^s(\mathbb{R}^3)$, with $s>5/2$ (remember that $H^{5/2}(\mathbb{R}^3)$ is the critical space for proving local existence for the Euler equations). For Inviscid limits for non-smooth vorticity see also Constantin and Wu [**55**]. We also wish to quote the nice review papers by Beirão da Veiga [**20**, **21**]. Related perturbation results have been also proved in Constantin [**51**], while a different approach -based on fractional powers of the Stokes operator- is also presented in [**54**]. In presence of boundaries none of this convergence results is known and we expect also that they are not correct, recall the condition (11) proved by Kato.

Moreover, it is known since long time that (with appropriate boundary conditions) the 2D problem in a smooth and bounded domain can be treated successfully. In fact, results of Lions [**101**] and of Bardos [8] can be used to show existence of weak solutions for the Euler equations, with a perturbation argument in terms of the viscosity. Anyway, there is need of using the Navier's-type slip-without-friction boundary conditions.

7.1 - *A two dimensional result*

In this section I give some details concerning the 2D case, to explain why it is considerably less difficult than the 3D one. We first recall that the Cauchy problem has been studied by Golovkin [**73**] and McGrath [**104**]. We also observe that in 2D the curl is the same as the gradient after a rotation of $\pi/2$. Recall in fact that in 2D vorticity ω is a scalar and

$$\omega = \partial_1 u_2 - \partial_2 u_1.$$

We consider the 2D Navier-Stokes (for simplicity, with vanishing external force) in a smooth and bounded domain $\Omega \subset \mathbb{R}^2$, with Navier-type boundary conditions

$$\begin{aligned} u_t^\nu - \nu\Delta u^\nu + (u^\nu \cdot \nabla)\, u^\nu + \nabla p^\nu &= 0 \qquad \text{in } \Omega \times]0,T], \\ \nabla \cdot u^\nu &= 0 \qquad \text{in } \Omega \times]0,T], \\ u^\nu \cdot \underline{\mathrm{n}} &= 0 \qquad \text{on } \Gamma \times]0,T], \\ \omega^\nu &= 0 \qquad \text{on } \Gamma \times]0,T]. \end{aligned} \tag{45}$$

The boundary-initial value problem can be studied by defining the bilinear form

$$a(u,v) = \nu \int_\Omega \nabla u \nabla v \, dx + \nu \int_\Gamma k(s)\, u(s)\, v(s)\, dS$$

where k is the curvature of the domain. The weak formulation is then: find $u(t) \in V$ a.e., such that

$$\frac{d}{dt}\int_\Omega u^\nu(t)\, v\, dx + a_\nu(u^\nu(t), v) + \int_\Omega (u^\nu \cdot \nabla)\, u^\nu \cdot v\, dx = 0 \qquad \forall\, v \in V.$$

By using the trace-type estimate

$$\int_\Gamma k(s)|u^\nu(s)|^2\, dS \le \varepsilon\|\nabla u^\nu\|^2 + C_\varepsilon\|u^\nu\|^2,$$

one obtains directly the energy balance

$$\frac{d}{dt}\|u^\nu\|^2 + \nu\|\nabla u^\nu\|^2 \le \nu C\|u\|^2.$$

This shows the following bounds, uniformly in $\nu > 0$,

$$u^\nu \in L^\infty(0,T;L^2(\Omega)) \qquad \text{and} \qquad \sqrt{\nu}\, u^\nu \in L^2(0,T;H^1(\Omega)).$$

Next, by taking the curl one gets the scalar equation (the two dimensional equation does not have the vortex stretching term in the right-hand side)

$$\omega_t^\nu + (u^\nu \cdot \nabla)\, \omega^\nu - \nu\Delta\omega^\nu = 0.$$

Next, testing with ω and standard integration by parts (since $\omega^\nu = 0$ on Γ) give

$$\frac{1}{2}\frac{d}{dt}\|\omega^\nu\|^2 + \nu\|\nabla\omega^\nu\|^2 = 0.$$

We obtain, again with a bound independent of $\nu > 0$, that

$$\omega^\nu \in L^\infty(0, T; L^2(\Omega)).$$

The next step is to show that the bound on ω implies the same on ∇u. In fact, by considering the elliptic system (which is the 2D counterpart of (32))

$$\begin{cases} -\Delta u^\nu = \nabla^\perp \omega^\nu \\ u^\nu \cdot n = 0 \\ \quad \omega^\nu = 0 \end{cases}$$

where $\nabla^\perp = (\partial_2, -\partial_1)$, by using u itself as test function one obtains immediately

$$\|\nabla u^\nu\|^2 \leq C(\Omega)\big(\|u^\nu\|^2 + \|\omega^\nu\|^2\big).$$

With this one proves that

$$u^\nu \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \text{uniformly in } \nu > 0.$$

Hence, we can extract (one can use the Aubin-Lions argument, see [**101**] or the Friederichs inequality, see Hopf [**75**]) a sub-sequence $\{u^{\nu_n}\}_{n\in\mathbb{N}}$ strongly converging to u^E in $L^2(\Omega \times (0, T))$, in such a way that

$$\int_0^T\int_\Omega (u^{\nu_n} \cdot \nabla)\, u^{\nu_n} \cdot v\, dx\, dt \to \int_0^T\int_\Omega (u^E \cdot \nabla)\, u^E \cdot v\, dx\, dt.$$

The function u^E turns out to be a (possibly non-unique) weak solution of the 2D Euler equations. The use of the vorticity equation is crucial here, but what is crucial is also the fact that the boundary condition prevents from

...generation of vorticity on the boundary.

If the initial datum is smoother (say with bounded vorticity) one can prove uniqueness, while the critical (non-fractional) Hilbert space for existence and uniqueness of smooth solutions is $H^2(\Omega)$.

Convergence in stronger norms, for strong solutions, have been proved recently in [**25**]. In particular, in this last reference the authors use the same technique introduced by Xiao and Xin [**148**] of taking successive powers of the operator curl, and multiplying by appropriate test functions, in order to obtain estimates on higher-order derivatives. In the sequel we will use the following notation:

$$\omega = \operatorname{curl} u, \qquad \zeta = \operatorname{curl} \omega = \operatorname{curl}^2 u, \quad \text{and} \quad \chi = \operatorname{curl} \zeta = \operatorname{curl}^3 u.$$

The notation is the same in both 2D and 3D case, and one has also to remember that in 3D they are all vector fields, while in 2D they have several vanishing components and these functions are considered as two dimensional vectors (u and ζ) or scalars (ω and χ). In particular, with the above quantities one can define equivalent norms, due to the following result.

