Riv. Mat. Univ. Parma (8) **2** (2009), 91-99

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On symmetric modules

Abstract. Let α be an endomorphism of an arbitrary ring R with identity and let M be a right R-module. We introduce the notion of α-symmetric modules as a generalization of α-reduced modules. A module M is called *α-symmetric* if, for any $m \in M$ and any $a, b \in R$, $mab = 0$ implies $mba = 0$; $ma = 0$ if and only if $m\alpha(a) = 0$. We show that the class of α-symmetric modules lies strictly between classes of α-reduced modules and α-semicommutative modules. We study characterizations of α-symmetric modules and their related properties including module extensions. For a rigid module M, M is α-reduced if and only if M is α-symmetric. For a module M, it is proved that $M[x]_{R[x]}$ is α-symmetric if and only if $M[x, x^{-1}]_{R[x,x^{-1}]}$ is α-symmetric.

Mathematics Subject Classification (2000): 16U80.

1 - Introduction

Throughout this paper all rings have an identity, all modules considered are unital right modules and all ring homomorphisms are unital (unless explicitly stated otherwise), **1** is the identity endomorphism. Let α be an endomorphism of an arbitrary ring R and let M be an R-module.

Recall that a ring R is *reduced* if it has no nonzero nilpotent elements. Reduced rings have been studied for over forty years (see [**10**]), and the reduced ring $R_{red} = R/Nil(R)$ associated with a commutative ring R has been of interest to commutative algebraists. Recently the reduced ring concept was extended to modules by Lee and Zhou in [**7**], that is, a module M is called *α-compatible* if, for any $m \in M$ and any $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$. An α-compatible module M is called *α-reduced* if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$.

Received: March 18, 2009; accepted in revised form: May 19, 2009.

The module M is called *reduced* if it is **1**-reduced. Hence a ring R is a reduced ring if and only if the right R-module R is a reduced module.

According to Lambek [**6**], a ring R is called *symmetric* if whenever $a, b, c \in R$ satisfy $abc = 0$, we have $bac = 0$; it is easily seen that this is a left-right symmetric concept. In [**5**], for an endomorphism α of a ring R, α is said to be *right symmetric* if $abc = 0$ implies $ac\alpha(b) = 0$ for $a, b, c \in R$. A ring R is called *right α-symmetric* if α is a right symmetric endomorphism of R. A module M is called *symmetric* ([**6**] and [8]), if whenever $a, b \in R$, $m \in M$ satisfy $mab = 0$, we have $mba = 0$.

A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. A module M is called α-semicommutative if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR\alpha(a) = 0$. The module M is called *semicommutative* if it is **1**-semicommutative. Buhphang and Rege in [**2**] studied basic properties of semicommutative modules. Agayev and Harmanci continued further investigations for semicommutative rings and modules in [**1**] and focused on the semicommutativity of subrings of matrix rings.

In this paper, we introduce the notion of α-symmetric modules as a generalization of α-reduced modules. It is shown that the class of α-symmetric modules lies strictly between classes of α-reduced modules and α-semicommutative modules. We study characterizations of α-symmetric modules and their related properties including module extensions.

2 - α-symmetric modules

Symmetric rings and symmetric modules are introduced in [**6**] and [**8**] and continued investigating in [**5**]. In this section we extend symmetric module notion to α-symmetric one by emphasizing α.

Definition 2.1. A symmetric module M is called *α-symmetric* if M is α-compatible. A ring R is said to be *right α-symmetric* if the right R-module R is α-symmetric.

Note that **1**-symmetric modules are exactly the symmetric modules. If the right R-module R is α-symmetric, then R is right α-symmetric ring in the sense of Kwak [**5**]. We now give some classes of modules which are symmetric or α-symmetric.

Example 2.1. (1) By [**9**, Proposition 2.2], all reduced modules are symmetric modules.

(2) All modules over commutative rings are symmetric.

(3) Let R denote the ring of integers $\mathbb{Z}$ and $\mathbb{Z}_{12}$ the ring of integers modulo 12.

Consider $M = \mathbb{Z}_{12}$ as an R-module. Then M is a symmetric module. Note that M is not reduced from [7, Example 1.3], the fact will be used in the sequel.

(4) Let $\mathbb{Z}$ denote the ring of integers. Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ and the right R-module $M = \left\{ \begin{pmatrix} 0 & m \\ 0 & n \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}$ and the homomorphism $\alpha : R \to R$ is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$ where $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. It is easy to check that M is α-symmetric.

