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On mortality in a two-sex population

Abstract. In this paper we consider the mathematical model of the dynamics of a two-sex population with gestation period presented by Busoni and Palczewski [**1**]. We solve a system of two differential equations with delay coupled by boundary conditions to obtain the densities of males and nonpregnant females. Then, we invert the problem to obtain the male mortality coefficient, assuming the density and the mortality coefficient of females known; we possibly get a unique solution. We also look for the female mortality coefficient, assuming the density and the mortality coefficient of males known. We use the method of successive approximations which leads to the uniqueness of the solution if it converges.

Keywords. Population dynamics, Partial differential equations with delay, Coefficient of mortality, Inverse problem.

Mathematics Subject Classification (2000): 92D25, 35R30.

1 - Introduction

In this work we initially report the mathematical model of dynamics of a two-sex population with gestation period developed by L. Teglielli (see [**10**]) and G. Busoni and A. Palczewski (see [**1**]) with the modification proposed by C. Fregoso (see [**3**]) concerning the expression of the density of mating. In this model we assume that individuals are characterized by sex (male or female) and age. Two individuals of opposite sex can mate and conceive a new individual. The births of new individuals

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take place after a gestation period. In this model the formation of pairs is neglected: every nonpregnant female can mate any male.

The model consists of a system of two partial differential equations, which describe the evolution of the densities of males and nonpregnant females, coupled by boundary and initial conditions. In Section 2 we report briefly the solutions of this direct problem. In Section 3 we study the inverse problem: we possibly get the unique coefficient of male mortality, letting the densities of male population at a certain time and the density and the coefficient of female mortality known (see also [**2, 6, 7, 8, 9**]). Finally, assuming the density of male population and the coefficient of male mortality known, we solve the inverse problem for the coefficient of female mortality using the method of successive approximations which may lead to a solution if the coefficients appearing in the model are suitably related each other. We do not examine this item here.

We can start the description of the model as in [**1**]. Let the numerical densities of individuals (s_m for males, s_f for females) be functions of their own age (a for males, b for females), of the (virtual) age u of the male parent and v of the female parent and of time t. The age of parents is virtual, as they could be dead: u and v represent their ages if they were still alive. Therefore, $s_m(a, u, v, t)$ is the distribution at time t of males of age a, born by a father whose age at time t would be u and by a mother whose age at time t would be v. The same holds for the female distribution $s_f(b, u, v, t)$. Of course the ages of males and females are bounded: $a \leq a_l < \infty$ and $b \leq b_l < \infty$. Note that not only the virtual age of parents is necessarily greater than a and b, $u > a$, $v > a, u > b, v > b$, but also it has a greater upper bound, that is $u \leq 2a_l$ and $v \leq 2b_l$. The total number of males and of females at time t and of a given age (a or b) are respectively:

$$M(a,t) = \int_0^{a_l}\int_0^{b_l} s_m(a,u,v,t)\,du\,dv\,, \tag{1}$$

$$F(b,t) = \int_0^{a_l}\int_0^{b_l} s_f(b,u,v,t)\,du\,dv\,. \tag{2}$$

Let $\mu_m(a)$ and $\mu_f(b)$ be the nonnegative mortalities of males and females respectively. Let g be the gestation time. These functions are also defined for negative value of the age, taking into account the mortality of foetuses. Note that we consider $\mu_m(a) = 0$ for $a < -g$ and similarly $\mu_f(b) = 0$ for $b < -g$, if necessary.

To model the mating process, we distinguish between two types of females: say $H(b, t)$ the distribution of pregnant females and $N(b, t)$ that of nonpregnant ones. Of course $H(b, t) + N(b, t) = F(b, t)$. Since in the following sections we will consider a nonlinear system for the distributions $M(a,t)$ and $N(b,t)$, we remark that the distributions s_m and s_f are introduced in order to construct a meaningful model. Let $k(a,b)$ be the mating factor (it describes the probability that a conception happens, depending on the parents' age). Note that $k(a,b) \geq 0$ for $a \in (0,a_l]$ and $b \in (0,b_l]$, while $k(a,b) = 0$ for $a<0$ or $b<0$. According to [**3**], we assume that the density of mating $\phi(a,b,t)$ has the following expression:

$$\phi(a,b,t) = 2k(a,b)M(a,t)N(b,t)\,. \tag{3}$$

Obviously, $\phi = 0$ if a or b are negative. We assume the density of mating (3) because it is the simplest nonlinear function of $M(a,t)$ and $N(b,t)$, while in literature, see [**5**], a homogeneous of order 1 function is assumed in order to have a bounded nonlinear term.

The expression for the distribution of pregnant females of age b at time t is:

$$H(b,t) = \int_{t-g}^{t}\int_{0}^{a_l} \phi(u, b-(t-s), s)\exp\left(-\int_{s}^{t}\mu_f(b-t+\tau)d\tau\right)du\,ds. \tag{4}$$

The model is described by the following equations:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)M(a,t) = -\mu_m(a)M(a,t) \quad \text{for } a\in(0,a_l]\,,\ \ t>0\,, \tag{5}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right)N(b,t) &= -\mu_f(b)N(b,t) - \int_0^{a_l}\phi(a,b,t)\,da\\ &+\exp\left(-\int_{-g}^{0}\mu_f(b+\tau)d\tau\right)\int_0^{a_l}\phi(a,b-g,t-g)\,da \quad \text{for } b\in(0,b_l]\,,\ t>0,\end{aligned} \tag{6}$$

coupled by the boundary conditions, for $t > 0$:

$$M(0,t) = \int_g^{a_l}\int_g^{b_l} s_m(0,u,v,t)\,du\,dv\,, \tag{7}$$

$$N(0,t) = F(0,t) = \int_g^{a_l}\int_g^{b_l} s_f(0,u,v,t)\,du\,dv \tag{8}$$

where $s_m(0,u,v,t)$ and $s_f(0,u,v,t)$ represent the numbers of newborn males and females, respectively, conceived at time $t-g$ by a father of age $u-g$ and a mother of age $v-g$ and survived the period of gestation $[t-g,t]$. The expressions of

$s_m(0,u,v,t)$ and $s_f(0,u,v,t)$ are given by:

$$s_m(0,u,v,t) = \beta_m(u-g, v-g)\,\phi(u-g, v-g, t-g) \times \exp\left(-\int_{t-g}^{t} (\mu_m(s-t) + \mu_f(v-t+s))ds\right), \tag{9}$$

$$s_f(0,u,v,t) = \beta_f(u-g, v-g)\,\phi(u-g, v-g, t-g) \times \exp\left(-\int_{t-g}^{t} (\mu_f(s-t) + \mu_f(v-t+s))ds\right), \tag{10}$$

where $\beta_m(a,b)$ and $\beta_f(a,b)$ represent the number of males and females, respectively, which have been conceived in the mating of a male of age a and a nonpregnant female of age b. We assume $\beta_m = 0$ and $\beta_f = 0$ when $a<0$ or $b<0$. Thus the boundary conditions (7) and (8) at $a = 0$ and $b = 0$ become:

$$M(0,t+g) = \int_0^{a_l-g}\int_0^{b_l-g} \beta_m(u,v)\,\phi(u,v,t) \times \exp\left(-\int_0^{g} (\mu_m(s-g) + \mu_f(v+s))ds\right) du\,dv \quad \text{for } t > -g, \tag{11}$$

$$N(0,t+g) = \int_0^{a_l-g}\int_0^{b_l-g} \beta_f(u,v)\,\phi(u,v,t) \times \exp\left(-\int_0^{g} (\mu_f(s-g) + \mu_f(v+s))ds\right) du\,dv \quad \text{for } t > -g. \tag{12}$$

Note that (6) is an equation with delay. This implies that the initial distributions for $M(a,t)$ and $N(b,t)$ have to be given on the whole interval $[-g, 0]$. Therefore, the model is completed by the two following initial conditions:

$$M(a,t) = \widehat{M}_0(a,t) \quad \text{for } t \in [-g,0],\ a \in [0,a_l], \tag{13}$$

$$N(b,t) = \widehat{N}_0(b,t) \quad \text{for } t \in [-g,0],\ b \in [0,b_l] \tag{14}$$

where $\widehat{M}_0$ and $\widehat{N}_0$ are two nonnegative given functions.

Therefore, our model is constituted by equations (5), (6), (11), (12), (13) and (14).

We make the following assumptions, which allow us to consider the evolution problem for $a \in (-g, a_l)$ and $b \in (-g, b_l)$:

(a) the biological parameters $\mu_m(a)$, $\mu_f(b)$, $\beta_m(a,b)$, $\beta_f(a,b)$ and $k(a,b)$ are nonnegative measurable functions;

(b) the initial functions $\widehat{M}_0(a,t)$ and $\widehat{N}_0(b,t)$
1. are nonnegative;
2. belong to $L^\infty(0,a_l)$ and $L^\infty(0,b_l)$, respectively, for $t \in [-g, 0]$;
3. are continuous in the L^∞-norm with respect to t: it means that $[-g,0] \ni t \mapsto \widehat{M}_0(\cdot,t) \in L^\infty(0,a_l)$ is continuous in the norm of $L^\infty(0,a_l)$ and $[-g,0] \ni t \mapsto \widehat{N}_0(\cdot,t) \in L^\infty(0,b_l)$ is continuous in the norm of $L^\infty(0,b_l)$;

(c) the biological parameters are such that $k \in L^\infty((0,a_l)\times(0,b_l))$, $\beta_m \in L^\infty(0,a_l)$ and $\beta_f \in L^\infty(0,b_l)$; for sake of simplicity, we use the same bound K for the functions $2k\beta_m$, $2k\beta_f$ and $2k$:

$$0 \le \operatorname*{ess\,sup}_{a\in(0,a_l),b\in(0,b_l)} 2k(a,b)\beta_m(a,b) \le K\,, \tag{15}$$

$$0 \le \operatorname*{ess\,sup}_{a\in(0,a_l),b\in(0,b_l)} 2k(a,b)\beta_f(a,b) \le K\,, \tag{16}$$

$$0 \le \operatorname*{ess\,sup}_{a\in(0,a_l),b\in(0,b_l)} 2k(a,b) \le K\,; \tag{17}$$

(d) the mortality coefficients are such that:

$$\mu_m \in L^1(-g,a) \text{ for } a \in (-g,a_l), \quad \lim_{a\to a_l^-} \int_{-g}^{a} \mu_m(s)\,ds = +\infty\,, \tag{18}$$

$$\mu_f \in L^1(-g,b) \text{ for } b \in (-g,b_l), \quad \lim_{b\to b_l^-} \int_{-g}^{b} \mu_f(s)\,ds = +\infty\,; \tag{19}$$

(e) the mating factor k is uniformly continuous with respect to $b \in (0,b_l)$ for every $a \in (0,a_l)$;

(f) the initial function $\widehat{M}_0(a,0)$ for $t=0$ is uniformly continuous with respect to $a \in [0,a_l)$; the initial function $\widehat{N}_0(b,t)$ for $t \in [-g,0]$ is uniformly continuous with respect to $b \in [0,b_l)$.