Lemma 7.1. *Let $u \in W^{s,q}(\Omega)$ a vector valued function. Then, we have the following inequalities, for $s \in \mathbb{N}$*

$$\|u\|_{s,q} \leq C\big(\|\operatorname{curl} u\|_{s-1,q} + \|\nabla \cdot u\|_{s-1,q} + \|u \cdot \underline{\mathrm{n}}\|_{s-1/q,q,\Gamma} + \|u\|_{s-1,q}\big),$$

$$\|u\|_{s,q} \leq C\|\operatorname{curl} u\|_{s-1,q} \qquad \text{if } u \cdot \underline{\mathrm{n}} = 0 \text{ on } \Gamma,$$

$$\|u\|_{s,q} \leq C\big(\|\operatorname{curl} u\|_{s-1,q} + \|\nabla \cdot u\|_{s-1,q} + \|u \times \underline{\mathrm{n}}\|_{s-1/q,q,\Gamma} + \|u\|_{s-1,q}\big).$$

For the proof and links also with the topological (Betti numbers) properties of Ω, see Bourguignon and Brezis [**41**], Xiao and Xin [**148**], and von Wahl [**146**]. In [**25**] the following convergence result is proved.

Theorem 7.1. *Let be given a smooth, bounded, and simply connected $\Omega \subset \mathbb{R}^2$ and let be given a divergence-free $u_0 \in W^{2,q}(\Omega)$, with $q > 3/2$ satisfying the slip boundary conditions $u \cdot \underline{\mathrm{n}} = \omega = 0$ on Γ. Then, for all positive $T > 0$ there exists a unique solution to both the Euler and Navier Stokes equations and*

$$u^\nu \to u^E \quad \textit{weakly}^* \textit{ in } \quad L^\infty(0,T;W^{2,q}(\Omega)).$$

In particular, in [**25**] the authors derived suitable a-priori estimates, by using as test function in the equation satisfied by ω and ζ, the quantities $|\omega|^{q-2}\omega$ and $|\zeta|^{q-2}\zeta$, respectively. The *sharp* convergence results have been recently proved in [**35**], by using the same approach with the curl operator, together with suitable approximation of the initial datum, see also [**12, 106, 99**]. For the sake of completeness, this is one of the results which will appear in [**35**].

Theorem 7.2. *Under the same hypotheses of the previous theorem it holds*

$$u^\nu \to u^E \quad \textit{in} \quad L^\infty(0,T;W^{2,q}(\Omega)).$$

Moreover, in the 2D case the fact that the domain is flat is not relevant, and we do not give further details, but in the next section we study a little bit the 3D case.

7.2 - *Differential inequalities: the* 3*D flat case*

In this section, we explain how to use (in some cases) the estimates on ω and ζ, to show high-order estimates on the solution. In particular, we try to focus on the main formulas of integration by parts which are needed. From now on we set in the 3D case and we do not give any further detail on the simpler 2D case, where flat and non-flat can be treated in the same way. This idea has been used in [**148**] in the $L^2(\Omega)$ case ($q = 2$) and in [**25**, **26**] in the general case. The first estimate is obtained by writing the vorticity equation and by using as test function $|\omega|^{q-2}\omega$. With suitable integration by parts and by using that $\nabla \cdot u = 0$ and u is tangential on the boundary one gets

$$\frac{1}{q}\frac{d}{dt}\|\omega\|_q^q + \nu\,\frac{2(2q-3)}{q^2}\int_\Omega |\nabla|\omega|^{\frac{q}{2}}|^2\,dx$$
$$\leq \int_\Omega |\omega|^q|\nabla u|\,dx + \nu\left|\int_\Gamma |\omega|^{q-2}\partial_i\omega_j\underline{\mathrm{n}}_i\omega_j\,dS\right|.$$

In the same way one obtains the estimate for $\zeta = \operatorname{curl}^2 u$

$$\frac{1}{q}\frac{d}{dt}\|\zeta\|_q^q + \nu\frac{2(2q-3)}{q^2}\int_\Omega |\nabla|\zeta|^{\frac{q}{2}}|^2\,dx$$
$$\leq \int_\Omega |\nabla u||\nabla\omega||\zeta|^{q-1}|\nabla u|\,dx + \nu\left|\int_\Gamma |\zeta|^{q-2}\partial_i\zeta_j\underline{\mathrm{n}}_i\zeta_j\,dS\right|.$$

A similar estimate can be also obtained for χ, but here we consider just the first two inequalities and these can be employed to study existence of strong solutions in $W^{2,q}(\Omega)$, with $q > 3$. As one can understand the main difficulty concerns the boundary integrals since the other integrals are more or less the usual ones that one has to control also in the space-periodic case. One first result, which makes relevant the difference between the flat and non-flat case, is the following:

Lemma 7.2. *Let us assume that* $u_3 = \omega_1 = \omega_2 = 0$ *on the boundary* $\Gamma = \{x_3 = 0\}$ *of* $\mathbb{R}^3_+$*(in the flat case they are exactly the slip-without-friction conditions* (4)). *Then, as we have previously seen,* $\partial_3\omega_3 = 0$ *on* Γ *and moreover*

$$\partial_i\,\omega_j\,\underline{\mathrm{n}}_i\,\omega_j = 0 \qquad \text{on } \Gamma,$$
$$\zeta_3 = 0 \qquad \text{on } \Gamma.$$

In addition, if ω *satisfies the vorticity equation for the Navier-Stokes equations,* *with any positive viscosity, then*

$$[\mathrm{curl}\,\zeta]_1 = [\mathrm{curl}\,\zeta]_2 = 0 \qquad \text{on } \Gamma,$$
$$\partial_i \zeta_j \,\underline{\mathrm{n}}_i\, \zeta_j = 0 \qquad \text{on } \Gamma.$$

Proof. The first two conditions are checked by a direct computation, while the result on curl ζ is obtained by observing that

$$\nu \,\mathrm{curl}\, \zeta = -\nu \Delta\omega = -\omega_t - (u\cdot\nabla)\,\omega + (\omega\cdot\nabla)\,u.$$

Hence, restricting this equality on the boundary, taking the first two components, and using the previously proved properties of the boundary values of ω, one gets also the second set of conditions. □

By using standard tools, especially the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ applied to the function $|\zeta|^{\frac{q}{2}}$ and the cancellation of all the boundary integrals, one easily gets the following result.

Proposition 7.1. *Assume that $q > 3$, then*

$$\frac{d}{dt}\|\zeta\|_q^q + c\,\nu\|\zeta\|_{3q}^q \le c\|\zeta\|_q^{q+1} + c\nu\|\zeta\|_q^q,$$

with constants independent of the viscosity.