We begin with a simple observation.

Lemma 2.1. *Let M be a module. Then the following are equivalent:*

(1) *M is α-symmetric.*

(2) *$mab = 0$ if and only if $mb\alpha(a) = 0$, where $m \in M$ and $a, b \in R$.*

Proof. (1) Clear from definitions. (2) The stated condition implies that for any $m \in M$ and a,$b \in R$, $mab = 0$ if and only if $mba = 0$. The rest is clear. □

By Definition 2.1, it is clear that α-symmetric modules are symmetric. Example 2.2 reveals that not all symmetric modules are α-symmetric for some α.

Example 2.2. Let $\mathbb{Z}$ denote the ring of integers. Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ and the right R-module $M = \left\{ \begin{pmatrix} 0 & m \\ n & k \end{pmatrix} \mid m, n, k \in \mathbb{Z} \right\}$ and α an homomorphism defined on R by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ where $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$. R is commutative and so M is a symmetric module. For $m = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in M$, $r = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in R$, we have $mrs = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0$. But $ms\alpha(r) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0$. Therefore M is not α-symmetric.

Proposition 2.1. *Let R be a ring and α an endomorphism of R. The class of α-symmetric modules is closed under submodules, direct products and so direct sums.*

Recall that a module M is called *cogenerated* by R if it is embedded in a direct product of copies of R, and M is *faithful* if the only $a \in R$ such that $Ma = 0$ is $a = 0$.

Proposition 2.2. *The following conditions are equivalent:*
(1) *R is an α-symmetric ring.*
(2) *Every cogenerated R-module is α-symmetric.*
(3) *Every submodule of a free R-module is α-symmetric.*
(4) *There exists a faithful α-symmetric R-module.*

Proof. It is a direct result of definitions and Proposition 2.1. □

A ring R is called α-rigid if $a\alpha(a) = 0$ implies $a = 0$ for any $a \in R$ (see [**3**]). A module M is called *α-rigid* if $ma\alpha(a) = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$. The module M is called *rigid* if it is **1**-rigid (see [**4**]). Hence M is *rigid* if and only if for any $m \in M$ and $a \in R$ $ma^2 = 0$ implies $ma = 0$. The fact that a ring R is α-rigid if and only if the right R-module R is α-rigid is just mentioned and proved implicitly in [**4**]. But we use this result in this note and so we prove it explicitly.

Lemma 2.2. *Let α be a homomorphism of a ring R. Then the right R-module R is α-rigid if and only if R is α-rigid.*

Proof. Necessity : let a, $b \in R$ with $a\alpha(b) = 0$. Then $1a\alpha(b) = 0$ implies $1ab = ab = 0$, where 1 is the identity of R. Sufficiency : let R be an α-rigid ring. We first show that R is a reduced ring. For if $a^2 = 0$ for $a \in R$, then $a\alpha(a)(\alpha(a\alpha(a)) = a\alpha(a^2)\alpha^2(a) = 0$ implies $a\alpha(a) = 0$ and so $a = 0$. To complete the proof we show that $ab\alpha(b) = 0$ for a, $b \in R$ implies $ab = 0$. Since R is reduced, from $ab\alpha(b) = 0$ we have $bab\alpha(bab) = 0$. Hence $bab = 0$ and $(ab)^2 = 0$. Thus $ab = 0$. □

Theorem 2.1. *For a module M we have the following:*
(1) *If M is α-reduced, then M is α-symmetric. The converse holds if M is rigid.*
(2) *If M is α-symmetric, then M is α-semicommutative. The converse holds if M is α-rigid.*

Proof. (1) Let $mab = 0$ where $m \in M, a, b \in R$. By [**7**, Lemma 1.2 (2)(a)] any α-reduced module is α-semicommutative, then $mR\alpha(ab) = 0$ and by the condition (2) of α-reduced module, $mRab = 0$. Hence $mbab = 0$ and so $m(ba)^2 = 0$. By [**7**, Lemma 1.2 (2)(c)], we have $mba = 0$. So M is α-symmetric. For the converse, assume that M is α-symmetric and rigid module and $ma = 0$ for $m \in M$ and $a \in R$. Let $mr_1 = m_1a \in mR \cap Ma$ where $r_1 \in R$, $m_1 \in M$. Being M α-symmetric we have $0 = mar_1 = mr_1a = m_1a^2$. Since M is rigid, $m_1a = 0$. The rest is clear.

