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M. K. Bose and R. Tiwari (*)

On (ω)topological spaces (**)

1 - Introduction

In [**1**], we introduced the notion of (ω)topological spaces. There we defined and studied some notions of separation axioms ((ω)Hausdorffness, (ω)regularity and (ω)normality) and compactness ((ω)compactness, local (ω)compactness and (ω)paracompactness). It was proved that a (ω)Hausdorff (ω)paracompact space is (ω)normal. In this paper, we introduce (ω^*)normality which is stronger than (ω)normality. We show that (*i*) a (ω)Hausdorff (ω)paracompact space is (ω^*)normal (Theorem 3.1), and (*ii*) if a space is (ω^*)normal, then every point finite (ω)open cover is shrinkable (Theorem 3.3). As promised in [**1**], a (ω)topological version of Michael's theorem— a characterization of (ω)paracompactness (Theorem 3.2) is given. Theorems 3.4 and 3.5 together reflect the celebrated Stone's theorem on paracompactness.

2 - Preliminaries

We denote the set of natural numbers by N. The following definitions were introduced in Bose and Tiwari [**1**].

(*) M. K. Bose: Dept. of Mathematics, University of North Bengal, Siliguri, W. Bengal - 734013, India. e-mail: manojkumarbose@yahoo.com; R. Tiwari: Dept. of Mathematics, St. Joseph's College, Darjeeling, W. Bengal - 734104, India. e-mail: tiwarirupesh1@yahoo.co.in
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Definition 2.1. *If* $\{\mathcal{J}_n\}$ *is a sequence of topologies on a set* X *with* $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ *for all* $n \in N$, *then the pair* $(X, \{\mathcal{J}_n\})$ *is called a* (ω)*topological space.*

In the sequel, the (ω)topological space $(X, \{\mathcal{J}_n\})$ is simply denoted by X. The closure of a set $A \subset X$ with respect to a topology $\mathcal{J}$ on X is denoted by $(\mathcal{J})clA$.

Definition 2.2. *Any set* $G \in \cup_n \mathcal{J}_n$ *is called a* (ω)*open set. A set* F *is called* (ω)*closed if* $X - F$ *is* (ω)*open. A set is* $(\sigma\omega)$*open (resp.* $(\delta\omega)$*closed) if it is the union (resp. intersection) of a countable number of* (ω)*open (resp.* (ω)*closed) sets.*

Definition 2.3. X *is said to be* (ω)*Hausdorff if for any two distinct points* x, y *of* X, *there exists an* n *such that for some* $U, V \in \mathcal{J}_n$, *we have* $x \in U$, $y \in V$ *and* $U \cap V = \emptyset$.

Definition 2.4. X *is said to be* (ω)*regular if given a* (ω)*closed set* F *and a point* $x \in X$ *with* $x \notin F$, *there exists an* n *such that for some* $U, V \in \mathcal{J}_n$, *we have* $x \in U$, $F \subset V$ *and* $U \cap V = \emptyset$.

Definition 2.5. X *is said to be* (ω)*normal if given two* (ω)*closed sets* A *and* B *with* $A \cap B = \emptyset$, *there exists an* n *such that for some* $U, V \in \mathcal{J}_n$, *we have* $A \subset U$, $B \subset V$ *and* $U \cap V = \emptyset$.

Definition 2.6. *A collection* $\mathcal{C}$ *of subsets of* X *is said to be* $(\mathcal{J}_n)$*locally finite if each* $x \in X$ *has a* $(\mathcal{J}_n)$*open nbd intersecting atmost finitely many sets* $\in \mathcal{C}$.

Definition 2.7. X *is said to be* (ω)*paracompact if every* (ω)*open cover of* X *has, for some* n, *a* $(\mathcal{J}_n)$*locally finite* $(\mathcal{J}_n)$*open refinement.*

In [**1**], (ω)paracompactness was defined for (ω)Hausdorff spaces. We require the following theorem [**1**].

Theorem 2.1. *If* X *is* (ω)*Hausdorff and* (ω)*paracompact, then* X *is* (ω)*regular.*

3 - (ω)paracompactness

We introduce the following definitions.