Standard results on Bernoulli ordinary differential equations applied to the differential inequality satisfied by $Y(t) = \|\zeta(t)\|_q^q$, imply then local boundedness of $\|\zeta\|_q$. Local boundedness means that $Y(t)$ is bounded in some interval $[0, T_0]$, which is non-vanishing and whose size is independent of ν (and depends mainly on $Y(0)$). By using this estimate together with Lemma 7.1, one obtains a uniform bound for u^ν in $L^\infty(0, T_0; W^{2,q}(\Omega))$ and standard arguments of compactness imply the following result (cf. [**25**]).

Theorem 7.3. *Assume that $q > 3$ Then, it follows*

$$u^\nu \overset{*}{\rightharpoonup} u^E \qquad \text{in } L^\infty(0, T_0; W^{2,q}(\Omega))$$
$$u^\nu \to u^E \qquad \text{in } C(0, T_0; W^{s,q}(\Omega)), \quad \text{for } 0 \le s < 2,$$

where u^E is the unique solution to the Euler equation with the same initial datum of the Navier-Stokes equations.

This result is not completely satisfactory, since the sharp convergence is missing. The sharp results has been recently obtained by myself and Spirito [**35**].

Theorem 7.4. *Assume that* $u_0 \in W^{2,q}(\Omega)$ *is divergence-free and satisfies the boundary conditions* (4). *Then*

$$u^\nu \to u^E \qquad \text{in } L^\infty(0, T_0; W^{2,q}(\Omega).$$

Full details of this result (whose proof is too long to be reproduced here) will appear in a forthcoming paper.

Remark 7.1. *The same* (*sharp*) *convergence results can be obtained also by assuming the initial datum is in* $W^{3,q}(\Omega)$, *with* $q > 3/2$ (*now the critical space is* $W^{3,3/2}(\Omega)$) *with the corresponding strong convergence in* $L^\infty(0, T_0; W^{3,q})$.

7.3 - *The* 3*D generic case*

The main difference between the flat case and the non-flat case is that in the non-flat case the various boundary integrals do not vanish identically and one has to work much more to have results similar (in any case weaker) to those of Lemma 7.2. We explain the main differences and we show some preliminary results.

Remark 7.2. *In the* 2*D case such a difference does not hold, hence one can freely integrate by parts also in the non-flat case, making the two dimensional problem much easier to be handled* (*in addition to the others reasons which are well-known*). *This is the reason why we are skipping most of the details in the two dimensional case.*

In this section we show some formulas needed in the integration by parts. Results are taken mainly from [**24, 25, 26, 148**]. First recall the elementary vector identity which holds for each smooth enough (say in $W^{1,q}(\Omega)$) vector field v

$$v = (\underline{n} \cdot v)\, \underline{n} + (\underline{n} \times v) \times \underline{n} \qquad \text{on } \Gamma. \tag{46}$$

Hence $(\underline{n} \times v)_{|\Gamma}$ determines completely the projection of $v_{|\Gamma}$ on the (local) tangent plane to Γ. Therefore, we shall call $(\underline{n} \times v)_{|\Gamma}$ the tangential component of $v_{|\Gamma}$.

We start by stating the counterpart of Lemma 7.2.

Lemma 7.3. *Let* Ω *be smooth and bounded and let be given a smooth* u *satisfying the boundary conditions* (4). *Then, concerning the term* $-\dfrac{\partial \omega}{\partial \underline{n}} \cdot \omega$ *we have the identity* (25), *which allows integration by parts as in* (26). *Concerning* $\zeta = \operatorname{curl} \omega$ *we have the following result*

$$\zeta \cdot \underline{n} = 0.$$

Proof. The proof of the latter inequality, follows from a change of coordinates. Let us fix $x_0 \in \Gamma$ and set a reference frame, such that e_1 and e_2 are tangential to Γ, while $e_3 = \underline{\mathrm{n}}$ and they are a right-hand triple of unit vectors. The curl is invariant by change of coordinates and consequently we have that

$$\zeta \cdot \underline{\mathrm{n}} = (\operatorname{curl} \omega) \cdot n = D_{\underline{\tau}}(\underline{\mathrm{n}} \times \omega) = 0,$$

where $D_{\underline{\tau}}$ is a linear combination of derivatives in the tangential directions (more precisely $\partial_1 \omega_2 - \partial_2 \omega_1$ in the new reference frame). □

By using this lemma the situation seems still similar to that we encountered in the flat case: The boundary integrals are vanishing or they can be estimated by standard trace inequalities, since there is a way to reduce the order of leading terms. The next lemma reveals the subtle difference between the two situations.

Lemma 7.4. *Let the boundary Γ be a surface of class C^k, with $k \geq 2$. Then, for any point $x_0 \in \Gamma$, the component of $(u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u$ along any tangential direction $\underline{\tau}$ has the form*

$$\big((u \cdot \nabla)\,\omega) - (\omega \cdot \nabla)\,u\big)(x_0) \cdot \underline{\tau}(x_0) = a_{ij}(x_0) u_i(x_0)\,\omega_j(x_0), \qquad x_0 \in \Gamma,$$

where the coefficients a_{ij} are of class C^k on Γ. Consequently,

$$\nu \operatorname{curl} \zeta \cdot \underline{\tau} = -\nu(\Delta\omega) \cdot \underline{\tau} = a_{ij} u_i\, \omega_j \qquad on\ \Gamma. \tag{47}$$

Proof. This result has been proved in [**26**]. I give here a slightly different proof, since I think it will be useful to understand it in order to try to find additional results for a problem which still presents some relevant open questions. First, let us observe that since u is a tangential vector field and $\omega \times \underline{\mathrm{n}}$ is identically zero on the boundary, then

$$\frac{\partial(\omega \times \underline{\mathrm{n}})}{\partial u} = 0 \qquad \text{on } \Gamma.$$

We write this with the summation convention to obtain

$$0 = u_l\, \partial_l(\varepsilon_{ijk}\, \omega_j\, n_k) = \varepsilon_{ijk}\, u_l\, (\partial_l \omega_j)\, n_k + \varepsilon_{ijk}\, u_l\, \omega_j\, (\partial_l n_k),$$

and we observe now that, by the previous formula

$$\big[\big((u \cdot \nabla)\,\omega\big) \times \underline{\mathrm{n}}\big]_i = \varepsilon_{ijk}\, u_l\, (\partial_l \omega_j)\, n_k = -\varepsilon_{ijk}\, u_l\, \omega_j\, (\partial_l n_k) \qquad \text{on } \Gamma.$$