(2) Necessity: let $ma = 0$. Then for any $r \in R$, $mar = 0$. Since M is α-symmetric, $mra = 0$. By the definition of α-symmetric module, $mr\alpha(a) = 0$. Then $mR\alpha(a) = 0$. Therefore M is α-semicommutative.

Sufficiency: note first that for any $m \in M$ and $a \in R$ with $ma = 0$, by hypothesis $mR\alpha(a) = 0$, and $ma = 0$ if and only if $ma\alpha(a) = 0$. Let $m \in M$ and a, $b \in R$ with $mab = 0$, we prove $mb\alpha(a) = 0$. We apply these facts to $mab = 0$ in turn to have $0 = mab = ma\alpha(b) = m\alpha(b)a\alpha(b)a = m(\alpha(b)a)\alpha(\alpha(b)a) = m\alpha(b)a = m\alpha(b)\alpha(\alpha(a))$ $= m\alpha(b\alpha(a)) = mb\alpha(a)$. Hence M is α-symmetric. □

The next example shows that the converse implication of the first statement in Theorem 2.1(1) is not true in general.

Example 2.3. Let $\mathbb{Z}_4$ denote the ring of integers modulo 4. Consider the ring $R = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{Z}_4\right\}$ and the right R-module $M = \left\{\begin{pmatrix} 0 & m \\ n & k \end{pmatrix} | m, n, k \in \mathbb{Z}_4\right\}$ and a homomorphism $\alpha : R \to R$ is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. We prove that M is an α-symmetric module but not α-reduced. For if $m = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \in M$ and $r = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in R$. Then $mr = 0$ but $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in mR \cap Mr$. Hence M is not an α-reduced module. We show M is α-symmetric. Since R is commutative for any $m \in M$ and $r, s \in R$, $mrs = 0$ implies $msr = 0$. To complete the proof we check, for any $m \in M$ and $r \in R$, $mr = 0$ if and only if $m\alpha(r) = 0$. We prove one way implication. The other way is similar. So let $m = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} \in M$, $r = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$. Assume that $mr = 0$ and m and r are non-zero. Then we have the equalities

$$(*) \qquad xa = 0, \quad ya = 0, \quad yb + za = 0.$$

If $y = 0$, then an easy calculation reveals that $m\alpha(r) = 0$. Suppose $y \neq 0$. If $a = 0$ then $m\alpha(r) = 0$. Assume $a \neq 0$. In this case the only solution in $\mathbb{Z}_4$ of this equality $ya = 0$ in (*) is that $y = 2$ and $a = 2$. Hence $m\alpha(r) = 0$.

Similarly, the converse implication of the first statement in Theorem 2.1(2) is not true in general, that is, there are α-semicommutative modules which are not α-symmetric.

Example 2.4. Let F be any field and consider the ring R and the module M as

$$R = \left\{\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in F\right\} \quad \text{and} \quad M = \begin{pmatrix} 0 & 0 & F \\ 0 & F & F \\ F & F & F \end{pmatrix}.$$

Define $\alpha : R \to R$, $\alpha\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. Let $m = \begin{pmatrix} 0 & 0 & x \\ 0 & y & z \\ k & l & u \end{pmatrix} \in M$ and $r = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in R$ with $mr = 0$. Then it is easy to check that if $a = 0$ then $\alpha(r) = 0$, and if $a \neq 0$, then $m = 0$. Hence $mR\alpha(r) = 0$. Let e_{ij} denote the 3×3 matrix units having a lone 1 as its (i,j)-entry and all other entries 0, and $m = e_{31} \in M$, $a = e_{23}$, $b = e_{12} + e_{13} + e_{23} \in R$. Then $mab = 0$. But $mba = e_{33}$ is a nonzero element of M. Let $m = e_{31} \in M$ and $a = e_{13} \in R$. Then $ma \neq 0$ but $\alpha(a) = 0$ and $m\alpha(a) = 0$. So M does satisfy neither first condition nor the second condition of the definition of α-symmetric module.

Recall that *singular submodule* $Z(M)$ of a module M consists of all elements having right annihilator in R essential as a right ideal. A module M is called *nonsingular* if $Z(M) = 0$.

