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Spanned vector bundles with canonical determinant on special curves

Abstract. Let C be a smooth curve of genus g. Here we construct (under geometric restrictions, like C hyperelliptic or a complete intersection) spanned rank n vector bundles E on C with canonical determinant and with a $(2n+1)$-dimensional linear subspace $W \subseteq H^0(E)$ such that the natural wedge map $\bigwedge^n (W) \rightarrow H^0(\det(E))$ is injective. The motivation came from a paper by Pirola and Rizzi, who used (E, W) to get certain non-trivial higher cycle maps on the relative jacobian of an n-dimensional family of curves $\mathcal{C} \rightarrow S$ with C as a fiber.

Keywords. Spanned vector bundle; canonical determinant; higher cycle map; Jacobian; Griffiths group.

Mathematics Subject Classification (2000): 14H60, 14H40, 14C25, 14M15.

1 - Introduction

Let C be a smooth and connected complex projective curve of genus $g \geq 3$. Let $A(n)$ be the set of all rank n spanned vector bundle E on C such that $\det(E) \cong \omega_C$, $h^0(C, E) \geq 2n+1$, and $h^0(C, E^*) = 0$. Since E is assumed to be spanned, the last condition is equivalent to assuming that $\mathcal{O}_C$ is not a direct factor of E. For any $E \in A(n)$ and any linear subspace $W \subseteq H^0(C, E)$ let $\phi_W : \bigwedge^n (W) \rightarrow H^0(C, \omega_C)$ denote the determinantal map. Let $B(n)$ denote the set of all pairs (E, W) such that $E \in A(n)$, W is a linear subspace of E, $\dim(W) = 2n+1$, and W spans E. Set

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$D(n) := \{(E, W) \in B(n) : \phi_W$ is injective$\}$. Since $\dim(W) = 2n+1$ and $\dim(\bigwedge^n (W)) = \binom{2n+1}{n}$, $g \geq \binom{2n+1}{n}$ if $D(n) \neq \emptyset$.

Theorem 1. *Assume C hyperelliptic. $D(n) \neq \emptyset$ if and only if $g \geq \binom{2n+1}{n}$.*

Theorem 2. *Assume $2g - 2 \geq 3 \cdot \binom{2n+1}{n}$ and that C is trigonal with Maroni invariant $(g-1)/4$ ([7], §1). Then $D(n) \neq \emptyset$.*

To get results for other curves the following definition ([2], [3]).

Definition 1. A line bundle L on C is said to be *primitive* if both L and $\omega_C \otimes L^*$ are spanned.

Theorem 3. *Fix an integer $n \geq 2$ and set $d := \binom{2n+1}{n} - 1$. Assume the existence of a spanned $R \in Pic(C)$ such that $R^{\otimes d}$ is primitive. Then $B(n) \neq \emptyset$.*

The primitivity of $R^{\otimes d}$ in the statement of Theorem 3 implies $d \cdot \deg(R) \leq 2g - 2$. We recall two classical cases which satisfy the assumptions of Theorem 3.

Example 1. Fix integers $r \geq 2$ and $d_i \geq 2$, $1 \leq i \leq r-1$. Let $C \subset \mathbb{P}^r$ be a smooth complete intersection of hypersurfaces of degree $d_1, \ldots, d_{r-1}$. The adjunction formula gives $\omega_C \cong \mathcal{O}_C(d_1 + \cdots + d_{r-1} - r - 1)$. Hence C satisfies the assumptions of Theorem 3 taking $R := \mathcal{O}_C(1)$ if $d \leq d_1 + \cdots + d_{r-1} - r - 1$.

Example 2. Let A be a rank 2 vector bundle on $\mathbb{P}^3$ such that there is $s \in H^0(\mathbb{P}^3, E)$ whose zero-locus $(s)_0$ is a smooth and connected curve C. Then C has degree $c_2(E)$ and $\omega_C \cong \mathcal{O}_C(c_1(E) - 4)$ ([4], proof of Proposition 2.1). Hence C satisfies the assumptions of Theorem 3 taking $R := \mathcal{O}_C(1)$ if $d \leq c_1(E) - 4$.

The motivation for these results came from a paper of G. P. Pirola and C. Rizzi in which they proved the geometric significance of the condition $D(n) \neq \emptyset$ ([8]). Since we will not need their set-up, we just state as a corollary the following immediate consequence of Theorems 1 and 2 and of [8], Theorem 2.2.

Corollary 1. *Fix integers $g, n \geq 2$ and a curve C as in the statements of Theorems* 1, 2 *or* 3. *Then there are an n-dimensional variety S, $s \in S$, a family $f : \mathcal{C} \to S$ of smooth curves such that $f^{-1}(s) \cong C$ and the adjunction map (equation* 3 *of* [8]) *is not trivial.*

Theorems 1, 2 and 3 also give non-trivial elements in the Griffiths group $\mathcal{W}_s^n(\mathcal{J}_s)$ (see [**6**] and [8], Theorem 5.5). Of course, everything in this paper depends from [8]. As in [8] all pairs $(E, W) \in D(n)$ we construct have as E a direct sum of line bundles.