Let us treat now the term $(\omega \cdot \nabla)\,u$: We start observing that since u is a tangential vector field and $u \cdot \underline{\mathrm{n}}$ is identically zero on the boundary, then

$$\frac{\partial(u \cdot \underline{\mathrm{n}})}{\partial \underline{\tau}} = 0 \qquad \text{on } \Gamma,$$

for any tangential vector $\underline{\tau}$. Since $\omega \parallel \underline{\mathrm{n}}$ we also obtain

$$\omega \times \nabla(u \cdot \underline{\mathrm{n}}) = 0 \qquad \text{on } \Gamma.$$

We write again the latter equality in coordinates

$$0 = \varepsilon_{ijk}\, \omega_j\, \partial_k(u_l n_l) = \varepsilon_{ijk}\, \omega_j\, (\partial_k u_l)\, n_l + \varepsilon_{ijk}\, \omega_j\, u_l\, (\partial_k n_l).$$

By using (24) and by observing that $\omega \parallel \underline{\mathrm{n}}$ we can write that

$$\varepsilon_{ijk}\, \omega_j\, (\partial_k u_l)\, n_l = \varepsilon_{ijk}\, \omega_j\, (\partial_l u_k)\, n_l = \varepsilon_{ijk}\, n_j\, (\partial_l u_k)\, \omega_l,$$

and next we observe that on Γ

$$-\big[\big((\omega \cdot \nabla)\, u\big) \times \underline{\mathrm{n}}\big]_i = -\varepsilon_{ijk}\, \omega_l\, (\partial_l u_j)\, n_k = \varepsilon_{ijk}\, \omega_l\, (\partial_l u_k)\, n_j = -\varepsilon_{ijk}\, \omega_j\, u_l\, (\partial_k n_l).$$

This finally proves that

$$\big[\big((u \cdot \nabla)\, \omega - (\omega \cdot \nabla)\, u\big) \times \underline{\mathrm{n}}\big]_i = -\,\varepsilon_{ijk}\, u_l\, [(\partial_l n_k) + (\partial_k n_l)]\, \omega_j \qquad \text{on } \Gamma. \tag{48}$$

□

The main effect of the result of Lemma 7.4 is that we can reduce the order of relevant terms in the boundary integral (in a way similar to what we have previously used in 5.1), but this time the identity (47) implies that *we lost the multiplication by* ν. By using the same type of identities used in the proof of Lemma 5.1 (now it is clear why it is becoming so important) and using the crucial fact that ζ is tangential to the boundary we obtain that

$$-\,\nu \int_\Omega \Delta\zeta \cdot \zeta \, dx = \nu \int_\Omega |\nabla \zeta|^2 \, dx - \nu \int_\Gamma (\zeta \times \underline{\mathrm{n}}) \cdot \zeta \, dS + \nu \int_\Gamma \zeta \cdot \nabla \underline{\mathrm{n}} \cdot \zeta \, dS.$$

Next, by multiplying by $|\zeta|^{q-2}\zeta$ the equation satisfied by ζ we get the following differential inequality, for all $\nu > 0$

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\|\zeta\|_q^q + \nu \frac{2(2q-3)}{q^2} \int_\Omega |\nabla|\zeta|^{\frac{q}{2}}|^2 \, dx \\ &\qquad \le \int_\Omega |\nabla u||\nabla\omega||\zeta|^{q-1}|\nabla u| \, dx + \nu \left| \int_\Gamma |\zeta|^{q-2}(\partial_i \underline{\mathrm{n}}_i)\, \zeta_i\, \zeta_j \, dS \right| \\ &\qquad + \left| \int_\Gamma |\zeta|^{q-1} A(x)|u|\, |\omega| \, dS \right|, \end{aligned}$$

with a smooth function $A(x)$. Observe again that there is no ν to multiply this boundary term, which nevertheless does not contain derivatives of ζ. Clearly, by a trace inequality the first boundary integral can be estimated by

$$\begin{aligned}\nu\left|\int_\Gamma |\zeta|^{q-2}(\partial_i \underline{\mathrm{n}}_i)\,\zeta_i\,\zeta_j\,dS\right| &\leq C\nu\|\zeta\|^q_{q,\Gamma} = C\nu\||\zeta|^{\frac{q}{2}}\|^2_{2,\Gamma}\\ &\leq C\nu\||\zeta|^{\frac{q}{2}}\|^2_2 + \nu\|\nabla|\zeta|^{\frac{q}{2}}\|^2_2\\ &\leq C\nu\|\zeta\|^q_q + \nu\|\nabla|\zeta|^{\frac{q}{2}}\|^2_2.\end{aligned}$$

Concerning the other boundary integral, when we pass from the boundary to the interior of Ω with trace inequalities, we get (with the standard Sobolev machinery) a term involving $|\nabla|\zeta|^{\frac{q}{2}}|^2$ integrated over Ω. To absorb this term in the left-hand side, we need to multiply and divide by some power of ν and the resulting inequality is the following:

$$\frac{1}{q}\frac{d}{dt}\|\zeta\|^q_q + c\nu\|\zeta\|^q_{3q} \leq c\|\zeta\|^{q+1}_q + c\nu\|\zeta\|^q_q + \frac{c}{\nu^{\frac{q-1}{q+1}}}\|\zeta\|_q^{\frac{q(q+3)}{q+1}}.$$

This inequality allows us to prove existence of solutions bounded in $W^{2,q}(\Omega)$ for each given positive ν. Unfortunately the life-span turns out to be dependent of ν itself and a-priori it is not bounded from below in terms of ν. This inequality -beside being useful to prove existence of classical solutions for the Navier-Stokes equations- is of no use when studying the limits as $\nu \to 0$.

This is the main difference between the two cases and at present this seems to be a technical point very hard to be overcome, see also Beirão da Veiga [**22**]. Moreover, while in the flat case (by a suitable induction argument [**26**]) one can prove estimates for essentially all $\|\mathrm{curl}^k\, u\|_q$ (with $k \geq 2$) in the general case it seems difficult also to obtain the estimates for $\chi = \mathrm{curl}^3\, u$, even if we have some partial results in this direction.

I finally present another new result which I recently obtained in collaboration with S. Spirito in [**35**]. By using a perturbation argument we are able to show existence of strong solutions to the Navier-Stokes equations in $L^\infty(0, T_0; W^{2,q}(\Omega))$ under the assumptions that: a) the viscosity is small enough and b) the initial datum is more regular. Taking inspiration from for instance the work of Constantin [**51**], we have the following result.