In Section 2 we will prove existence and uniqueness of the solution of the integral formulation of the P.D.E. model. Section 3 is devoted to a first study of the inverse problem consisting of the determination of the functions μ_m and μ_f by assuming the knowledge of additional data. In this paper we consider $M(a,T)$ and $N(b,T)$ to be known at time $t = T > 0$.

The authors are aware of few papers on inverse problems in population dynamics (see [**2, 4, 6, 7, 8, 9**]), all of them being devoted to a one-sex population dynamics linear equation, i.e. an equation like that appearing in (5).

We are going to investigate the solution of the system of equations (5), (6), (11), (12), (13) and (14) in the following section.

2 - Existence and uniqueness of the solution

We consider the problem presented above as an evolution problem in the Banach space $L^1(0,a_l)\times L^1(0,b_l)$ for every $t>0$. In this work, a solution for the integral formulation of the problem is given, instead of a solution for the differential one. Since we are considering differential equations with delay, we are going to look for a solution on successive strips (intervals) in time $t\in(ig,(i+1)g]$. Indeed, it is possible to calculate $M(0,t)$ and $N(0,t)$ for t in every strip $(ig,(i+1)g]$ in function of the same quantities calculated in the preceding strip. In the following, the initial conditions for the i-th strip, that means for $t\in(ig,(i+1)g]$, will be denoted $\widehat{M}_i(a,t)$ and $\widehat{N}_i(b,t)$, where $t\in((i-1)g,ig]$. The solution of our problem will be found by integrating equations (5) and (6) along the characteristics, iterating on every strip. For male distribution, we obtain, in the i-th strip, that is for $t\in(ig,(i+1)g]$, $i\in\mathbb{N}\cup\{0\}$, and for $a\in(0,a_l]$:

$$M(a,t)=M(a_0,t_0)\exp\left(-\int_{a_0}^{a}\mu_m(s)\,ds\right) \tag{20}$$

where

$$(a_0,t_0)=(0,t-a)\qquad\text{for }\ a<t-ig\,, \tag{21}$$

and

$$(a_0,t_0)=(a-(t-ig),ig)\qquad\text{for }\ a\ge t-ig\,. \tag{22}$$

To obtain the solution for nonpregnant females, let us put equation (3) in (6), obtaining for $b\in(0,b_l]$ and $t>0$:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right)N(b,t)&=-\,\mu_f(b)N(b,t)-\int_0^{a_l}2k(a,b)M(a,t)\,da\,N(b,t)\\&\quad+\exp\left(-\int_{-g}^{0}\mu_f(b+\tau)d\tau\right)\int_0^{a_l}2k(a,b-g)M(a,t-g)daN(b-g,t-g).\end{aligned} \tag{23}$$

Putting

$$(24) \qquad A(b,t) =_{def} \int_0^{a_l} 2k(a,b)M(a,t)da \quad \text{and} \quad m(b,t) =_{def} \mu_f(b) + A(b,t),$$

equation (23) becomes, for $b \in (0, b_l]$ and $t > 0$:

$$(25) \qquad \begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right)N(b,t) &= -\, m(b,t)N(b,t) \\ &+ \exp\left(-\int_{-g}^{0} \mu_f(b+\tau)d\tau\right)A(b-g,t-g)N(b-g,t-g). \end{aligned}$$

Note that it results:

$$(26) \qquad A(b-g,t-g) = \int_0^{a_l} 2k(a,b-g)M(a,t-g)\,da = 0\,, \qquad \text{for} \quad b<g$$

because $k(a, b-g) = 0$ for $b - g < 0$. Introducing the following notation

$$(27) \qquad L(b; b-g, t-g) =_{def} \exp\left(-\int_{-g}^{0} \mu_f(b+\tau)\,d\tau\right) A(b-g,t-g)\,,$$

equation (25) becomes, for $b \in (0, b_l]$ and $t > 0$:

$$(28) \qquad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right)N(b,t) = -\, m(b,t)N(b,t) + L(b; b-g, t-g)N(b-g,t-g).$$

Obviously, see (26), it is

$$(29) \qquad L(b; b-g, t-g) = 0 \qquad \text{for} \;\; b<g\,.$$

Put also

$$(30) \qquad S(b,t) =_{def} L(b; b-g, t-g)N(b-g,t-g)\,.$$

Taking into account (29), it results

$$(31) \qquad S(b,t) = 0 \qquad \text{for} \;\; b<g\,.$$

Note that, since we are working on successive strips for $t \in (ig, (i+g)]$, $S(b,t)$ is a known function, as $L(b; b-g, t-g)$ is given and $N(b-g, t-g)$ comes from the preceding strip. The differential equation (28) for nonpregnant females becomes, for

$b \in (0, b_l]$ and $t > 0$:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right) N(b,t) = -m(b,t)N(b,t) + S(b,t)\,. \tag{32}$$

Integrating (32) along the characteristics, iterating on every strip, we obtain for $t \in (ig, (i+1)g]$, $i = 0, 1, \ldots$, and for $b \in (0, b_l]$:

$$N(b,t) = \begin{cases} N(0, t-b)\exp\left(-\int_0^b m(s, s+t-b)ds\right) & \text{for } b < t - ig, \\ \widehat{N}_i(b-t+ig, ig)\exp\left(-\int_0^{t-ig} m(s+b-t+ig, s+ig)ds\right) & \\ \quad + \int_0^{t-ig} \exp\left(-\int_s^{t-ig} m(\sigma+b-t+ig, \sigma+ig)\,d\sigma\right) & \\ \quad \times S(s+b-t+ig, s+ig)\,ds & \text{for } b \geq t - ig. \end{cases} \tag{33}$$

Note that the second addend of the solution for $b \geq t - ig$ is null if $b < g$. Indeed for $0 < s < t - ig \leq b < g$, it results $s + b - t + ig < t - ig + b - t + ig = b < g$, thus $S(s+b-t+ig, s+ig) = 0$ (see (31)). Therefore, if $t - ig \leq b < g$, the solution is constituted only by the first addend: $N(b,t) = \widehat{N}_i(b-t+ig, ig)\exp\left(-\int_0^{t-ig} m(s+b-t+ig, s+ig)\,ds\right)$.

We are going to enunciate the main results regarding the functions M and N obtained in (20) and (33). The proof of Lemma 2.1 is obvious.

Lemma 2.1. *The functions $M(a,t)$ and $N(b,t)$ obtained in* (20) *and* (33), *respectively, are nonnegative almost everywhere in the i-th strip, that is for $t \in (ig, (i+1)g]$, for every $i \in \mathbb{N} \cup \{0\}$.*

Lemma 2.2. *Let, for $t \in [-g, 0]$:*

$$\operatorname*{ess\,sup}_{a \in (0,a_l)} \widehat{M}_0(a,t) \leq \overline{M}_0 < +\infty, \quad \operatorname*{ess\,sup}_{b \in (0,b_l)} \widehat{N}_0(b,t) \leq \overline{N}_0 < +\infty. \tag{34}$$

Therefore, for $t \in (ig, (i+1)g]$, it follows:

$$\operatorname*{ess\,sup}_{a \in (0,a_l)} M(a,t) \leq \overline{M}_0 \prod_{s=0}^{i}\left(1 + K\overline{N}_s a_l b_l\right) < +\infty\,,$$

$$\operatorname*{ess\,sup}_{b \in (0,b_l)} N(b,t) \leq \overline{N}_i\left(1 + K\overline{M}_i a_l b_l + K\overline{M}_i a_l g\right) < +\infty\,,$$

where, for $i = 1, 2, \ldots,$ *it is:*

$$\overline{M}_i = \overline{M}_0 \prod_{s=0}^{i-1} \big(1 + K\overline{N}_s a_l b_l\big) < +\infty\,,$$

$$\overline{N}_i = \overline{N}_{i-1}\big(1 + K\overline{M}_{i-1} a_l b_l + K\overline{M}_{i-1} a_l g\big) < +\infty\,.$$

The proof may be read in the appendix.

Remark 2.1. *It results:* $\lim_{a\to a_l^-} M(a,t) = 0$ *and* $\lim_{b\to b_l^-} N(b,t) = 0$ *on every strip* $(ig, (i+1)g]$, *as it can be seen by considering* (20), (18), (33) *and* (19).

Remark 2.2. *Since* $\widehat{M}_0(a,t) \in L^\infty(0, a_l)$ *and* $\widehat{N}_0(b,t) \in L^\infty(0, b_l)$, *for* $t \in [-g, 0]$, *see hypothesis* (*b*)-2, *it also holds* $\widehat{M}_0(a,t) \in L^1(0, a_l)$ *and* $\widehat{N}_0(b,t) \in L^1(0, b_l)$, *for* $t \in [-g, 0]$. *Note that, for* $t \in [-g, 0]$,

$$\|\widehat{M}_0(a,t)\|_{L^1(0,a_l)} \le \overline{M}_0 a_l < +\infty \quad \textit{and} \quad \|\widehat{N}_0(b,t)\|_{L^1(0,b_l)} \le \overline{N}_0 b_l < +\infty. \tag{35}$$

Lemma 2.3. *If the quoted hypotheses hold, then functions* (20) *and* (33) *are such that* $M(\cdot,t) \in L^1(0, a_l)$ *and* $N(\cdot,t) \in L^1(0, b_l)$, *respectively, and they both are strongly continuous, for* $t \in (ig, (i+1)g]$, *for* $i \in \mathbb{N} \cup \{0\}$.