Theorem 2.2. *Let M be a nonsingular module. Then M is α-reduced if and only if M is α-symmetric.*

Proof. Necessity is clear from Theorem 2.1(1). Sufficiency follows from [**9**, Theorem 4.2]. □

Theorem 2.3. *A ring R is α-symmetric if and only if every flat module M_R is α-symmetric.*

Proof. Necessity: let M be a flat module over the α-symmetric ring R and $0 \to K \to F \to M \to 0$ a short exact sequence with F free right R-module. By Lemma 2.1 F is an α-symmetric module and we write $M = F/K$ and any element $\overline{y} = y + K \in M$ for $y \in F$. Let $\overline{y}ab = 0$ and $\overline{y}c = 0$, where $\overline{y} \in M$ and $a,b,c \in R$. We want to show $\overline{y}ba = 0$ and $\overline{y}\alpha(c) = 0$. Since $\overline{y}ab = 0$ and $\overline{y}c = 0$, $yab \in K$ and $yc \in K$. Since M is flat, there exists a homomorphism $\theta : F \to K$ with $\theta(yab) = yab$, $\theta(yc) = yc$. Set $u = \theta(y) - y \in F$. Then $uab = 0$ and $uc = 0$. Since F is α-symmetric, $uba = 0$ and $u\alpha(c) = 0$. Then $\theta(yba) = yba$ and $\theta(y\alpha(c)) = y\alpha(c)$. Since $\theta(y) \in K$, we have $yba \in K$ and $y\alpha(c) \in K$. Therefore, $\overline{y}ba = 0$ and $\overline{y}\alpha(c) = 0$. We use this same method to prove other implication. So assume that $\overline{y}\alpha(a) = 0$ for some $\overline{y} \in M$ and $a \in R$. Then $y\alpha(a) \in K$. There exists a homomorphism $\gamma : F \to K$ with $\gamma(y\alpha(a)) = y\alpha(a)$. Let $v = \gamma(y) - y$. Then $v \in F$ and $v\alpha(a) = 0$. Since F is α-symmetric, $va = 0$. Hence $\gamma(ya) = ya \in K$. Thus $\overline{y}a = 0$. Sufficiency is clear. □

A *regular element* of a ring R means a nonzero element which is not zero divisor.

Let S be a multiplicatively closed subset of R consisting of regular central elements. We may localize R and M at S and we may seek when the localization $S^{-1}M_{S^{-1}R}$ is α-symmetric. If $\alpha : R \to R$ is a homomorphism of the ring R, then $S^{-1}\alpha : S^{-1}R \to S^{-1}R$ defined by $S^{-1}\alpha(a/s) = \alpha(a)/s$ is a homomorphism of the ring $S^{-1}R$. Clearly this map extends α and we shall also denote this map by α.

Proposition 2.3. *Let S be a multiplicatively closed subset of R consisting of regular central elements. A module M_R is α-symmetric if and only if $S^{-1}M_{S^{-1}R}$ is α-symmetric.*

Proof. Assume that M_R is α-symmetric and $(m/s)(a/t)(b/r) = 0$ in $S^{-1}M$ where $m/s \in S^{-1}M, a/t, b/r \in S^{-1}R$. Then $mab = 0$ and by the assumption $mba = 0$ and $mb\alpha(a) = 0$. Therefore $(m/s)(b/r)(a/t) = 0$ and $(m/s)(b/r)\alpha(a/t) = (m/s)(b/r)(\alpha(a)/t) = 0$. The rest is clear. □

In [7] Lee and Zhou introduced the following notation. For a module M, we consider $M[x] = \left\{ \sum_{i=0}^{s} m_i x^i \ : \ s \geq 0, m_i \in M \right\}$, $M[x]$ is an Abelian group under an obvious addition operation. Moreover $M[x]$ becomes a right $R[x]$-module under the following scalar product operation:

For $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{i=0}^{t} a_i x^i \in R[x]$,

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i a_j \right) x^k.$$

By these operations $M[x]$ becomes a right module over $R[x]$. In the same way, the Laurent polynomial extension $M[x, x^{-1}]$ becomes a right module over $R[x, x^{-1}]$ with a similar scalar product.