Theorem 7.5. *Assume that* $u_0 \in W^{4,q}(\Omega)$ *is divergence-free and satisfies the boundary conditions* (4). *Let* $[0, T_0] > 0$ *the non-empty interval of existence for smooth solutions to the Euler equations with initial datum* u_0. *Then, there exists* $\nu_0 > 0$, *such that for all* $0 < \nu < \nu_0$

$$\|u^\nu(t) - u^E(t)\|_{W^{2,q}} \leq c\,\nu^{\frac{1}{2}}, \qquad \forall\, t \in [0, T_0].$$

We are requiring more regularity on the initial datum and this allows to have a smoother solution of the Euler equations. We use then the solution u^E to show existence of u^ν (with bounds independent of ν) by showing that they are "close." In fact, the difference $u^\nu - u^E$ satisfies a differential system, with initial datum equal to zero and which is similar to the Navier-Stokes system. These two facts allow us to derive suitable estimates, which nevertheless give also an order of convergence in terms of ν. Details of the proof, which requires some new tools with respect to the previous literature, will appear in the forthcoming paper [**35**].

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References

[**1**] Y. Achdou, O. Pironneau and F. Valentin, *Effective boundary conditions for laminar flows over periodic rough boundaries*, J. Comput. Phys. **147** (1998), 187-218.

[**2**] N. A. Adams and S. Stolz, *Deconvolution methods for subgrid-scale approximation in LES*, In *Modern Simulation Strategies for Turbulent Flow*, R.T. Edwards, Philadelphia 2001.

[**3**] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London 1975.

[**4**] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure Appl. Math. **12** (1959), 623-727.

[**5**] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech. **2** (2000), 16-98.

[**6**] K. Asano, *Zero-viscosity limit of the incompressible Navier-Stokes equation, II*, Mathematical analysis of fluid and plasma dynamics, I (Kyoto, 1986), Sūrikaisekikenkyūsho Kōkyūroku (1988), no. 656, 105-128.

[**7**] H.-O. Bae and B. J. Jin, *Regularity for the Navier-Stokes equations with slip boundary condition*, Proc. Amer. Math. Soc. **136** (2008), 2439-2443.

[**8**] C. Bardos, *Existence et unicité de la solution de l'équation d'Euler en dimension deux*, J. Math. Anal. Appl. **40** (1972), 769-790.

[**9**] G. K. Batchelor, *An introduction to fluid dynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge 1999.

[**10**] G. S. Beavers and D. D. Joseph, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech. **30** (1967), 197-207.

[11] C. Bègue, C. Conca, F. Murat and O. Pironneau, *Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression*, In *Nonlinear partial differential equations and their applications. Collège de France Seminar. Vol. IX*, Pitman Res. Notes Math. Ser., vol. 181, Longman Sci. Tech., Harlow 1988, pp. 179-264.

[12] H. Beirão da Veiga, *Singular limits in compressible fluid dynamics*, Arch. Rational Mech. Anal. **128** (1994), 313-327.

[13] H. Beirão da Veiga, *A new approach to the L^2-regularity theorems for linear stationary nonhomogeneous Stokes systems*, Portugal. Math. **54** (1997), 271-286.

[14] H. Beirão da Veiga, *Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions*, Adv. Differential Equations **9** (2004), 1079-1114.

[15] H. Beirão da Veiga, *On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions*, Comm. Pure Appl. Math. **58** (2005), 552-577.

[16] H. Beirão da Veiga, *Regularity of solutions to a non-homogeneous boundary value problem for general Stokes systems in $\mathbb{R}^n_+$*, Math. Ann. **331** (2005), 203-217.

[17] H. Beirão da Veiga, *Vorticity and regularity for flows under the Navier boundary condition*, Commun. Pure Appl. Anal. **5** (2006), 907-918.

[18] H. Beirão da Veiga, *Remarks on the Navier-Stokes evolution equations under slip type boundary conditions with linear friction*, Port. Math. (N.S.) **64** (2007), 377-387.

[19] H. Beirão da Veiga, *Vorticity and regularity for viscous incompressible flows under the Dirichlet boundary condition. Results and related open problems*, J. Math. Fluid Mech. **9** (2007), 506-516.

[20] H. Beirão da Veiga, *A review on some contributions to perturbation theory, singular limits and well-posedness*, J. Math. Anal. Appl. **352** (2009), 271-292.

[21] H. Beirão da Veiga, *On the sharp vanishing viscosity limit of viscous incompressible fluid flows*, In *New Directions in Mathematical Fluid Mechanics*, Birkhäuser, Basel 2010, pp. 113-122.

[22] H. Beirão da Veiga, *A challenging open problem: The inviscid limit under slip-type boundary conditions*, Discrete Contin. Dyn. Syst. Ser. S **3** (2010), 231-236.

[23] H. Beirão da Veiga and L. C. Berselli, *On the regularizing effect of the vorticity direction in incompressible viscous flows*, Differential Integral Equations **15** (2002), 345-356.

[24] H. Beirão da Veiga and L. C. Berselli, *Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary*, J. Differential Equations **246** (2009), 597-628.

[25] H. Beirão da Veiga and F. Crispo, *Sharp inviscid limit results under Navier type boundary conditions. An L^p theory*, J. Math. Fluid Mech. doi:10.1007/s00021-009-0295-4. Online First.

[26] H. Beirão da Veiga and F. Crispo, *Concerning the $W^{k,p}$-inviscid limit for 3-D flows under a slip boundary condition*, J. Math. Fluid Mech. doi:10.1007/s00021-009-0012-3. Online First.

[27] A. BENSOUSSAN and R. TEMAM, *Équations stochastiques du type Navier-Stokes*, J. Functional Analysis **13** (1973), 195-222.

[28] L. BERS, F. JOHN and M. SCHECHTER, *Partial differential equations*, with supplements by Lars Gårding and A. N. Milgram, with a preface by A. S. Householder, American Mathematical Society, Providence, R.I. 1979.

[29] L. C. BERSELLI, *Some geometric constraints and the problem of global regularity for the Navier-Stokes equations*, Nonlinearity **22** (2009), 2561-2581.

[30] L. C. BERSELLI, *An elementary approach to the 3D Navier-Stokes equations with Navier boundary conditions: Existence and uniqueness of various classes of solutions in the flat boundary case*, Discrete Contin. Dyn. Syst. Ser. S **3** (2010), 199-219.

[31] L. C. BERSELLI and D. CÓRDOBA, *On the regularity of the solutions to the 3D Navier-Stokes equations: a remark on the role of the helicity*, C. R. Math. Acad. Sci. Paris **347** (2009), 613-618.