The proof may be read in the appendix. Therefore, we have proved that:

Theorem 2.1. *Under hypotheses* (*a*)-(*f*), *for every* $t \in (0, T]$, *where* $T < +\infty$, *equations* (20) *and* (33) *give the unique strongly continuous solution* $t \mapsto (M(a,t), N(b,t)) \in C\big((0,T], L^1((0,a_l)\times(0,b_l))\big)$, *for the problem given by* (5), (6), (11), (12), (13) *and* (14).

3 - The inverse problem

In this section we want to invert in some sense the problem studied in the previous sections. First, we look for the expression of the mortality coefficient of males, $\mu_m(a)$, assuming the knowledge of the density $M(a,T)$ of male population at a certain time $t = T > 0$, the density $N(b,t)$ and the mortality coefficient of females, $\mu_f(b)$. Then, we look for $\mu_f(b)$ assuming $N(b,T)$, $M(a,t)$ and $\mu_m(a)$ known. We are able to recover μ_m and to show that it is unique.

Remark 3.1. *Since we are looking for* μ_m *and* μ_f *time independent, it appears that the time* T *may be anyone. We choose* $T \in (0, g]$ *and such that* $a_l = \bar{n}T$, *for* $\bar{n} \in \mathbb{N}$.

We note that:

Lemma 3.1. *The function* $M(a,t)$ *in* (20) *is almost everywhere decreasing when the coefficient of male mortality* $\mu_m(a)$ *increases, assuming* $\mu_f(b)$ *and* $N(b,t)$ *to be known. The function* $N(b,t)$ *in* (33) *is almost everywhere decreasing when the coefficient of female mortality* $\mu_f(b)$ *increases, assuming* $\mu_m(a)$ *and* $M(a,t)$ *to be known.*

3.1 - *The coefficient of male mortality*

We can evaluate the coefficient of male mortality μ_m, if we assume that M at a certain time $t = T \in (0, g]$, N and μ_f are known. We also assume $\mu_m(a)$ for $a \in [-g, 0]$ to be known.

Remark 3.2. *Remind that we are assuming* $\mu_m \in L^1(0,a)$ *for* $a < a_l$. *It implies that the derivative with respect to a of* $\int_{a-c}^{a} \mu_m(s)\,ds$ *exists almost everywhere in* $(0, a_l)$.

Time $T \in (0, g]$ represents the time at which we make a (non-null) census of male population. Thus,

$$M(a,T) =_{def} \psi_m(a) \tag{36}$$

is a known function which we assume to be positive, $\psi_m(a) > 0$, almost everywhere, and such that $\lim_{a \to a_l^-} \psi_m(a) = 0$. We first invert the problem for $0 < a < T$, then we proceed in successive strips of amplitude T.

If $a \in [0, T)$, it is, because of (20) and (36):

$$\psi_m(a) = M(0, T-a) \exp\left(- \int_0^a \mu_m(s)\,ds \right),$$

where $\psi_m(a)$ and $M(0, T-a)$ are known functions. From this relation the uniqueness of μ_m easily follows. In equation

$$\exp\left(\int_0^a \mu_m(s)\,ds \right) = \frac{M(0, T-a)}{\psi_m(a)},$$

if the right hand side is greater than 1 for almost every $a \in [0, T)$, then $\mu_m(a) \geq 0$. It results

$$\mu_m(a) = \frac{d}{da}\left(\ln \frac{M(0, T-a)}{\psi_m(a)}\right). \tag{37}$$

Note that $M(0, T-a)$ is known: indeed, it is

$$M(0, T-a) = \int_g^{a_l}\int_g^{b_l} \beta_m(u-g, v-g)\, 2k(u-g, v-g)\widehat{M}_0(u-g, T-a-g)$$
$$\widehat{N}_0(v-g, T-a-g)\exp\left(-\int_{T-a-g}^{T-a} (\,\mu_m(s-T+a) + \mu_f(v-T+a+s))\, ds\right) du\, dv,$$

where β_m, k, $\widehat{M}_0$, $\widehat{N}_0$, μ_f are known and μ_m is known in $[-g, 0]$.

If $a \geq T$, it results, because of (20) and (36):

$$\psi_m(a) = \widehat{M}_0(a-T, 0)\exp\left(-\int_{a-T}^{a} \mu_m(s)\, ds\right).$$

Again, the uniqueness of μ_m follows; $\mu_m \geq 0$ for almost every $a \in [T, 2T)$ if $\widehat{M}_0(a-T, 0)/\psi_m(a) \geq 1$ almost everywhere. We obtain:

$$\mu_m(a) = \mu_m(a-T) + \frac{d}{da}\left(\ln \frac{\widehat{M}_0(a-T, 0)}{\psi_m(a)}\right). \tag{38}$$

The second addend in the right hand side of (38) is a known term. The first addend is known when $a - T \in [0, T)$, because of (37), and therefore for $a \in [T, 2T)$. Hence, we proceed on successive intervals of amplitude T.

It means that, if $a \in [T, 2T)$, it is, see (37) and (38):

$$\begin{aligned}\mu_m(a) &= \frac{d}{da}\left(\ln \frac{M(0, T-(a-T))}{\psi_m(a-T)}\right) + \frac{d}{da}\left(\ln \frac{\widehat{M}_0(a-T, 0)}{\psi_m(a)}\right)\\ &= \frac{d}{da}\left(\ln \frac{M(0, 2T-a)}{\psi_m(a-T)} + \ln \frac{\widehat{M}_0(a-T, 0)}{\psi_m(a)}\right),\end{aligned}$$

and therefore:

$$\mu_m(a) = \frac{d}{da}\left(\ln \frac{M(0, 2T-a)\widehat{M}_0(a-T, 0)}{\psi_m(a-T)\psi_m(a)}\right). \tag{39}$$

If $a \in [2T, 3T)$, it results, see (38) and (39):

$$\begin{aligned}\mu_m(a) &= \mu_m(a-T) + \frac{d}{da}\left(\ln \frac{\widehat{M}_0(a-T,0)}{\psi_m(a)}\right)\\ &= \frac{d}{da}\left(\ln \frac{M(0,2T-(a-T))\widehat{M}_0((a-T)-T,0)}{\psi_m((a-T)-T)\psi_m(a-T)}\right) + \frac{d}{da}\left(\ln \frac{\widehat{M}_0(a-T,0)}{\psi_m(a)}\right)\\ &= \frac{d}{da}\left(\ln \frac{M(0,3T-a)\widehat{M}_0(a-2T,0)}{\psi_m(a-2T)\psi_m(a-T)} + \ln \frac{\widehat{M}_0(a-T,0)}{\psi_m(a)}\right),\end{aligned}$$

hence

$$\mu_m(a) = \frac{d}{da}\left(\ln \frac{M(0,3T-a)\widehat{M}_0(a-2T,0)\widehat{M}_0(a-T,0)}{\psi_m(a-2T)\psi_m(a-T)\psi_m(a)}\right).$$

By induction, it is easy to prove that if $a \in [nT, (n+1)T)$, for $n = 0, 1, \ldots$, the coefficient of male mortality is unique and given by

$$\mu_m(a) = \frac{d}{da}\left(\ln \frac{M(0,(n+1)T-a)\prod\limits_{j=1}^{n}\widehat{M}_0(a-jT,0)}{\prod\limits_{j=0}^{n}\psi_m(a-jT)}\right), \tag{40}$$

where, by convention, $\prod\limits_{j=1}^{0}\widehat{M}_0(a-jT,0) =_{def} 1$.

Remark 3.3. *The preceding formulas show that a unique μ_m can be obtained for almost every $a \in (0, a_l)$. Such a coefficient may be non positive for some census ψ_m; in order that μ_m is nonnegative it is necessary that the right hand side of* (40) *is nonnegative; this amounts to a relation between ψ_m, initial data and biological parameters β_m, k, μ_f being satisfied. We omit to introduce such a relation. For related problems and difficulties, see the pioneering works by Pilant and Rundell* [**6, 7, 8, 9**].

3.2 - *The coefficient of female mortality*

The procedure to obtain the female coefficient of mortality is more complicated. It consists of successive steps in order to obtain an approximation to μ_f in $[-g, b_l)$, say $\mu_0 = \mu_0(b)$, and in successive constructions of $\mu_n = \mu_n(b)$ in $[-g, b_l)$ which should converge to μ_f. It appears to be an arduous task to prove the convergence.

We assume that $N(b,T)$ at a certain time $t = T \in (0, g]$ is known. Let

$$N(b,T) =_{def} \psi_f(b), \tag{41}$$

where $\psi_f(b)$ is a known, positive almost everywhere function ($\psi_f(b) > 0$ almost everywhere), and such that $\lim_{b\to b_l^-} \psi_f(b) = 0$. We assume that M and μ_m are known and that μ_f is known in $[-g, 0]$. The following relations have to be considered true almost everywhere in the sets where they are defined.

The following steps are intended to eliminate b from the argument of the integrand functions, to make it appear in the extremes of integration.