Definition 3.1. X *is said to be* (ω^*)*normal if given two* $(\delta\omega)$*closed sets* A *and* B *with* $A \cap B = \emptyset$, *there exists an* n *such that for some* $U, V \in \mathcal{J}_n$, *we have* $A \subset U$, $B \subset V$ *and* $U \cap V = \emptyset$.

It is easy to see that X is (ω^*)normal iff for any $(\delta\omega)$closed set F and any $(\sigma\omega)$open set G with $F \subset G$, there exists an n such that for some $(\mathcal{J}_n)$open set U, we have $F \subset U \subset (\mathcal{J}_n)clU \subset G$.

Obviously (ω^*)normality is stronger than (ω)normality.

Example 3.1. *The (ω)topological space $(N, \{\mathcal{J}_n\})$, where $\mathcal{J}_n$ is the topology generated by the base*

$$\{\emptyset, N\} \cup \{\cup_{i=1}^{n}\{\{even\ integers \leq i\}, \{odd\ integers \leq i\}\}\}$$

is (ω)normal but not (ω^*)normal.

Definition 3.2. *A (ω)open cover $\mathcal{U} = \{U_a | a \in A\}$ of X is shrinkable if there exists a (ω)open cover $\mathcal{V} = \{V_a \mid a \in A\}$ such that for each $a \in A$, there exists an $n_a \in N$ such that $(\mathcal{J}_{n_a})clV_a \subset U_a$. In this case, $\mathcal{V}$ is said to be a shrinking of $\mathcal{U}$.*

Definition 3.3. *A (ω)open cover $\mathcal{U}$ of X is said to be point finite if each $x \in X$, belongs to finitely many sets $\in \mathcal{U}$.*

Theorem 3.1. *If X is (ω)Hausdorff and (ω)paracompact, then it is (ω^*)normal.*

Proof. Let A and B be two disjoint $(\delta\omega)$closed sets. Let $x \in A$. Then $x \notin B$, and so there exists a (ω)closed set F such that $B \subset F$ and $x \notin F$. Therefore by Theorem 2.1, there exists, for some $n_x \in N$, two disjoint $(\mathcal{J}_{n_x})$open sets U_x and V_x such that $x \in U_x, F \subset V_x$. The set $X - A$ is $(\sigma\omega)$open. Suppose $X - A = \cup_{k\in N} G_k$, where G_k are (ω)open sets. Then the family $\mathcal{U} = \{U_x \mid x \in A\} \cup \{G_k \mid k \in N\}$ is a (ω)open cover of X. Since X is (ω)paracompact, there exists, for some n, a $(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement $\mathcal{C}$ of $\mathcal{U}$. Let $U = \cup\{G \in \mathcal{C} \mid G \cap A \neq \emptyset\}$. Then $U \in \mathcal{J}_n$ and $A \subset U$. For each $y \in B$, there exists a $(\mathcal{J}_n)$open nbd W_y of y that intersects only a finite number of sets $U_1(y), ..., U_k(y) \in \mathcal{C}$ with $U_i(y) \cap A \neq \emptyset,\ i = 1, 2, ..., k$. Let $U_i(y) \subset U_{x_i}$, $i = 1, 2, ..., k$ and $H_y = W_y \cap \left(\cap_{i=1}^{k} V_{x_i}\right)$. Then $y \in H_y$, and $H_y \in \mathcal{J}_l$ where $l = \max(n, n_{x_1}, n_{x_2}, ..., n_{x_k})$. Also $U \cap H_y = \emptyset$. Suppose $X - B = \cup_{k\in N} D_k$, where D_k are (ω)open. Let $\mathcal{E}$ be a $(\mathcal{J}_m)$locally finite $(\mathcal{J}_m)$open refinement of the (ω)open cover $\mathcal{W} = \{H_y \mid y \in B\} \cup \{D_k \mid k \in N\}$. Let $W = \cup\{E \in \mathcal{E} \mid E \cap B \neq \emptyset\}$. Then W is $(\mathcal{J}_m)$open, $B \subset W$ and $U \cap W = \emptyset$. Also $U, W \in \mathcal{J}_r$, $r = \max(m, n)$. □

Now we provide a (ω)topological version of Michael's theorem (Michael [3]) on (ω)paracompactness. We call a collection $\mathcal{C}$ of subsets of X, σ-$(\mathcal{J}_n)$*locally finite* if $\mathcal{C} = \cup_{k=1}^{\infty} \mathcal{C}_k$, where each $\mathcal{C}_k$ is a $(\mathcal{J}_n)$locally finite collection.