Corollary 2.1. *For a module M, $M[x]_{R[x]}$ is α-symmetric if and only if $M[x, x^{-1}]_{R[x,x^{-1}]}$ is α-symmetric.*

Proof. Let $S = \{1, x, x^2, ...\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of regular central elements of $R[x]$. Since $S^{-1}M[x] = M[x, x^{-1}]$ and $S^{-1}R[x] = R[x, x^{-1}]$, the result is clear from Proposition 2.3. □

Proposition 2.4. *Let M be an α-symmetric module and $m \in M, a_i \in R$. Then we have the following:*

(1) $ma_1 \ldots a_n = 0$ *if and only if* $ma_{\sigma(1)} \ldots a_{\sigma(n)} = 0$, *where* $n \in \mathbb{N}$ *and* $\sigma \in S_n$,

(2) $ma_1a_2\dots a_n = 0$ *if and only if* $m\alpha^{i_1}(a_1)\alpha^{i_2}(a_2)\dots\alpha^{i_n}(a_n) = 0$ *for any* $i_1,\dots,i_n \in \mathbb{N}$.

Proof. (1) For $n = 1$ the claim is evident. The case $n = 2$ follows from M being α-symmetric. Let $n = 3$ and $ma_1a_2a_3 = 0$. Since M is α-symmetric, $ma_1a_2a_3 = m(a_1)(a_2a_3) = 0$ implies $m(a_2a_3)a_1 = 0$. Also, using α-symmetry of M, $(ma_2)(a_3)(a_1) = 0$ implies $(ma_2)(a_1)(a_3) = 0$. Therefore, our claim holds for $\sigma_1 = (123)$ and $\sigma_2 = (12)$. Any other element of S_3 is a composition of cycles σ_1 and σ_2, so the case $n = 3$ is completed. For $n > 3$ it is enough to note that $S_n = \langle(12), (12\dots n)\rangle$ and to apply associativity of multiplication in R.

(2) It is sufficient to prove that $ma_1\dots a_{i-1}a_ia_{i+1}\dots a_n = 0$ if and only if $ma_1\dots a_{i-1}\alpha(a_i)a_{i+1}\dots a_n = 0$ for any i. Since M is an α-symmetric module, using (1), it can be easily proved. □

Let $T(M)$ denote the set of all torsion elements of a module M, that is, $T(M) = \{m \in M \mid ma = 0 \text{ for some nonzero } a \in R\}$.

Theorem 2.4. *Let R be a ring with no zero-divisors. Then we have the following:*

(1) *If M is an α-symmetric module, then $T(M)$ is an α-symmetric submodule of M.*

(2) *M is an α-symmetric module if and only if $T(M)$ is an α-symmetric module.*

Proof. (1) First we show that $T(M)$ is a submodule of M. For if $m_1, m_2 \in T(M)$ and $r \in R$, then we prove that $m_1 - m_2$ and m_1r belong to $T(M)$. There exist t_1, $t_2 \in R$ with $m_1t_1 = 0$ and $m_2t_2 = 0$. Since any symmetric module is semicommutative, we have $m_1Rt_1 = 0$ and $m_2Rt_2 = 0$. In particular $m_1t_2t_1 = 0$ and $m_2t_2t_1 = 0$. Then $(m_1 - m_2)t_2t_1 = 0$ and so $m_1 - m_2 \in T(M)$. Assume that $m_1t_1 = 0$. Then $m_1Rt_1 = 0$. Hence $m_1r \in T(M)$ for all $\in R$. By Proposition 2.1, α-symmetric modules are closed under submodules, $T(M)$ is also an α-symmetric module.

(2) One way is clear from (1). For the other way, let $0 \neq m \in M$ and $0 \neq a, b \in R$ with $mab = 0$. Since $m \in T(M)$ and $T(M)$ is α-symmetric and R has no zero-divisors, we have $mba = 0$. It completes the proof. □

Theorem 2.5. *Let R be a ring with no nonzero zero-divisors. If M is an α-symmetric module, then $M/T(M)$ is an α-symmetric module.*

Proof. Let $\overline{m}$ be any element of $M/T(M)$ with $m \in M$. Suppose $\overline{m}ab = 0$, for any $a, b \in R$ and $m \in M$. So there exist $r \in R$ such that $mabr = 0$. By hypothesis and Lemma 2.4, we have $marb = mbar = 0$. Hence $\overline{m}ba = 0$. Now the proof of the rest is clear since to prove for any $\overline{m} \in M$ and $a \in R$, $\overline{m}a = 0$ if and only if $\overline{m}\alpha(a) = 0$ is routin. So $M/T(M)$ is an α-symmetric module. □

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