When $b \in [0, T)$, because of (33) and (41), for $i = 0$, we have:

$$\psi_f(b) = N(0, T-b) \exp\left(-\int_0^b m(s, s+T-b)\, ds \right)$$

that is

$$\begin{aligned}\psi_f(b) = &\int_g^{a_l}\int_g^{b_l} \beta_f(u-g, v-g) 2k(u-g, v-g) M(u-g, T-b-g) N(v-g, T-b-g)\\ &\times \exp\left(-\int_{T-b-g}^{T-b} (\mu_f(s-T+b) + \mu_f(v-T+b+s))\, ds \right) du\, dv\\ &\times \exp\left(-\int_0^b \mu_f(s)\, ds \right) \exp\left(-\int_0^b A(s, s+T-b)\, ds \right),\end{aligned}$$

see (8), (10), (3) and the second definition in (24). Note that $M(u-g, T-b-g) = \widehat{M}_0(u-g, T-b-g)$ and $N(v-g, T-b-g) = \widehat{N}_0(v-g, T-b-g)$ because $T-b-g \in [-g, 0]$, for $b \in [0, T)$ and $T \in (0, g]$. Therefore, it is:

$$\begin{aligned}\psi_f(b) = &\int_g^{a_l}\int_g^{b_l} \beta_f(u-g, v-g) 2k(u-g, v-g) \widehat{M}_0(u-g, T-b-g) \widehat{N}_0(v-g, T-b-g)\\ &\times \exp\left(-\int_{T-b-g}^{T-b} (\mu_f(s-T+b) + \mu_f(v-T+b+s)) ds \right) du\, dv\\ &\times \exp\left(-\int_0^b \mu_f(s)\, ds \right) \exp\left(-\int_0^b A(s, s+T-b)\, ds \right),\end{aligned} \tag{42}$$

where A, defined in (24), here is a known term. Put

$$l(u-g, v-g, t-b-g) =_{def} \beta_f(u-g, v-g) 2k(u-g, v-g) \times \widehat{M}_0(u-g, t-b-g)\widehat{N}_0(v-g, t-b-g), \tag{43}$$

$$p(b) =_{def} \exp\Big(-\int_0^b A(s, s+T-b)\, ds\Big) \tag{44}$$

and

$$r(v-g, t-b-g) =_{def} \int_g^{a_l} l(u-g, v-g, t-b-g)\, du\,; \tag{45}$$

here l, p and r are known functions. As a consequence, (42) becomes

$$\begin{aligned} \psi_f(b) = & \, p(b) \exp\Big(-\int_0^b \mu_f(s) ds\Big) \\ & \times \int_g^{b_l} r(v-g, T-b-g) \exp\Big(-\int_{T-b-g}^{T-b} \mu_f(s-T+b) ds\Big) \\ & \times \exp\Big(-\int_{T-b-g}^{T-b} \mu_f(v-T+b+s) ds\Big) dv. \end{aligned} \tag{46}$$

Note that $\exp\Big(-\int_{T-b-g}^{T-b} \mu_f(s-T+b)\, ds\Big) = \exp\Big(-\int_{-g}^{0} \mu_f(s')\, ds'\Big)$ is a known constant, as we are assuming μ_f to be known in $[-g, 0]$. Put

$$\overline{p} =_{def} \exp\Big(-\int_{-g}^{0} \mu_f(s')\, ds'\Big). \tag{47}$$

Equation (46) becomes

$$\begin{aligned} \psi_f(b) = & \, \overline{p}\; p(b) \exp\Big(-\int_0^b \mu_f(s) ds\Big) \\ & \times \int_g^{b_l} r(v-g, T-b-g) \exp\Big(-\int_{T-b-g}^{T-b} \mu_f(v+s-T+b) ds\Big) dv, \end{aligned}$$

and therefore we obtain

$$\exp\left(\int_0^b \mu_f(s)ds\right) = \frac{\overline{p}\,p(b)}{\psi_f(b)}\int_g^{b_l} r(v-g, T-b-g)\exp\left(-\int_{-g}^{0}\mu_f(v+z)dz\right)dv.$$

Hence, it results

$$\mu_f(b) = \frac{d}{db}\left\{\ln\left[\frac{\overline{p}\,p(b)}{\psi_f(b)}\int_g^{b_l} r(v-g, T-b-g)\exp\left(-\int_{-g}^{0}\mu_f(v+z)dz\right)dv\right]\right\} \tag{48}$$

when $b \in [0, T)$. Note that $\mu_f(v+z)$ is unknown for $v \in (g, b_l)$ and $z \in (-g, 0)$.

It is worth to put

$$T_1 =_{def} \min\{2T, g\}\,. \tag{49}$$

When $b \in [T, T_1)$, because of (33) and (41), for $i = 0$, we have:

$$\psi_f(b) = \widehat{N}_0(b-T, 0)\,\exp\left(-\int_0^T m(s+b-T, s)\,ds\right) + \int_0^T \exp\left(-\int_s^T m(\sigma+b-T, \sigma)\,d\sigma\right)S(s+b-T, s)\,ds\,.$$

Putting

$$q(b, s) =_{def} \exp\left(-\int_s^T A(\sigma+b-T, \sigma)\,d\sigma\right) \tag{50}$$

(it is a known function), we obtain

$$\begin{aligned}\psi_f(b) = {}& \widehat{N}_0(b-T, 0)\,\exp\left(-\int_0^T \mu_f(s+b-T)\,ds\right)q(b, 0)\\ &+ \int_0^T \exp\left(-\int_s^T \mu_f(\sigma+b-T)\,d\sigma\right)q(b, s)\\ &\times L(s+b-T; s+b-T-g, s-g)N(s+b-T-g, s-g)\,ds\,.\end{aligned}$$

Note that $N(s+b-T-g, s-g) = \widehat{N}_0(s+b-T-g, s-g)$ because $s-g \in [-g, 0]$

for $s \in (0,T)$ and $T \le g$. Then, reminding (27), we can write

$$\begin{aligned}\psi_f(b) &= \widehat{N}_0(b-T,0)\,\exp\left(-\int_0^T \mu_f(s+b-T)\,ds\right) q(b,0)\\ &+\int_0^T \exp\left(-\int_s^T \mu_f(\sigma+b-T)d\sigma\right) q(b,s)\exp\left(-\int_{-g}^0 \mu_f(s+b-T+\tau)d\tau\right)\\ &\quad\times A(s+b-T-g,s-g)\,\widehat{N}_0(s+b-T-g,s-g)\,ds\,.\end{aligned}$$

We put

$$\overline{q}(b,s) =_{def} q(b,s)A(s+b-T-g,s-g)\,\widehat{N}_0(s+b-T-g,s-g) \tag{51}$$

(it is a known function). Note that it is $s+b-T-g<0$ and $s+b-T>0$ for $0<s<T$, $0<T\le g$ and $T\le b<T_1$. Therefore, we can write:

$$\begin{aligned}\psi_f(b) &= \widehat{N}_0(b-T,0)\exp\left(-\int_0^T \mu_f(s+b-T)ds\right) q(b,0)\\ &+\int_0^T \exp\left(-\int_s^T \mu_f(\sigma+b-T)d\sigma\right)\exp\left(-\int_{s+b-T-g}^0 \mu_f(\tau')d\tau'\right)\\ &\quad\times\exp\left(-\int_0^{s+b-T}\mu_f(\tau')d\tau'\right)\overline{q}(b,s)ds.\end{aligned} \tag{52}$$

Note that $\exp\Big(-\int\limits_{s+b-T-g}^{0} \mu_f(\tau')\,d\tau'\Big)$ is a known function. The next step consists in rewriting (52) as follows:

$$\begin{aligned}\psi_f(b) &= \widehat{N}_0(b-T,0)\exp\left(-\int_{b-T}^b \mu_f(s')ds'\right) q(b,0)\\ &+\int_0^T \exp\left(-\int_{s+b-T}^b \mu_f(\sigma')d\sigma'\right)\exp\left(-\int_{s+b-T-g}^0 \mu_f(\tau')d\tau'\right)\\ &\quad\times\exp\left(-\int_0^{s+b-T}\mu_f(\tau')\,d\tau'\right)\overline{q}(b,s)\,ds\,.\end{aligned} \tag{53}$$

We put

$$\widehat{q}(b) =_{def} \widehat{N}_0(b-T,0)\, q(b,0)\,, \tag{54}$$

then equation (53) becomes

$$\begin{aligned}\psi_f(b) &= \widehat{q}(b)\exp\left(-\int_{b-T}^{b}\mu_f(s')ds'\right)\\ &+\exp\left(-\int_0^b\mu_f(\tau')d\tau'\right)\int_0^T\exp\left(-\int_{s+b-T-g}^{0}\mu_f(\tau')d\tau'\right)\overline{q}(b,s)ds.\end{aligned} \tag{55}$$

Now, we can rewrite (55) as follows:

$$\begin{aligned}\psi_f(b) = \widehat{q}(b)\, &\exp\left(-\int_{b-T}^{T}\mu_f(s')\,ds'\right)\exp\left(-\int_T^b\mu_f(s')\,ds'\right)\\ &+\exp\left(-\int_0^T\mu_f(\tau')\,d\tau'\right)\exp\left(-\int_T^b\mu_f(\tau')\,d\tau'\right)\\ &\times\int_0^T\exp\left(-\int_{s+b-T-g}^{0}\mu_f(\tau')\,d\tau'\right)\overline{q}(b,s)\,ds.\end{aligned}$$

We obtain

$$\begin{aligned}\exp\left(\int_T^b\mu_f(\tau')\,d\tau'\right) = \frac{1}{\psi_f(b)}&\left[\widehat{q}(b)\exp\left(-\int_{b-T}^{T}\mu_f(s')\,ds'\right)\right.\\ &\left.+\int_0^T\exp\left(-\int_{s+b-T-g}^{T}\mu_f(\tau')\,d\tau'\right)\overline{q}(b,s)\,ds\right],\end{aligned}$$

where in the right hand side μ_f is evaluated at ages in the preceding intervals $[-g,0]$ and $(0,T)$. It follows

$$\begin{aligned}\mu_f(b) = \frac{d}{db}\ln\Bigg\{\frac{1}{\psi_f(b)}&\left[\widehat{q}(b)\exp\left(-\int_{b-T}^{T}\mu_f(s)\,ds\right)\right.\\ &\left.+\int_0^T\exp\left(-\int_{s+b-T-g}^{T}\mu_f(\tau)\,d\tau\right)\overline{q}(b,s)\,ds\right]\Bigg\},\end{aligned} \tag{56}$$

we obtain the mortality for the ages in the interval $[T, T_1)$. It depends on the coefficient of female mortality in the preceding intervals, $[-g, 0]$ and $(0, T)$.