[32] L. C. BERSELLI, T. ILIESCU and W. J. LAYTON, *Mathematics of Large Eddy Simulation of turbulent flows*, Scientific Computation, Springer-Verlag, Berlin 2006.

[33] L. C. BERSELLI and R. LEWANDOWKSI, *Convergence of approximate deconvolution models to the filtered Navier-Stokes Equations*, arXiv:0912.4121v1 (2009).

[34] L. C. BERSELLI and M. ROMITO, *On the existence and uniqueness of weak solutions for a vorticity seeding model*, SIAM J. Math. Anal. **37** (2006), 1780-1799 (electronic).

[35] L. C. BERSELLI and S. SPIRITO, *On the vanishing viscosity limit for the Navier-Stokes equations under slip boundary conditions in general domains*, Tech. Report 2010/6, Dip. Mat. Applicata "U.Dini", Università di Pisa.

[36] L. C. BERSELLI and S. SPIRITO, *A remark on the Euler equations in dimension two*, (2010). In preparation.

[37] H. BESSAIH and F. FLANDOLI, *2-D Euler equation perturbed by noise*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), 35-54.

[38] H. BESSAIH and F. FLANDOLI, *Weak attractor for a dissipative Euler equation*, J. Dynam. Differential Equations **12** (2000), 713-732.

[39] A. BORSELLI, *On the Stokes problem: Existence of various classes of solutions*, Master's thesis, University of Pisa (2009). (Italian).

[40] J. BOURGAIN and H. BREZIS, *On the equation* $\operatorname{div} Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16** (2003), 393-426 (electronic).

[41] J. P. BOURGUIGNON and H. BREZIS, *Remarks on the Euler equation*, J. Functional Analysis **15** (1974), 341-363.

[42] H. BREZIS, *Analyse fonctionnelle*, *Théorie et applications*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris 1983.

[43] F. BREZZI and M. FORTIN, *Mixed and hybrid finite element methods*, Springer Series in Computational Mathematics, vol. 15, Springer-Verlag, New York 1991.

[44] D. BUCUR, E. FEIREISL and Š. NEČASOVÁ, *Influence of wall roughness on the slip behaviour of viscous fluids*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 957-973.

[45] D. Bucur, E. Feireisl, Š. Nečasová and J. Wolf, *On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries*, J. Differential Equations **244** (2008), 2890-2908.

[46] M. Cannone and S. Friedlander, *Navier: blow-up and collapse*, Notices Amer. Math. Soc. **50** (2003), 7-13.

[47] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova **31** (1961), 308-340.

[48] A. J. Chorin and J. E. Marsden, *A mathematical introduction to fluid mechanics*, 3rd edition, Springer-Verlag, New York 1993.

[49] T. Clopeau, A. Mikelicç and R. Robert, *On the vanishing viscosity limit for the* 2D *incompressible Navier-Stokes equations with the friction type boundary conditions*, Nonlinearity **11** (1998), 1625-1636.

[50] L. Consiglieri, *A nonlocal friction problem for a class of non-Newtonian flows*, Port. Math. (N.S.) **60** (2003), 237-252.

[51] P. Constantin, *Note on loss of regularity for solutions of the* 3*-D incompressible Euler and related equations*, Comm. Math. Phys. **104** (1986), 311-326.

[52] P. Constantin, *On the Euler equations of incompressible fluids*, Bull. Amer. Math. Soc. (N.S.) **44** (2007), 603-621.

[53] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), 775-789.

[54] P. Constantin and C. Foiaș, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL 1988.

[55] P. Constantin and J. Wu, *The inviscid limit for non-smooth vorticity*, Indiana Univ. Math. J. **45** (1996), 67-81.

[56] Leonardo da Vinci, *Codice atlantico*, Piumati, fo. 74 (1894), 1510 ca.

[57] C. R. Doering and J. D. Gibbon, *Applied analysis of the Navier-Stokes equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge 1995.

[58] G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*, Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219, Springer-Verlag, Berlin-New York 1976.

[59] D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) **92** (1970), 102-163.

[60] F. Flandoli, *An introduction to* 3*D stochastic fluid dynamics*, Lecture Notes in Math., vol. 1942, Springer, Berlin 2008, pp. 51-150, Proceedings of the C.I.M.E.-E.M.S. Summer School held in Cetraro, Aug. 29 - Sep. 3, 2005, on *SPDE in hydrodynamics: recent progress and prospects*.

[61] C. Foiaș, O. Manley, R. Rosa and R. Temam, *Navier-Stokes equations and turbulence*, Encyclopedia of Mathematics and its Applications, vol. 83, Cambridge University Press, Cambridge 2001.

[62] C. Foiaș and R. Temam, *Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 28-63.

[63] U. Frisch, *Turbulence, The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge 1995.

[**64**] H. FUJITA, *A coherent analysis of Stokes flows under boundary conditions of friction type*, J. Comput. Appl. Math. **149** (2002), 57-69.

[**65**] G. P. GALDI, *An introduction to the mathematical theory of the Navier-Stokes equations, Vol. I, Linearized steady problems*, Springer Tracts in Natural Philosophy, vol. 38, Springer-Verlag, New York 1994.

[**66**] G. P. GALDI, *An introduction to the mathematical theory of the Navier-Stokes equations, Vol. II, Nonlinear steady problems*, Springer Tracts in Natural Philosophy, vol. 39, Springer-Verlag, New York 1994.

[**67**] G. P. GALDI, *An introduction to the Navier-Stokes initial-boundary value problem*, In *Fundamental directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech., Birkhäuser, Basel 2000, pp. 1-70.

[**68**] G. P. GALDI and W. J. LAYTON, *Approximation of the larger eddies in fluid motions, II, A model for space-filtered flow*, Math. Models Methods Appl. Sci. **10** (2000), 343-350.

[**69**] G. P. GALDI, C. G. SIMADER and H. SOHR, *A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$*, Math. Ann. **331** (2005), 41-74.

[**70**] B. J. GEURTS, *Elements of Direct and Large Eddy Simulation*, Edwards Publishing, Flourtown, PA 2003.

[**71**] V. GIRAULT, *Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in $\mathbf{R}^3$*, Math. Comp. **51** (1988), 55-74.

[**72**] V. GIRAULT and P.-A. RAVIART, *Finite element approximation of the Navier-Stokes equations*, Lecture Notes in Mathematics, vol. 749, Springer-Verlag, Berlin-New York 1979.

[**73**] K. K. GOLOVKIN, *Vanishing viscosity in the Cauchy problem for equations of hydrodynamics*, Trudy Mat. Inst. Steklov. **92** (1966), 31-49.