Remark 1. Almost everything here works without any modification if we drop the assumption $\det(E) \cong \omega_C$ (here only the degree d of E is an important datum) and we allow W of an arbitrary, but fixed, dimension m). We get lower bounds on g depending from g, d, m and (if d is low) the geometry of C (see [**8**], §1 and §2, for this set-up). We leave this easy extension to the interested reader. If we don't require that W spans E, then the construction of examples are easier. Similarly, if we assume that E is spanned, but drop the condition "$\det(E) \cong \omega_C$" (just requiring $h^1(\det(E)) > 0$), we get example for all k-gonal curves in Theorem 3 without assuming the primitivity of $R^{\otimes d}$. We only need to require that $R^{\otimes d}$ is special.

2 - The proofs

In the next lemma we will use the geometric interpretation of the wedge map for curves in Grassmannian ([**10**], [**1**]).

Lemma 1. *Fix integers $m > n > 0$, and $d > 0$. Let $G(n, m)$ denote the Grassmannian of all $(m-n)$-dimensional linear subspaces of $\mathbb{C}^{\oplus m}$. Let*

$$0 \to S \to \mathcal{O}_{G(m-n,n)}^{\oplus m} \to Q \to 0 \tag{1}$$

be the Euler sequence of $G(m-n, m)$. Hence Q is the tautological quotient bundle, S is the tautological subbundle, rank$(Q) = n$, rank$(S) = m - n$, and $\det(Q) \cong \det(S^*) \cong \mathcal{O}_{G(m-n,m)}(1)$. *Let $j : G(m-n, n) \hookrightarrow \mathbb{P}^N$, $N := \binom{m}{n} - 1$, be the Plücker embedding. There is a curve $X \subset G(m-n, n)$ such that $X \cong \mathbb{P}^1$, $\deg(X) = d$, the linear span $\langle j(X) \rangle$ of $j(X)$ has dimension* $\min\{d, N\}$, *and the vector bundle $E := Q|X$ is rigid, i.e. its splitting type $a_1 \geq \cdots \geq a_n$ satisfies $a_1 := \lceil d/n \rceil$ and $a_n = \lfloor d/n \rfloor$.*

Proof. The case $d = 1$ is obvious, because $G(n-m, m)$ contains lines. Hence we will assume $d \geq 2$. Let $A(n, d, 0)$ be the set of all isomorphism classes of rank n vector bundles on $\mathbb{P}^1$ with degree d. Any rank n vector bundle E on $\mathbb{P}^1$ is uniquely determined by its splitting type $a_1(E) \geq \cdots \geq a_n(E)$. $E \in A(n, d, 0)$ if and only if

$a_n \geq 0$ and $a_1 + \cdots + a_n = d$. Note that $h^0(\mathbb{P}^1, E) = d + n$ and $h^1(\mathbb{P}^1, E) = 0$ for all $E \in A(n, d, 0)$. Hence the set $G(n-m, H^0(E))'$ of all $(n-m)$-dimensional linear subspaces of $H^0(\mathbb{P}^1, E)$ spanning E is a non-empty open subset of the Grassmannian $G(m-n, d+n)$. Hence $\dim(G(m-n, H^0(E))')$ does not depend from the choice of $E \in A(n, d, 0)$. Let $T(n, m, d)$ denote the set of all degree d maps $\mathbb{P}^1 \to G(m-n, n)$. Any $\phi \in T(n, m, d)$ is uniquely determined by the choice of $E \in A(n, d, 0)$ and an n-dimensional linear subspace V of $H^0(\mathbb{P}^1, E)$ spannig E. Let $D \subset G(m-n, m)$ be the line. Since $Q|D$ has splitting type $1 \geq 0 \geq \cdots \geq 0$, $S^*|D$ has splitting type $1 \geq 0 \geq \cdots \geq 0$, and $TG(m-n, m) \cong Q \otimes S^*$, the vector bundle $TG(n-m, m)|D$ has splitting type $2 \geq 1 \geq \cdots \geq 0$ in which 1 appears $n-2$ times and 0 appears $(m-n-1)(m-1)$ times. Hence the normal bundle $N_{D,G(m-n,m)}$ of D in $G(m-n, m)$ has splitting type $1 \geq \cdots \geq 0$ in which 1 appears $n-2$ times and 0 appears $(m-n-1)(m-1)$ times. A chain $T \subset G(n-m, m)$ of d lines is a nodal union $D_1 \cup \cdots \cup D_d$ of d distinct lines such that $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Note that $\deg(T) = d$ and $p_a(T) = 0$ for any such T. Let $C(n, m, d)$ be the set of all chains of d lines. Set $C'(n, m, d) := \{T \in C(n, m, d) : \dim(\langle j(T) \rangle) = \min\{N, d\}\}$. $C'(n, m, d)$ is a non-empty open subset of the irreducible variety $C(n, m, d)$. Fix any $T = A_1 \cup \cdots \cup A_d \in C'(n, m, d)$. The given ordering of the lines of T has the property that each curve $T_i := D_1 \cup \cdots \cup D_i$, $2 \leq i < d$, is a chain of i lines and D_{i+1} intersects transversally T_i at a unique point, P, which belongs to D_i. Since $h^1(D_i, N_{D_i,G(n-m,m)}(-P)) = 0$, we see by induction on d that each chain of lines is smoothable ([**5**] or [**9**]). Fix any $\phi \in T(n, m, d)$ such that $\phi(\mathbb{P}^1)$ is a small deformation of $T \in C'(n, m, d)$. Hence $Y := \phi(\mathbb{P}^1)$ is a smooth and rational degree d curve. By semicontinuity we may also assume that $\dim(\langle j(Y) \rangle) \geq \dim(\langle j(T) \rangle) = \min\{d, N\}$. Since $h^0(Y, \mathcal{O}_Y(1)) = d + 1$, we get $\dim(\langle j(Y) \rangle) = \min\{d, N\}$. We may deform $Q|Y$ to a rigid vector bundle. Since $h^0(\mathbb{P}^1, E) = d + n$ and $h^1(\mathbb{P}^1, E) = 0$ for all $E \in A(n, d, 0)$, there is an embedding $X \subset G(n-m, m)$ of $\mathbb{P}^1$ with image near Y (and hence with $\dim(\langle j(X) \rangle) = \min\{d, N\}$) such that $Q|X$ is rigid. □