The following step consists in studying the coefficient of female mortality for $b \in [T_1, T_2)$, where

$$T_2 =_{def} T + T_1 \,. \tag{57}$$

We can write again that, see (33) and (41) for $i = 0$:

$$\psi_f(b) = \widehat{N}_0(b-T, 0)\ \exp\left(-\int_0^T m(s+b-T, s)\, ds\right) + \int_0^T \exp\left(-\int_s^T m(\sigma+b-T, \sigma)\, d\sigma\right) S(s+b-T, s)\, ds \,.$$

We follow the steps of the preceding case, when it was $b \in [T, T_1)$, and we get, see (52):

$$\psi_f(b) = \widehat{N}_0(b-T, 0) \exp\left(-\int_0^T \mu_f(s+b-T) ds\right) q(b, 0) + \int_0^T \exp\left(-\int_s^T \mu_f(\sigma+b-T) d\sigma\right) \exp\left(-\int_{s+b-T-g}^{s+b-T} \mu_f(\tau') d\tau'\right) \overline{q}(b, s) ds$$

or rather

$$\psi_f(b) = \widehat{N}_0(b-T, 0) \exp\left(-\int_{b-T}^{b} \mu_f(s') ds'\right) q(b, 0) + \int_0^T \exp\left(-\int_{s+b-T}^{b} \mu_f(\sigma') d\sigma'\right) \exp\left(-\int_{s+b-T-g}^{s+b-T} \mu_f(\tau') d\tau'\right) \overline{q}(b, s) ds$$

or equivalently

$$\begin{aligned} \psi_f(b) = \widehat{N}_0(b-T, 0)\ \exp\left(-\int_{b-T}^{b} \mu_f(s')\, ds'\right) q(b, 0) \\ + \int_0^T \exp\left(-\int_{s+b-T-g}^{b} \mu_f(\tau')\, d\tau'\right) \overline{q}(b, s)\, ds \,. \end{aligned} \tag{58}$$

Note that it is $s+b-T-g<T_1$ for $0<s<T\le g$ and $T_1\le b<T_2=T+T_1 \le g+T_1$. Indeed, $s<T$ implies $s-T<0$. Since $b<g+T_1$, it is also $b-g<T_1$ and summing the negative quantity $s-T$ to $b-g$, we obtain $b-g+s-T<b-g<T_1$. Therefore, we can write (58) as follows

$$\psi_f(b)=\widehat{N}_0(b-T,0)\exp\left(-\int_{b-T}^{T_1}\mu_f(s')ds'\right)\exp\left(-\int_{T_1}^{b}\mu_f(s')ds'\right)q(b,0)$$
$$+\int_0^T\exp\left(-\int_{s+b-T-g}^{T_1}\mu_f(\tau')d\tau'\right)\exp\left(-\int_{T_1}^{b}\mu_f(\tau')d\tau'\right)\overline{q}(b,s)ds$$

and so it results

$$\exp\left(\int_{T_1}^{b}\mu_f(s)\,ds\right)=\frac{1}{\psi_f(b)}\left[\widehat{N}_0(b-T,0)\,\exp\left(-\int_{b-T}^{T_1}\mu_f(s)\,ds\right)q(b,0)\right.$$
$$\left.+\int_0^T\exp\left(-\int_{s+b-T-g}^{T_1}\mu_f(\tau)\,d\tau\right)\overline{q}(b,s)\,ds\right].$$

Because of (54), we can write

$$\int_{T_1}^{b}\mu_f(s)ds$$
$$=\ln\left\{\frac{1}{\psi_f(b)}\left[\widehat{q}(b)\exp\left(-\int_{b-T}^{T_1}\mu_f(s)ds\right)+\int_0^T\exp\left(-\int_{s+b-T-g}^{T_1}\mu_f(\tau)d\tau\right)\overline{q}(b,s)ds\right]\right\}.$$

Finally we obtain the coefficient of female mortality for $b\in[T_1,T_2)$:

$$\mu_f(b)=\frac{d}{db}\ln\left\{\frac{1}{\psi_f(b)}\left[\widehat{q}(b)\exp\left(-\int_{b-T}^{T_1}\mu_f(s)ds\right)\right.\right.$$
$$\left.\left.+\int_0^T\exp\left(-\int_{s+b-T-g}^{T_1}\mu_f(\tau)d\tau\right)\overline{q}(b,s)ds\right]\right\}. \tag{59}$$

We can repeat the procedure on successive intervals, $b \in [T_k, T_{k+1})$, $k = 0, 1, \ldots, z$, where z is the last index for which $T_z < b_l$ and where

$$
\begin{cases}
T_0 & =_{def}\ T \\
T_1 & =_{def}\ \min\{2T, g\} \\
T_{k+1} & =_{def}\ T_k + T\,, \quad \text{for } k = 1, 2, \ldots, z-1 \\
T_{z+1} & =_{def}\ b_l\,,
\end{cases}
\tag{60}
$$

obtaining

$$
\begin{aligned}
\mu_f(b) = \frac{d}{db} \ln \Bigg\{ \frac{1}{\psi_f(b)} \Bigg[\widehat{q}(b) \exp\Bigg(- \int_{b-T}^{T_k} \mu_f(s) ds \Bigg) \\
+ \int_0^T \exp\Bigg(- \int_{s+b-T-g}^{T_k} \mu_f(\tau) d\tau \Bigg) \overline{q}(b, s) ds \Bigg] \Bigg\}.
\end{aligned}
\tag{61}
$$

Now we can proceed using the method of successive approximations: let $\mu_0 \in L^1[-g, b_l)$ be assigned, and $\mu_0[-g, 0] = \mu_f[-g, 0]$, and satisfy assumptions in (19). By using (48) we can build, for $b \in [-g, T)$, the following function $\mu_{1,1}(b)$:

$$
\mu_{1,1}(b) = \begin{cases}
\mu_0(b) \quad \text{for } b \in [-g, 0) \\
\dfrac{d}{db} \Bigg\{ \ln \Bigg[\dfrac{\overline{p}\, p(b)}{\psi_f(b)} \displaystyle\int_g^{b_l} r(v - g, T - b - g) \exp\Big(- \int_{-g}^{0} \mu_0(v + z) dz \Big) dv \Bigg] \Bigg\} \\
\qquad \text{for } b \in [0, T).
\end{cases}
$$

Once we know $\mu_{1,1}[-g, T)$, we can build $\mu_{1,2}(b)$ for $b \in [-g, T_1)$, using (56):

$$
\mu_{1,2}(b) = \begin{cases}
\mu_{1,1}(b) \quad \text{for } b \in [-g, T) \\
\dfrac{d}{db} \ln \Bigg\{ \dfrac{1}{\psi_f(b)} \Bigg[\widehat{q}(b) \exp\Bigg(- \displaystyle\int_{b-T}^{T} \mu_{1,1}(s) ds \Bigg) \\
\qquad + \displaystyle\int_0^T \exp\Bigg(- \int_{s+b-T-g}^{T} \mu_{1,1}(\tau) d\tau \Bigg) \overline{q}(b, s) ds \Bigg] \Bigg\} \\
\qquad \text{for } b \in [T, T_1)\,.
\end{cases}
$$

Now we build $\mu_{1,3}(b)$ for $b \in [-g, T_2)$, using (59):

$$\mu_{1,3}(b) = \begin{cases} \mu_{1,2}(b) \quad \text{for } \ b \in [-g, T_1) \\ \dfrac{d}{db} \ln \left\{ \dfrac{1}{\psi_f(b)} \left[\widehat{q}(b) \exp\left(- \int\limits_{b-T}^{T_1} \mu_{1,2}(s) ds \right) \right.\right. \\ \qquad\qquad \left.\left. + \int\limits_0^T \exp\left(- \int\limits_{s+b-T-g}^{T_1} \mu_{1,2}(\tau) d\tau \right) \overline{q}(b,s) ds \right] \right\} \\ \qquad\qquad \text{for } \ b \in [T_1, T_2)\,. \end{cases}$$

We can go on this way till we reach b_l, obtaining $\mu_{1,z+2}(b)$ for $b \in [-g, b_l)$, see (61):

$$\mu_{1,z+2}(b) = \begin{cases} \mu_{1,z+1}(b) \quad \text{for } \ b \in [-g, T_z) \\ \dfrac{d}{db} \ln \left\{ \dfrac{1}{\psi_f(b)} \left[\widehat{q}(b) \exp\left(- \int\limits_{b-T}^{T_z} \mu_{1,z+1}(s) ds \right) \right.\right. \\ \qquad\qquad \left.\left. + \int\limits_0^T \exp\left(- \int\limits_{s+b-T-g}^{T_z} \mu_{1,z+1}(\tau) d\tau \right) \overline{q}(b,s) ds \right] \right\} \\ \qquad\qquad \text{for } b \in [T_z, b_l). \end{cases}$$

Let us put, for $b \in [-g, b_l)$:

(62)
$$\mu_{1,z+2}(b) =_{def} \mu_1(b)\,.$$

Now we can restart the approximations using μ_1 in (48): for $b \in [-g, T)$, we obtain the following function $\mu_{2,1}(b)$

$$\mu_{2,1}(b) = \begin{cases} \mu_1(b) \quad \text{for } \ b \in [-g, 0) \\ \dfrac{d}{db} \left\{ \ln \left[\dfrac{\overline{p}\, p(b)}{\psi_f(b)} \int\limits_g^{b_l} r(v-g, T-b-g) \exp\left(- \int\limits_{-g}^{0} \mu_1(v+z) dz \right) dv \right] \right\} \\ \qquad\qquad \text{for } \ b \in [0, T), \end{cases}$$

and following the preceding steps, we also obtain $\mu_{2,2}[-g, T_1)$, $\mu_{2,3}[-g, T_2), \ldots,$

$\mu_{2,z+2}[-g, b_l)$. Let us put

$$\mu_{2,z+2}(b) =_{def} \mu_2(b)\,. \tag{63}$$

We repeat the same steps to build $\mu_n(b) \in L^1[-g, b_l)$, $n \in \mathbb{N}$. If

$$\lim_{n\to+\infty} \mu_n(b) < +\infty\,,$$

we have the unique mortality coefficient of females

$$\mu_f(b) = \lim_{n\to+\infty} \mu_n(b)\,,$$

in $L^1[-g, b_l)$. To prove the convergence of μ_n one should know the mutual relationships between the biological parameters, but this item is not examined in the present work.

4 - Appendix

This appendix is devoted to the proof of two lemmas which we have used in Section 2.

Proof [Lemma 2.2]. We first consider $i = 0$, therefore $t \in (0, g]$.