Theorem 3.2. *Suppose, for each* $n \in N$, $(X, \mathcal{J}_n)$ *is a regular topological space. Then the following statements are equivalent.*

(*a*) *X is* (ω)*paracompact.*

(*b*) *Each* (ω)*open cover of X has, for some n, a* σ-($\mathcal{J}_n$)*locally finite* ($\mathcal{J}_n$)*open refinement.*

(*c*) *Each* (ω)*open cover* $\mathcal{U}$ *of X has, for some n, a* ($\mathcal{J}_n$)*open refinement* $\mathcal{V}$ *and* $\mathcal{V}$ *has a* ($\mathcal{J}_n$)*locally finite refinement (not necessarily* ($\mathcal{J}_n$)*open).*

(*d*) *Each* (ω)*open cover* $\mathcal{U}$ *of X has, for some m, a* ($\mathcal{J}_m$)*open refinement* $\mathcal{V}$ *and* $\mathcal{V}$ *has, for some n, a* ($\mathcal{J}_n$)*locally finite* ($\mathcal{J}_n$)*closed refinement.*

Proof. $(a) \Longrightarrow (b)$: Follows easily.

$(b) \Longrightarrow (c)$: Similar to the proof of the part $(2) \Longrightarrow (3)$ of Theorem 2.3 (Dugundji [**2**], p. 163) .

$(c) \Longrightarrow (d)$: Let $\mathcal{U}$ be a (ω)open cover of X. Then by (*c*), $\mathcal{U}$ has, for some m, a ($\mathcal{J}_m$)open refinement $\mathcal{V}$. Consider $x \in X$ and $V_x \in \mathcal{V}$ such that $x \in V_x$. Then by the regularity of $(X, \mathcal{J}_m)$, there exists a ($\mathcal{J}_m$)open set W_x such that $x \in W_x \subset (\mathcal{J}_m)clW_x \subset V_x$. Then $\mathcal{W} = \{W_x \mid x \in X\}$ is a ($\mathcal{J}_m$)open, and hence (ω)open cover of X. Therefore by (*c*), $\mathcal{W}$ has, for some l, a ($\mathcal{J}_l$)locally finite refinement $\mathcal{A}$. For $A \in \mathcal{A}$, there exists a W_x such that $A \subset W_x$. So if $n = \max(l, m)$, then $(\mathcal{J}_n)clA \subset (\mathcal{J}_n)clW_x \subset (\mathcal{J}_m)clW_x \subset V_x$. Since $\mathcal{A}$ is ($\mathcal{J}_l$)locally finite and $\mathcal{J}_l \subset \mathcal{J}_n$, it is also ($\mathcal{J}_n$)locally finite. Thus, by 9.2 (Dugundji [**2**], p. 82), $\{(\mathcal{J}_n)clA \mid A \in \mathcal{A}\}$ is a ($\mathcal{J}_n$)locally finite ($\mathcal{J}_n$)closed refinement of $\mathcal{V}$.

$(d) \Longrightarrow (a)$: Let $\mathcal{U}$ be a (ω)open cover of X. Then by (*d*), it has, for some l, a ($\mathcal{J}_l$)open refinement $\mathcal{V}$ and $\mathcal{V}$ has, for some m, a ($\mathcal{J}_m$)locally finite ($\mathcal{J}_m$)closed refinement $\mathcal{W}$. For each point $x \in X$, there exists some ($\mathcal{J}_m$)open nbd intersecting a finite number of elements of $\mathcal{W}$. The collection $\mathcal{G}$ of all these nbds forms a ($\mathcal{J}_m$)open cover, and hence a (ω)open cover of X. Therefore by (*d*), $\mathcal{G}$ has, for some n, a ($\mathcal{J}_n$)locally finite ($\mathcal{J}_n$)closed refinement $\mathcal{F}$. For $W \in \mathcal{W}$, we define