Proof of Theorem 1. We saw in the introduction that if $D(n) \neq \emptyset$, then $g \geq \binom{2n+1}{n}$. Assume $g \geq \binom{2n+1}{n}$. Let $j : G(n+1, 2n+1) \hookrightarrow \mathbb{P}^N$, $N := \binom{2n+1}{n} - 1$, be the Plücker embedding. Let $T \subset G(n+1, 2n+1)$ be a smooth rational curve such that $\deg(T) = \binom{2n+1}{n}$ and $\langle j(T) \rangle = \mathbb{P}^N$ (Lemma 1). Since $h^0(T, \mathcal{O}_T(1)) = N + 1$, $j(X)$ is linearly normal. Hence there is a $(2n+1)$-dimensional linear subspace V of $H^0(T, Q|T)$ such that $\phi_V : \bigwedge^n(V) \to H^0(T, \mathcal{O}_T(1)) = 0$. Let $a_1 \geq \cdots \geq a_n$ be the splitting type of $Q|T$. Lemma 1 gives $a_n = \lfloor \binom{2n+1}{n}/n \rfloor$. We only need $a_n > 0$. Let $R \in \mathrm{Pic}^2(C)$ be the hyperelliptic line bundle. Hence the linear system $|R|$ induces a degree 2 morphism $u : X \to T$. Set $F := u^*(Q|T)$ and $M := u^*(V) \subseteq H^0(E)$. Since ϕ_V is injective, $\phi_M : \bigwedge^n (M) \to H^0(\det(E))$ is injective. M spans E. Since $F \cong \oplus_{i=1}^n R^{\otimes a_i}$, $\det(F) \cong R^{\otimes x}$, where $x := \binom{2n+1}{n}$. We assumed $x \leq g-1$. Set $b_1 := a_i$ for $1 \leq i \leq n-1$, and $b_n := a_n + (g-1-x)$ and take any fixed $D \in |R^{\otimes(g-1-x)}|$ to see $R^{\otimes a_n}$ as a subsheaf of $R^{\otimes b_n}$. D also gives an inclusion $F \cong \oplus_{i=1}^n R^{\otimes a_i} \hookrightarrow \oplus_{i=1}^n R^{\otimes b_i} =: E$. This inclusion allows M to be seen as a linear subspace M' of $H^0(E)$. Since $\sum_{i=1}^n = g-1$, $\det(E) \cong \omega_C$. Since $b_n \geq a_n > 0$, $h^0(E^*) = 0$. Since ϕ_M is injective, $\phi_{M'}$ is injective. M' does not span E if $g - 1 \neq$. However, the injectivity of $\phi_{M'}$ implies the injectivity of ϕ_W for a general $(2n+1)$-dimensional linear subspace of $H^0(E)$. Since E is spanned, W is general, and $\dim(W) > \mathrm{rank}(E)$, W spans E. □

Proof of Theorem 2. Let $R \in \mathrm{Pic}^3(C)$ be the trigonal line bundle. By assumption $g - 1 \equiv 0 \bmod 3$ and $\omega_C \cong R^{\otimes(2g-2)/3}$. By assumption $(2g-2)/3 \geq \binom{2n+1}{3}$. Take $T \subset G(n+1, 2n+1)$ as in the proof of Theorem 1 and copy that proof taking as $u : X \to T$ the degree 3 morphism associated to $|R|$. □

Proof of Theorem 3. Use R as in the proofs of Theorems 1 and 2. □

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