1. Consider (20) for the male distribution. If $a < t$, the solution is bounded by

$$M(a,t) = M(0, t-a)\exp\left(-\int_0^a \mu_m(s)ds\right) \le M(0, t-a) \le K\overline{M}_0\overline{N}_0 a_l b_l$$

according to (11), (15), (13), (14) and (34).
If $a \ge t$, the solution for males is

$$M(a,t) = M(a-t, 0)\exp\left(-\int_{a-t}^a \mu_m(s)ds\right) \le M(a-t, 0) = \widehat{M}_0(a-t, 0) \le \overline{M}_0.$$

Therefore, for $t \in (0, g]$,

$$M(a,t) \le K\overline{M}_0\overline{N}_0 a_l b_l + \overline{M}_0 = \overline{M}_0\big(1 + K\overline{N}_0 a_l b_l\big) =_{def} \overline{M}_1 < +\infty.$$

2. Then consider solution (33) for the female distribution. If $b < t$, it results, according to (11), (12), (16) and (34),

$$N(b,t) = N(0, t-b)\exp\left(-\int_0^b m(s, s+t-b)ds\right) \le N(0, t-b) \le K\overline{M}_0\overline{N}_0 a_l b_l.$$

If $b \geq t$, it is

$$N(b,t) = \widehat{N}_0(b-t,0)\,\exp\left(-\int_0^t m(s+b-t,s)\,ds\right) + \int_0^t \exp\left(-\int_s^t m(\sigma+b-t,\sigma)\,d\sigma\right) S(s+b-t,s)\,ds$$
$$\leq \widehat{N}_0(b-t,0) + \int_0^t S(s+b-t,s)ds \leq \overline{N}_0 + \int_0^t S(s+b-t,s)ds,$$

because of (34). Taking into account that $s-g \in (-g,0]$ for $s \in (0,t]$ and $t \in (0,g]$, owing to (17), one obtains $N(b,t) \leq \overline{N}_0 + K\overline{M}_0\overline{N}_0 a_l g$.
To conclude the estimate for $N(b,t)$ in the first strip, one has:

$$N(b,t) \leq \overline{N}_0\big(1 + K\overline{M}_0 a_l b_l + K\overline{M}_0 a_l g\big) =_{def} \overline{N}_1 < +\infty.$$

Now consider the second strip, for $i=1$, therefore $t \in (g,2g]$. According to the steps for the preceding case, one has, more briefly, the following results.

1. Consider again (20) for the male distribution. If $a < t-g$, it is

$$M(a,t) \leq K \int_g^{a_l}\int_g^{b_l} M(u-g,t-a-g)N(v-g,t-a-g)\,du\,dv.$$

In the second strip it results $t \in (g,2g]$, then $t-a-g \in (0,g]$. As a consequence, for $M(u-g,t-a-g)$ and $N(v-g,t-a-g)$ one can use the estimates obtained for $i=0$. Therefore, $M(a,t) \leq K\overline{M}_0\big(1+K\overline{N}_0 a_l b_l\big)\overline{N}_1 a_l b_l$.
If $a \geq t-g$, one has, using the estimate obtained for M in the first strip: $M(a,t) \leq M(a-t+g,g) \leq \overline{M}_0\big(1+K\overline{N}_0 a_l b_l\big)$.
Therefore, for $t \in (g,2g]$,

$$M(a,t) \leq \overline{M}_0\big(1+K\overline{N}_0 a_l b_l\big)\big(1+K\overline{N}_1 a_l b_l\big) =_{def} \overline{M}_2 < +\infty.$$

2. Now, consider solution (33) for the female distribution. If $b < t-g$, one can use the estimates obtained for $i=0$ and it results: $N(b,t) \leq K\overline{M}_1\overline{N}_1 a_l b_l$. If $b \geq t-g$, it is $N(b,t) \leq \overline{N}_1 + K\overline{M}_1\overline{N}_1 a_l g$.
To conclude the estimate for $N(b,t)$ in the second strip, for $i=1$, one has:

$$N(b,t) \leq \overline{N}_1\big(1 + K\overline{M}_1 a_l b_l + K\overline{M}_1 a_l g\big) =_{def} \overline{N}_2 < +\infty.$$

The proof can be easily concluded by induction.

Proof [Lemma 2.3]. We first consider $i = 0$, therefore $t \in (0, g]$, as usual. The proof will go on by induction.

1. Consider equation (20). We want to prove that $M(a,t)$ given in (20) belongs to $L^1(0, a_l)$ for any $t \in (0, g]$, see item 1.1, and that it is strongly continuous with respect to t, see item 1.2. In what follows it is taken into account that M and N are nonnegative.

1.1 For $a < t$, because of (35) and (15),

$$\int_0^t M(a,t)\,da \leq \int_0^t M(0, t-a)\,da \leq K\overline{M}_0\,a_l\,\overline{N}_0\,b_l\,g < +\infty.$$

For $a \geq t$, it is:

$$\int_t^{a_l} M(a,t)da \leq \int_t^{a_l} M(a-t, 0)da = \int_t^{a_l} \widehat{M}_0(a-t, 0)da \leq \overline{M}_0 a_l < +\infty.$$

Therefore, $M(a,t) \in L^1(0, a_l)$ for every $t \in (0, g]$.

1.2 Consider the following limit:

$$\lim_{h\to 0} \|M(a, t+h) - M(a,t)\|_{L^1(0,a_l)} = \lim_{h\to 0} \int_0^{a_l} \big|M(a,t+h) - M(a,t)\big|da$$

$$= \lim_{h\to 0}\left(\int_0^t \big|M(a,t+h) - M(a,t)\big|da + \int_t^{a_l} \big|M(a,t+h) - M(a,t)\big|da \right).$$

One evaluates the first addend:

$$\int_0^t \big|M(a,t+h) - M(a,t)\big|da \leq \int_0^t \big|M(0, t+h-a) - M(0, t-a)\big|da$$

$$= \int_0^t \Bigg| \int_g^{a_l}\int_g^{b_l} \beta_m(u-g, v-g)2k(u-g, v-g)$$

$$\times M(u-g, t+h-a-g)N(v-g, t+h-a-g)$$

$$\times \exp\left[-\int_{t+h-a-g}^{t+h-a} (\mu_m(s-t-h+a) + \mu_f(v-t-h+a+s))ds \right] dudv$$

$$-\int_g^{a_l}\int_g^{b_l}\beta_m(u-g,v-g)2k(u-g,v-g)$$

$$\times M(u-g,t-a-g)N(v-g,t-a-g)$$

$$\times\exp\left[-\int_{t-a-g}^{t-a}(\mu_m(s-t+a)+\mu_f(v-t+a+s))ds\right]du\,dv\Bigg|da$$

$$\leq K\int_0^t\int_g^{a_l}\int_g^{b_l}\big|\widehat{M}_0(u-g,t+h-a-g)\widehat{N}_0(v-g,t+h-a-g)$$

$$-\widehat{M}_0(u-g,t-a-g)\widehat{N}_0(v-g,t-a-g)\big|dudvda$$

$$\leq K\int_0^t\int_g^{a_l}\int_g^{b_l}\widehat{M}_0(u-g,t+h-a-g)$$

$$\times\big|\widehat{N}_0(v-g,t+h-a-g)-\widehat{N}_0(v-g,t-a-g)\big|dudvda$$

$$+K\int_0^t\int_g^{a_l}\int_g^{b_l}\widehat{N}_0(v-g,t-a-g)$$

$$\times\big|\widehat{M}_0(u-g,t+h-a-g)-\widehat{M}_0(u-g,t-a-g)\big|dudvda$$

$$<K\,\varepsilon\,a_l\,b_l\,g(\overline{M}_0+\overline{N}_0),$$

for $|h|<\delta(\varepsilon)$, because of the hypothesis of strong continuity of $\widehat{M}_0(\cdot,t)$ and $\widehat{N}_0(\cdot,t)$, see (b)-3, and because of (35).
Now, one evaluates the second addend:

$$\int_t^{a_l}|M(a,t+h)-M(a,t)|\,da$$

$$=\int_t^{a_l}\Bigg|M(a-t-h,0)\exp\left(-\int_{a-t-h}^{a}\mu_m(s)ds\right)$$

$$- M(a-t,0)\exp\left(-\int_{a-t}^{a}\mu_m(s)ds\right)\Big|da$$

$$\leq \int_t^{a_l}\widehat{M}_0(a-t-h,0)\Big|\exp\left(-\int_{a-t-h}^{a}\mu_m(s)ds\right)-\exp\left(-\int_{a-t}^{a}\mu_m(s)ds\right)\Big|da$$

$$+\int_t^{a_l}\big|\widehat{M}_0(a-t-h,0)-\widehat{M}_0(a-t,0)\big|\,da<\varepsilon\overline{M}_0\,a_l+\varepsilon\,a_l,$$

for $|h|<\delta(\varepsilon)$, because of the continuity of the integrals in the exponentials, to (35) and to the uniform continuity of $\widehat{M}_0(a,0)$ with respect to $a\in[0,a_l)$, see (f).
Therefore, it results $\lim_{h\to 0}\|M(a,t+h)-M(a,t)\|_{L^1(0,a_l)}=0$.

2. Consider equation (33) for nonpregnant females. We are going to prove that $N(b,t)$ given in (33) belongs to $L^1(0,b_l)$, see item 2.1, and that it is strongly continuous, see item 2.2.

2.1 For $b<t$, because of (16) and (35),

$$\int_0^t N(b,t)\,db\leq\int_0^t N(0,t-b)db$$

$$=\int_0^t\int_g^{a_l}\int_g^{b_l}\beta_f(u-g,v-g)2k(u-g,v-g)$$

$$\times\widehat{M}_0(u-g,t-b-g)\widehat{N}_0(v-g,t-b-g)$$

$$\times\exp\left(-\int_{t-b-g}^{t-b}(\mu_f(s-t+b)+\mu_f(v-t+b+s))ds\right)dudvdb$$

$$\leq K\overline{M}_0\,a_l\overline{N}_0\,b_l\,g<+\infty.$$

For $b\geq t$, it results, because of (17) and (35),

$$\int_t^{b_l}N(b,t)db=\int_t^{b_l}\left[\widehat{N}_0(b-t,0)\exp\left(-\int_0^t m(s+b-t,s)ds\right)\right.$$

$$\left.+\int_0^t\exp\left(-\int_s^t m(\sigma+b-t,\sigma)d\sigma\right)S(s+b-t,s)ds\right]db$$

$$\leq \int_t^{b_l} \widehat{N}_0(b-t,0)\,db + \int_t^{b_l}\int_0^t S(s+b-t,s)\,ds\,db$$

$$\leq \overline{N}_0 b_l + \int_t^{b_l}\int_0^t A(s+b-t-g,s-g)\widehat{N}_0(s+b-t-g,s-g)ds\,db$$

$$\leq \overline{N}_0 b_l + K\int_t^{b_l}\int_0^t\int_0^{a_l} \widehat{M}_0(a,s-g)da\widehat{N}_0(s+b-t-g,s-g)ds\,db$$

$$\leq \overline{N}_0\, b_l + K\overline{M}_0\, a_l\overline{N}_0\, b_l\, g < +\infty.$$

Therefore, $N(b,t) \in L^1(0,b_l)$ for every $t \in (0,g]$.