$$\mathcal{F}(W) = \{F \in \mathcal{F} \mid F \cap W = \emptyset\},$$

$$F(W) = \cup \{F \mid F \in \mathcal{F}(W)\},$$

$$D(W) = X - F(W).$$

Then $\mathcal{F}(W)$ is a ($\mathcal{J}_n$)locally finite collection of ($\mathcal{J}_n$)closed sets. Therefore by 9.2 (Dugundji [**2**], p. 82), $F(W)$ is ($\mathcal{J}_n$)closed. Hence $D(W)$ is ($\mathcal{J}_n$)open. Since $\mathcal{W}$ is a cover of X, it follows that the collection $\mathcal{D} = \{D(W) \mid W \in \mathcal{W}\}$ is a ($\mathcal{J}_n$)open cover of X. It is also ($\mathcal{J}_n$)locally finite. In fact, if B_x is a ($\mathcal{J}_n$)open nbd of x intersecting the sets $F_1, F_2, ..., F_k$ of $\mathcal{F}$, then $B_x \cap D(W) \neq \emptyset$ implies that for some $i = 1, 2, ..., k$, $F_i \cap D(W) \neq \emptyset$ and so $F_i \cap W \neq \emptyset$. Since each F_i is contained in some member of $\mathcal{G}$, it can intersect a finite number of members W of $\mathcal{W}$, and so it follows that B_x can intersect a finite number of sets $D(W)$ of $\mathcal{D}$.

Now for each $W \in \mathcal{W}$, we choose a $V \in \mathcal{V}$ such that $W \subset V$. If $r = \max(l, n)$, then the collection $\{V \cap D(W) \mid W \in \mathcal{W}\}$ is a $(\mathcal{J}_r)$locally finite $(\mathcal{J}_r)$open refinement of $\mathcal{V}$ and hence of $\mathcal{U}$. Hence X is (ω)paracompact. □

Theorem 3.3. *If X is (ω^*)normal, then every point finite (ω)open cover of X is shrinkable.*

Proof. Suppose X is (ω^*)normal and $\mathcal{U} = \{U_a \mid a \in \mathcal{A}\}$ is a point finite (ω)open cover of X. We well-order the set A and let $A = \{1, 2, 3, ..., a, ...\}$. We now construct a collection of (ω)open sets $\mathcal{V} = \{V_a \mid a \in A\}$, by transfinite induction, as follows. Let $F_1 = X - \cup\{U_a \mid a > 1\}$. Since for a fixed n, an arbitrary union of $(\mathcal{J}_n)$open sets is a $(\mathcal{J}_n)$open set, it follows that the union $\cup\{U_a \mid a > 1\}$ is $(\sigma\omega)$open, and hence F_1 is $(\delta\omega)$closed. Also $F_1 \subset U_1$. So by (ω^*)normality, there exists an $n_1 \in N$ such that for some $(\mathcal{J}_{n_1})$open set V_1, $F_1 \subset V_1 \subset (\mathcal{J}_{n_1})clV_1 \subset U_1$. We suppose V_β has been defined for each $\beta < a$. Let $F_a = X - H_a$, where $H_a = (\cup\{V_\beta \mid \beta < a\}) \cup (\cup\{U_\gamma \mid \gamma > a\})$. Then F_a is $(\delta\omega)$closed and $F_a \subset U_a$. Therefore there exists an $n_a \in N$ such that for some $(\mathcal{J}_{n_a})$open set V_a, we have $F_a \subset V_a \subset (\mathcal{J}_{n_a})clV_a \subset U_a$. Now we show that $\mathcal{V} = \{V_a \mid a \in A\}$ is a cover of X. If $x \in X$, then x belongs to only finitely many elements $U_{a_1}, U_{a_2}, ..., U_{a_n}$ of $\mathcal{U}$. If $a = \max(a_1, a_2, ..., a_n)$, then $x \notin U_\gamma$ for $\gamma > a$. Therefore $x \in V_\beta$ for some $\beta \leq a$, since if $x \notin V_\beta$ for all $\beta < a$, then x must belong to F_a and hence to V_a. Thus $\mathcal{V}$ is a cover of X and hence it is a shrinking of $\mathcal{U}$. □

For $x \in X, A \subset X$ and a (ω)open cover $\mathcal{U}$ of X. We write

$$St(x, \mathcal{U}) = \cup\{U \mid x \in U \in \mathcal{U}\}$$

$$St(A, \mathcal{U}) = \cup\{U \mid U \in \mathcal{U},\ U \cap A \neq \phi\}.$$

A cover $\mathcal{V}$ of X is called a *barycentric refinement* (resp. *star refinement*) (Dugundji [**2**], p. 167) of $\mathcal{U}$ if for each $x \in X$ (resp. $V \in \mathcal{V}$), $St(x, \mathcal{V})$ (resp. $St(V, \mathcal{V})$) is contained in some $U \in \mathcal{U}$.