[**74**] J. G. HEYWOOD, R. RANNACHER and S. TUREK, *Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations*, Internat. J. Numer. Methods Fluids **22** (1996), 325-352.

[**75**] E. HOPF, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213-231.

[**76**] W. JÄGER and A. MIKELIĆ, *On the boundary conditions at the contact interface between a porous medium and a free fluid*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23** (1996), 403-465.

[**77**] W. JÄGER and A. MIKELIĆ, *On the interface boundary condition of Beavers, Joseph, and Saffman*, SIAM J. Appl. Math. **60** (2000), 1111-1127 (electronic).

[**78**] V. JOHN, *Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equations—numerical tests and aspects of the implementation*, J. Comput. Appl. Math. **147** (2002), 287-300.

[**79**] V. JOHN, *Large eddy simulation of turbulent incompressible flows*, Analytical and numerical results for a class of LES, Lecture Notes in Computational Science and Engineering, vol. 34, Springer-Verlag, Berlin 2004.

[**80**] V. JOHN, W. J. LAYTON and N. SAHIN, *Derivation and analysis of near wall models for channel and recirculating flows*, Comput. Math. Appl. **48** (2004), 1135-1151.

[81] T. KATO, *Nonstationary flows of viscous and ideal fluids in* $\mathbf{R}^3$, J. Functional Analysis **9** (1972), 296-305.

[82] T. KATO, *Quasi-linear equations of evolution, with applications to partial differential equations*, In *Spectral theory and differential equations* (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens), Lecture Notes in Math., Vol. 448, Springer, Berlin 1975, pp. 25-70.

[83] T. KATO, *Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary*, In *Seminar on nonlinear partial differential equations* (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York 1984, pp. 85-98.

[84] T. KATO and C. Y. LAI, *Nonlinear evolution equations and the Euler flow*, J. Funct. Anal. **56** (1984), 15-28.

[85] J. P. KELLIHER, *Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane*, SIAM J. Math. Anal. **38** (2006), 210-232 (electronic).

[86] J. P. KELLIHER, *On Kato's conditions for vanishing viscosity*, Indiana Univ. Math. J. **56** (2007), 1711-1721.

[87] J. P. KELLIHER, *Vanishing viscosity and the accumulation of vorticity on the boundary*, Commun. Math. Sci. **6** (2008), 869-880.

[88] H. KIM, *Existence and regularity of very weak solutions of the stationary Navier-Stokes equations*, Arch. Rational Mech. Anal. **193** (2009), 117-152.

[89] A. N. KOLMOGOROV, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers*, Proc. Roy. Soc. London Ser. A **434** (1991), no. 1890, 9-13. Translated from: C. R. (Dokl.) Acad. Sci. URSS (N.S.) **30** (1941), 301-305 (Russian).

[90] S. G. KREĬN and G. I. LAPTEV, *On the problem of the motion of a viscous fluid in an open vessel*, Funkcional. Anal. i Priložen. **2** (1968), 40-50.

[91] O. A. LADYŽHENSKAYA, *New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them*, Proc. Stek. Inst. Math. **102** (1967), 95-118.

[92] O. A. LADYŽHENSKAYA, *The mathematical theory of viscous incompressible flow*, 2nd English edition, Mathematics and its Applications, vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris 1969.

[93] H. LAMB, *Hydrodynamics*, reprint of the 1932 sixth edition, with a foreword by R. A. Caflisch, Cambridge Mathematical Library, Cambridge University Press, Cambridge 1993.

[94] L. D. LANDAU and E. M. LIFSHITZ, *Fluid Mechanics*, Course of Theoretical Physics, vol. 6, Pergamon Press, London 1959.

[95] B. E. LAUNDER and D. B. SPALDING, *Lectures in Mathematical Models of Turbulence*, Academic Press, London 1972.

[96] W. J. LAYTON, *Advanced models for large eddy simulation*, In *Computational Fluid Dynamics-Multiscale Methods*, Vol. VKI LS 2002-06 Von Karman Institute for Fluid Dynamics, Rhode-St-Genèse, Belgium 2002.

[97] J. LERAY, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193-248.

[98] M. LESIEUR, O. MÉTAIS and P. COMTE, *Large-eddy simulations of turbulence*,

with a preface by James J. Riley, Cambridge University Press, New York 2005.

[**99**] R. Lewandowski, *Vorticities in a LES model for 3D periodic turbulent flows*, J. Math. Fluid Mech. 8 (2006), 398-422.

[**100**] A. Liakos, *Discretization of the Navier-Stokes equations with slip boundary condition*, Numer. Methods Partial Differential Equations **17** (2001), 26-42.

[**101**] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier–Villars, Paris 1969.

[**102**] C. Liu, L. Sakell and T. Beutner, editors, *Numerical analysis of large eddy simulation*, Greyden Press, Columbus 2001.

[**103**] M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas, *On the inviscid limit for two-dimensional incompressible flow with Navier friction condition*, SIAM J. Math. Anal. **36** (2005), 1130-1141 (electronic).

[**104**] F. J. McGrath, *Nonstationary plane flow of viscous and ideal fluids*, Arch. Rational Mech. Anal. **27** (1967), 329-348.

[**105**] J. Málek and K. R. Rajagopal, *Mathematical issues concerning the Navier-Stokes equations and some of its generalizations*, In *Evolutionary equations, Vol. II*, Handb. Differ. Equ. Elsevier/North-Holland, Amsterdam 2005, pp. 371-459.

[**106**] N. Masmoudi, *Remarks about the inviscid limit of the Navier-Stokes system*, Comm. Math. Phys. **270** (2007), 777-788.

[**107**] J. C. Maxwell, *On stresses in rarified gases arising from inequalities of temperature*, Philos. Trans. Roy. Soc. London **170** (1879), 231-256.

[**108**] A. L. Mazzucato, *On the zero viscosity limit in incompressible fluids*, Physica Scripta (2008), T132, 014002.

[**109**] A. S. Monin and A. M. Yaglom, *Statistical fluid mechanics: mechanics of turbulence, Vol. I and Vol. II*, English edition updated, augmented and revised by the authors, Dover Publications Inc., Mineola, NY 2007.

[**110**] P. B. Mucha, *On the inviscid limit of the Navier-Stokes equations for flows with large flux*, Nonlinearity **16** (2003), 1715-1732.

[**111**] C. L. M. H. Navier, *Mémoire sur les lois du mouvement des fluides*, Mém. Acad. Sci. Inst. de France (2) **6** (1823), 389-440.

[**112**] J. Nečas, *Équations aux dérivées partielles*, Presses de l'Université de Montréal, Montréal 1965.