2.2 Consider the following limit:

$$\lim_{h\to 0} \|N(b,t+h) - N(b,t)\|_{L^1(0,b_l)} = \lim_{h\to 0}\int_0^{b_l} \big|N(b,t+h) - N(b,t)\big|db$$

$$= \lim_{h\to 0}\left(\int_0^t \big|N(b,t+h) - N(b,t)\big|db + \int_t^{b_l} \big|N(b,t+h) - N(b,t)\big|db\right).$$

Evaluating the first addend, one obtains:

$$\int_0^t \big|N(b,t+h) - N(b,t)\big|db$$

$$= \int_0^t \Bigg|N(0,t+h-b)\exp\left(-\int_0^b m(s,s+t+h-b)\,ds\right)$$

$$-N(0,t-b)\,\exp\left(-\int_0^b m(s,s+t-b)\,ds\right)\Bigg|\,db$$

$$\leq \int_0^t N(0,t+h-b)\Bigg|\exp\left(-\int_0^b m(s,s+t+h-b)\,ds\right)$$

$$-\exp\left(-\int_0^b m(s,s+t-b)\,ds\right)\Bigg|db$$

$$+\int_0^t \big|N(0,t+h-b) - N(0,t-b)\big|\,db\,.$$

One observes that, according to (24):

$$\left|\exp\left(-\int_0^b m(s,s+t+h-b)ds\right)-\exp\left(-\int_0^b m(s,s+t-b)ds\right)\right|$$

$$\leq\left|\exp\left(-\int_0^b m(s,s+t+h-b)ds\right)\exp\left(\int_0^b m(s,s+t-b)ds\right)-1\right|$$

$$=\left|\exp\left(-\int_0^b [\mu_f(s)+A(s,s+t+h-b)-\mu_f(s)-A(s,s+t-b)]ds\right)-1\right|$$

$$\leq\left|\exp\left(-K\int_0^b\int_0^{a_l}[M(a,s+t+h-b)-M(a,s+t-b)]dads\right)-1\right|$$

$$<\left|\exp(-K\,(-\ \bar{\varepsilon})\,a_l\,g)-1\right|\leq\left|\exp\left(K\bar{\varepsilon}\,a_l\,g\right)-1\right|<\varepsilon,$$

because of (17) and the strong continuity of $M(a,t)$ for $t\in(0,g]$ proved in item 1.2 (note that for $0\leq s\leq b<t\leq g$ and for a suitably small h, it results $s+t-b\in(0,g]$ and $s+t+h-b\in(0,g]$). Therefore,

$$\int_0^t\left|N(b,t+h)-N(b,t)\right|db$$

$$<\int_0^t N(0,t+h-b)\,\varepsilon\,db+\int_0^t\left|N(0,t+h-b)-N(0,t-b)\right|db$$

$$=\varepsilon\int_0^t\int_g^{a_l}\int_g^{b_l}\beta_f(u-g,v-g)2k(u-g,v-g)$$

$$\times\widehat{M}_0(u-g,t+h-b-g)\widehat{N}_0(v-g,t+h-b-g)$$

$$\times\exp\left(-\int_{t+h-b-g}^{t+h-b}(\mu_f(s-t-h+b)+\mu_f(v-t-h+b+s))ds\right)dudvdb$$

$$+\int_0^t\left|N(0,t+h-b)-N(0,t-b)\right|db$$

$$\leq \varepsilon K \overline{M}_0\, a_l\, \overline{N}_0\, b_l\, g$$

$$+ \int_0^t \left| \int_g^{a_l} \int_g^{b_l} \beta_f(u-g, v-g) 2k(u-g, v-g) \right.$$

$$\times \exp\left(- \int_{-g}^{0} (\mu_f(s') + \mu_f(v+s')) ds' \right)$$

$$\times \Big[\widehat{M}_0(u-g, t+h-b-g) \widehat{N}_0(v-g, t+h-b-g)$$

$$\left. - \widehat{M}_0(u-g, t-b-g) \widehat{N}_0(v-g, t-b-g) \Big] dudv \right| db$$

$$\leq \varepsilon K \overline{M}_0\, a_l\, \overline{N}_0\, b_l\, g$$

$$+ K \int_0^t \int_g^{a_l} \int_g^{b_l} \Big| \widehat{M}_0(u-g, t+h-b-g) \Big[\widehat{N}_0(v-g, t+h-b-g)$$

$$- \widehat{N}_0(v-g, t-b-g) \Big]$$

$$+ \widehat{N}_0(v-g, t-b-g) \Big[\widehat{M}_0(u-g, t+h-b-g)$$

$$- \widehat{M}_0(u-g, t-b-g) \Big] \Big| dudvdb$$

$$< \varepsilon K \overline{M}_0\, a_l\, \overline{N}_0\, b_l\, g + K \overline{M}_0\, \varepsilon\, a_l\, b_l\, g + K \overline{N}_0\, \varepsilon\, a_l\, b_l\, g,$$

for $|h| < \delta(\varepsilon)$. In this chain of inequalities one has used (16), (35) and hypothesis (b)-3.

Consider the second addend in the initial limit. According to (33), it results:

$$\int_t^{b_l} \big| N(b, t+h) - N(b, t) \big|\, db$$

$$\leq \int_t^{b_l} \left| \widehat{N}_0(b-t-h, 0) \exp\left(- \int_{b-t-h}^{b} m(s, s-b+t+h) ds \right) \right.$$

$$\left. - \widehat{N}_0(b-t, 0) \exp\left(- \int_{b-t}^{b} m(s, s-b+t) ds \right) \right| db$$

$$+ \int_t^{b_l} \left| \int_0^{t+h} \exp\left(- \int_{s+b-t-h}^{b} m(\sigma, \sigma-b+t+h) d\sigma \right) S(s+b-t-h, s) ds \right.$$

$$
\begin{aligned}
& - \int_0^t \exp\left(- \int_{s+b-t}^{b} m(\sigma, \sigma - b + t) d\sigma \right) S(s+b-t, s) ds \Bigg| db \\
\le & \int_t^{b_l} \widehat{N}_0(b-t-h, 0) \Bigg| \exp\left(- \int_{b-t-h}^{b} m(s, s-b+t+h) ds \right) \\
& \qquad - \exp\left(- \int_{b-t}^{b} m(s, s-b+t) ds \right) \Bigg| db \\
& + \int_t^{b_l} \exp\left(- \int_{b-t}^{b} m(s, s-b+t) ds \right) \left| \widehat{N}_0(b-t-h, 0) - \widehat{N}_0(b-t, 0) \right| db \\
& + \int_t^{b_l} \Bigg| \int_0^{t+h} \Bigg[\exp\left(- \int_{s+b-t-h}^{b} m(\sigma, \sigma - b + t + h) d\sigma \right) S(s+b-t-h, s) \\
& \qquad - \exp\left(- \int_{s+b-t}^{b} m(\sigma, \sigma - b + t) d\sigma \right) S(s+b-t, s) \Bigg] ds \Bigg| db \\
& + \int_t^{b_l} \Bigg| \int_t^{t+h} \exp\left(- \int_{s+b-t}^{b} m(\sigma, \sigma - b + t) d\sigma \right) S(s+b-t, s) ds \Bigg| db \\
< & \int_t^{b_l} \widehat{N}_0(b-t-h, 0) \Bigg| \exp\left(- \int_{b-t-h}^{b} m(s, s-b+t+h)\, ds \right) \\
& \qquad - \exp\left(- \int_{b-t}^{b} m(s, s-b+t) ds \right) \Bigg| db + \varepsilon b_l \\
& + \int_t^{b_l} \int_0^{t+h} \Bigg| \exp\left(- \int_{s+b-t-h}^{b} m(\sigma, \sigma - b + t + h) d\sigma \right) S(s+b-t-h, s) \\
& \qquad - \exp\left(- \int_{s+b-t}^{b} m(\sigma, \sigma - b + t) d\sigma \right) S(s+b-t, s) \Bigg| ds db \\
& + \int_t^{b_l} \Bigg| \int_t^{t+h} S(s+b-t, s) ds \Bigg| db,
\end{aligned}
$$

for $|h| < \delta(\varepsilon)$, because of the uniform continuity of $\widehat{N}_0(b,0)$ with respect to b, see hypothesis (f). Going on with the chain of inequalities, one has:

$$\int_t^{b_l} \left|N(b,t+h) - N(b,t)\right| db$$

$$< \int_t^{b_l} \widehat{N}_0(b-t-h,0)\left| \exp\left(-\int_{b-t-h}^{b} m(s,s-b+t+h)ds\right) - \exp\left(-\int_{b-t}^{b} m(s,s-b+t)ds\right)\right| db + \varepsilon b_l$$

$$+ \int_t^{b_l}\int_0^{t+h} \left| \exp\left(-\int_{s+b-t-h}^{b} m(\sigma,\sigma-b+t+h)d\sigma\right) S(s+b-t-h,s) - \exp\left(-\int_{s+b-t}^{b} m(\sigma,\sigma-b+t)d\sigma\right) S(s+b-t,s)\right| ds db + \varepsilon\, b_l,$$

for $|h| < \delta(\varepsilon)$, because of the continuity of integrals. Moreover, one has, also reminding that $\widehat{N}_0$ belongs to $L^\infty(0,b_l)$:

$$\int_t^{b_l} \left|N(b,t+h) - N(b,t)\right| db$$

$$< 2\varepsilon b_l + \int_t^{b_l} \widehat{N}_0(b-t-h,0)\left| \exp\left(-\int_{b-t-h}^{b} m(s,s-b+t+h)ds\right) - \exp\left(-\int_{b-t-h}^{b} m(s,s-b+t)ds\right)\right| db$$

$$+ \int_t^{b_l} \widehat{N}_0(b-t-h,0)\left| \exp\left(-\int_{b-t-h}^{b} m(s,s-b+t)ds\right) - \exp\left(-\int_{b-t}^{b} m(s,s-b+t)ds\right)\right| db$$

$$+\int_t^{b_l}\int_0^{t+h}\exp\left(-\int_{s+b-t-h}^{b} m(\sigma,\sigma-b+t+h)d\sigma\right)$$

$$\times\left|S(s+b-t-h,s)-S(s+b-t,s)\right|dsdb$$

$$+\int_t^{b_l}\int_0^{t+h}\left|S(s+b-t,s)\right|\left|\exp\left(-\int_{s+b-t-h}^{b} m(\sigma,\sigma-b+t+h)d\sigma\right)\right.$$

$$\left.-\exp\left(-\int_{s+b-t}^{b} m(\sigma,\sigma-b+t)d\sigma\right)\right|dsdb$$

$$<2\varepsilon\, b_l+2\varepsilon\overline{N}_0\, b_l+\int_t^{b_l}\int_0^{t+h}\left|S(s+b-t-h,s)-S(s+b-t,s)\right|dsdb$$

$$+\int_t^{b_l}\int_0^{t+h}\left|S(s+b-t,s)\right|\left|\exp\left(-\int_{s+b-t-h}^{b} m(\sigma,\sigma-b+t+h)d\sigma\right)\right.$$

$$\left.-\exp\left(-\int_{s+b-t-h}^{b} m(\sigma,\sigma-b+t)d\sigma\right)\right|dsdb$$

$$+\int_t^{b_l}\int_0^{t+h}\left|S(s+b-t,s)\right|\left|\exp\left(-\int_{s+b-t-h}^{b} m(\sigma,\sigma-b+t)d\sigma\right)\right.$$

$$\left.-\exp\left(-\int_{s+b-t}^{b} m(\sigma,\sigma-b+t)d\sigma\right)\right|dsdb,$$

for $|h|<\delta(\varepsilon)$. One evaluates the term $\int_t^{b_l}\int_0^{t+h}\left|S(s+b-t-h,s)\right.$ $\left.-S(s+b-t,s)\right|ds\,db$, according to (30), (29) and (24); it results:

$$\int_t^{b_l}\int_0^{t+h}\left|S(s+b-t-h,s)-S(s+b-t,s)\right|ds\,db$$

$$=\int_t^{b_l}\int_0^{t+h}\left|\exp\left(-\int_{-g}^{0}\mu_f(s+b-t-h+\tau)d\tau\right)\widehat{N}_0(s+b-t-h-g,s-g)\right.$$

$$\times \left[\int_0^{a_l} 2k(a, s+b-t-h-g)\widehat{M}_0(a, s-g)da \right.$$

$$\left. - \int_0^{a_l} 2k(a, s+b-t-g)\widehat{M}_0(a, s-g)da \right]$$

$$+ \int_0^{a_l} 2k(a, s+b-t-g)\widehat{M}_0(a, s-g)da$$

$$\times \left[\exp\left(- \int_{-g}^{0} \mu_f(s+b-t-h+\tau)d\tau \right) \widehat{N}_0(s+b-t-h-g, s-g) \right.$$

$$\left. \left. - \exp\left(- \int_{-g}^{0} \mu_f(s+b-t+\tau)d\tau \right) \widehat{N}_0(s+b-t-g, s-g) \right] \right| dsdb$$

$$\leq \int_t^{b_l} \int_0^{t+h} \exp\left(- \int_{-g}^{0} \mu_f(s+b-t-h+\tau)d\tau \right) \widehat{N}_0(s+b-t-h-g, s-g)$$

$$\times \left| \int_0^{a_l} 2[k(a, s+b-t-h-g) - k(a, s+b-t-g)]\widehat{M}_0(a, s-g)da \right| dsdb$$

$$+ \int_t^{b_l} \int_0^{t+h} \int_0^{a_l} 2k(a, s+b-t-g)\widehat{M}_0(a, s-g)da$$

$$\times \left| \exp\left(- \int_{-g}^{0} \mu_f(s+b-t-h+\tau)d\tau \right) \widehat{N}_0(s+b-t-h-g, s-g) \right.$$

$$\left. - \exp\left(- \int_{-g}^{0} \mu_f(s+b-t+\tau)d\tau \right) \widehat{N}_0(s+b-t-g, s-g) \right| dsdb$$

$$\leq \int_t^{b_l} \int_0^{t+h} \widehat{N}_0(s+b-t-h-g, s-g)$$

$$\times \int_0^{a_l} 2\big|k(a, s+b-t-h-g) - k(a, s+b-t-g)\big|\widehat{M}_0(a, s-g)dadsdb$$

$$+K\int_t^{b_l}\int_0^{t+h}\int_0^{a_l}\widehat{M}_0(a,s-g)da\Big|\widehat{N}_0(s+b-t-h-g,s-g)$$
$$-\widehat{N}_0(s+b-t-g,s-g)\Big|dsdb$$

$$+K\int_t^{b_l}\int_0^{t+h}\int_0^{a_l}\widehat{M}_0(a,s-g)da\widehat{N}_0(s+b-t-g,s-g)$$

$$\times\left|\exp\left(-\int_{-g}^{0}\mu_f(s+b-t-h+\tau)d\tau\right)-\exp\left(-\int_{-g}^{0}\mu_f(s+b-t+\tau)d\tau\right)\right|dsdb$$

$$< 2\varepsilon\overline{M}_0\,a_l\,\overline{N}_0\,b_l\,g+K\,\overline{M}_0\,a_l\,\varepsilon\,b_l\,g+K\,\overline{M}_0\,a_l\,\varepsilon\,\overline{N}_0\,b_l\,g\,,$$

for $|h|<\delta(\varepsilon)$, according to (17), to the uniform continuity of $k(a,b)$ with respect to $b\in[0,b_l)$ for $a\in[0,a_l)$, see hypothesis (e), to hypothesis (b), see also (34), to the uniform continuity of $\widehat{N}_0(b,t)$ with respect to b for $t\in[-g,0]$ and to the continuity of integrals.
Resuming the chain of inequalities considered before, one can write, for $|h|<\delta(\varepsilon)$:

$$\int_t^{b_l}\big|N(b,t+h)-N(b,t)\big|db$$
$$<2\varepsilon b_l+2\varepsilon\overline{N}_0b_l+2\varepsilon\overline{M}_0a_l\overline{N}_0b_lg+K\overline{M}_0a_l\varepsilon b_lg+K\overline{M}_0a_l\varepsilon\overline{N}_0b_lg$$
$$+\int_t^{b_l}\int_0^{t+h}\big|S(s+b-t,s)\big|\left|\exp\left(-\int_{s+b-t-h}^{b}m(\sigma,\sigma-b+t+h)d\sigma\right)\right.$$
$$\left.-\exp\left(-\int_{s+b-t-h}^{b}m(\sigma,\sigma-b+t)d\sigma\right)\right|dsdb$$
$$+\int_t^{b_l}\int_0^{t+h}\big|S(s+b-t,s)\big|\left|\exp\left(-\int_{s+b-t-h}^{b}m(\sigma,\sigma-b+t)d\sigma\right)\right.$$
$$\left.-\exp\left(-\int_{s+b-t}^{b}m(\sigma,\sigma-b+t)d\sigma\right)\right|dsdb.$$

Being

$$\big|S(s+b-t,s)\big| = \Bigg|\exp\Bigg(-\int\limits_{-g}^{0}\mu_f(s+b-t+\tau)d\tau\Bigg)\int\limits_{0}^{a_l}2k(a,s+b-t-g)$$

$$\widehat{M}_0(a,s-g)da\widehat{N}_0(s+b-t-g,s-g)\Bigg|$$

$$\leq \int\limits_{0}^{a_l}2k(a,s+b-t-g)\widehat{M}_0(a,s-g)\,da\,\widehat{N}_0(s+b-t-g,s-g)$$

$$\leq K\overline{M}_0\,a_l\,\widehat{N}_0(s+b-t-g,s-g),$$

see (17) and (34), one obtains, for $|h|<\delta(\varepsilon)$:

$$\int\limits_{t}^{b_l}\big|N(b,t+h)-N(b,t)\big|db < 2\varepsilon b_l + 2\varepsilon\overline{N}_0 b_l + 2\varepsilon\overline{M}_0 a_l\overline{N}_0 b_l g$$

$$+K\overline{M}_0 a_l\varepsilon b_l g + K\overline{M}_0 a_l\varepsilon\overline{N}_0 b_l g + K\overline{M}_0 a_l\varepsilon\overline{N}_0 b_l g + K\overline{M}_0 a_l\varepsilon\overline{N}_0 b_l g.$$

One concludes that

$$\int\limits_{0}^{b_l}\big|N(b,t+h)-N(b,t)\big|db < \varepsilon K\overline{M}_0 a_l\overline{N}_0 b_l g + K\overline{M}_0\varepsilon a_l b_l g$$

$$+K\overline{N}_0\varepsilon a_l b_l g + 2\varepsilon b_l + 2\varepsilon\overline{N}_0 b_l + 2\varepsilon\overline{M}_0 a_l\overline{N}_0 b_l g + K\overline{M}_0 a_l\varepsilon b_l g$$

$$+K\overline{M}_0 a_l\varepsilon\overline{N}_0 b_l g + K\overline{M}_0 a_l\varepsilon\overline{N}_0 b_l g + K\overline{M}_0 a_l\varepsilon\overline{N}_0 b_l g,$$

for $|h|<\delta(\varepsilon)$, therefore

$$\lim_{h\to 0}\|N(b,t+h)-N(b,t)\|_{L^1(0,b_l)} = 0.$$

Following the steps in items 1 and 2, one can also prove that functions (20) and (33) belong to $L^1(0,a_l)$ and $L^1(0,b_l)$, respectively, and are strongly continuous for $t\in(g,2g]$, that is for $i=1$. By induction, the proof can be concluded.

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