[**113**] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris 1967.

[**114**] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115-162.

[**115**] C. Parés, *Existence, uniqueness and regularity of solution of the equations of a turbulence model for incompressible fluids*, Appl. Anal. **43** (1992), 245-296.

[**116**] U. Piomelli and E. Balaras, *Wall-layer models for Large-eddy simulations*, Annu. Rev. Fluid Mech., vol. 34, Annual Reviews, Palo Alto, CA 2002, pp. 349-374.

[**117**] S. B. Pope, *Turbulent flows*, Cambridge University Press, Cambridge 2000.

[**118**] A. Quarteroni, *Mathematical modelling of the cardiovascular system*, In *Proceedings of the International Congress of Mathematicians, Vol. III* (Beijing, 2002), Higher Ed. Press, Beijing 2002, pp. 839-849.

[**119**] R. RAUTMANN, *A direct approach to vorticity transport & diffusion*, In *Kyoto Conference on the Navier-Stokes Equations and their Applications*, RIMS Kôkyûroku Bessatsu, Kyoto 2007, pp. 305-329.

[**120**] L. F. RICHARDSON, *Weather prediction by numerical process*, Cambridge University Press, Cambridge 1922.

[**121**] P. SAGAUT, *Large eddy simulation for incompressible flows*, An introduction, with an introduction by Marcel Lesieur, translated from the 1998 French original by the author, Scientific Computation, Springer-Verlag, Berlin 2001.

[**122**] M. SAMMARTINO and R. E. CAFLISCH, *Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space, II, Construction of the Navier-Stokes solution*, Comm. Math. Phys. **192** (1998), 463-491.

[**123**] N. SAITO and H. FUJITA, *Regularity of solutions to the Stokes equations under a certain nonlinear boundary condition*, In *The Navier-Stokes equations: theory and numerical methods* (Varenna, 2000), Lecture Notes in Pure and Appl. Math., vol. 223, Dekker, New York 2002, pp. 73-86.

[**124**] P. SECCHI, *Nonstationary flows of viscous and ideal incompressible fluids in a half-plane*, Ricerche Mat. **34** (1985), 27-44.

[**125**] J. SERRIN, *Mathematical principles of classical fluid mechanics*, In *Handbuch der Physik* (*herausgegeben von S. Flügge*)*, Bd. 8/1, Strömungsmechanik I* (*Mitherausgeber C. Truesdell*), Springer-Verlag, Berlin 1959, pp. 125-263.

[**126**] C. G. SIMADER and H. SOHR, *The Dirichlet problem for the Laplacian in bounded and unbounded domains*, A new approach to weak, strong and $(2+k)$-solutions in Sobolev-type spaces, *Pitman Research Notes in Mathematics Series*, vol. 360, Longman, Harlow 1996.

[**127**] J. SMAGORINSKY, *General circulation experiments with the primitive equations*, Monthly Weather Review **91** (1963), 99-164.

[**128**] H. SOHR, *The Navier-Stokes equations*, An elementary functional analytic approach, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel 2001.

[**129**] V. A. SOLONNIKOV, *The Green's matrices for elliptic boundary value problems, II*, Boundary value problems of mathematical physics, 7, Trudy Mat. Inst. Steklov. **116** (1971), 181-216.

[**130**] V. A. SOLONNIKOV and V. E. ŠČADILOV, *A certain boundary value problem for the stationary system of Navier-Stokes equations*, Boundary value problems of mathematical physics, 8, Trudy Mat. Inst. Steklov. **125** (1973), 196-210.

[**131**] G. STOKES, *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids*, Trans. Cambridge Phil. Soc. 8 (1845), 287-305.

[**132**] S. STOLZ and N. A. ADAMS, *An approximate deconvolution procedure for large-eddy simulation*, Phys. Fluids **11** (1999), 1699-1701.

[**133**] H. S. G. SWANN, *The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in R_3*, Trans. Amer. Math. Soc. **157** (1971), 373-397.

[**134**] L. TARTAR, *Nonlinear partial differential equations using compactness method*, report #1584, Mathematics Research Center, Univ. of Wisconsin, Madison 1975.

[**135**] R. TEMAM, *On the Euler equations of incompressible perfect fluids*, In *Séminaire Goulaouic-Lions-Schwartz (1974-1975), Équations aux derivées partielles linéaires et non linéaires, Exp. No. 10*, Centre Math., École Polytech., Paris 1975, p. 15.

[**136**] R. TEMAM, *On the Euler equations of incompressible perfect fluids*, J. Functional Analysis **20** (1975), 32-43.

[**137**] R. TEMAM, *Navier-Stokes equations and nonlinear functional analysis*, 2nd edition, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66, SIAM, Philadelphia, PA 1995.

[**138**] R. TEMAM, *Navier-Stokes equations*, Theory and numerical analysis, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI 2001.

[**139**] H. TENNEKES and J. L. LUMLEY, *A first course in turbulence*, MIT Press, 1972.

[**140**] C. TRUESDELL, *The mechanical foundations of elasticity and fluid dynamics*, J. Rational Mech. Anal. **1** (1952), 125-171, 173-300.

[**141**] A. VASSEUR, *Regularity criterion for 3D Navier-Stokes equations in terms of the direction of the velocity*, Appl. Math. **54** (2009), 47-52.

[**142**] R. VERFÜRTH, *Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition*, Numer. Math. **50** (1987), 697-721.

[**143**] R. VERFÜRTH, *Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition, II*, Numer. Math. **59** (1991), 615-636.

[**144**] M. J. VISHIK and A. V. FURSIKOV, *Mathematical problems of statistical hydromechanics*, Kluwer, Dordrecht 1988.

[**145**] J. VON NEUMANN, *Discussion on the existence and uniqueness or multiplicity of solutions of the aerodynamical equations*, Bull. Amer. Math. Soc. **47** (2010), 145-154. Reprint of: Proc. Symposium on the Motion of Gaseous Masses of Cosmical Dimensions (Paris, August 16-19, 1949), Chap. 10, pp. 75-85.

[**146**] W. VON WAHL, *Estimating* ∇u *by* $\operatorname{div} u$ *and* $\operatorname{curl} u$, Math. Methods Appl. Sci. **15** (1992), 123-143.

[**147**] X. WANG, Y.-G. WANG and Z. XIN, *Boundary layers in incompressible Navier-Stokes equations with Navier boundary conditions for the vanishing viscosity limit*, Commun. Math. Sci. (2010), In Press.

[**148**] Y. XIAO and Z. XIN, *On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition*, Comm. Pure Appl. Math. **60** (2007), 1027-1055.

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