Theorem 3.4. *If X is (ω)Hausdorff and (ω)paracompact, then every (ω)open cover of X has, for some n, a $(\mathcal{J}_n)$open barycentric refinement.*

Proof. Suppose X is (ω)Hausdorff and (ω)paracompact. Then by Theorem 3.1, X is (ω^*)normal. Let $\mathcal{U}$ be a (ω)open cover of X. Then for some m, $\mathcal{U}$ has a $(\mathcal{J}_m)$locally finite $(\mathcal{J}_m)$open refinement $\mathcal{V} = \{V_a \mid a \in A\}$. Since X is (ω^*)normal and since each $(\mathcal{J}_n)$locally finite cover is point finite, by Theorem 3.3, there exists a shrinking

$\mathcal{W} = \{W_a \mid a \in A\}$ of $\mathcal{V}$. For each $a \in A$, there exists an $n_a \in N$ such that $(\mathcal{J}_{n_a})clW_a \subset V_a$, and so $\mathcal{W}$ is also $(\mathcal{J}_m)$locally finite. Since for all n, $\mathcal{J}_n \subset \mathcal{J}_{n+1}$, we may assume that $n_a \geq m$ for all a. Now let $x \in X$ and

$$G_x = \cap\{V_a \mid a \in A,\ x \in (\mathcal{J}_{n_a})clW_a\}.$$

Since $(\mathcal{J}_{n_a})clW_a \subset V_a$, each V_a in the above intersection contains x. So from the $(\mathcal{J}_m)$local finiteness of $\mathcal{V}$, it follows that G_x is the intersection of finite number of $(\mathcal{J}_m)$open sets V_a. Hence G_x is $(\mathcal{J}_m)$open. If

$$F_x = \cup\{(\mathcal{J}_{n_a})clW_a \mid x \notin (\mathcal{J}_{n_a})clW_a\},$$

then $X - F_x$ is $(\sigma\omega)$open : Let $y \in X - F_x$. Since $\mathcal{W}$ is $(\mathcal{J}_m)$locally finite, there exists $(\mathcal{J}_m)$open nbd D_y of y such that except for a finite number of a, we have $D_y \cap W_a = \emptyset \Longrightarrow D_y \cap (\mathcal{J}_m)clW_a = \emptyset$. Suppose D_y intersects the sets $(\mathcal{J}_m)clW_{a_1}, ..., (\mathcal{J}_m)clW_{a_k}$ for which $x \notin (\mathcal{J}_{n_a})clW_a$. Since for all a, $n_a \geq m$, D_y intersects at most $(\mathcal{J}_{n_{a_1}})clW_{a_1}, ..., (\mathcal{J}_{n_{a_k}})clW_{a_k}$. Therefore $B_y = D_y \cap (\cap_{i=1}^{k} (X - (\mathcal{J}_{n_{a_i}})clW_{a_i}))$ is a $(\mathcal{J}_l)$open nbd of y, where $l = \max(m, n_{a_1}, ..., n_{a_k})$. Also $B_y \subset X - F_x$. Therefore it follows that $X - F_x$ is $(\sigma\omega)$open. Hence $H_x = G_x \cap (X - F_x)$ is $(\sigma\omega)$open. So $\mathcal{H} = \{H_x \mid x \in X\}$ is a $(\sigma\omega)$open cover of X. For each x, replacing the $(\sigma\omega)$open set H_x of $\mathcal{H}$ by the (ω)open sets whose union is H_x, we get a (ω)open cover $\mathcal{C}$ of X. Let $\mathcal{P}$ be a $(\mathcal{J}_n)$open refinement of $\mathcal{C}$ for some n. We now show that $\mathcal{P}$ is a $(\mathcal{J}_n)$open barycentric refinement of $\mathcal{U}$. It is sufficient to show that $\mathcal{C}$ is a barycentric refinement of $\mathcal{V}$.

We take any $y \in X$ and an a such that $y \in (\mathcal{J}_{n_a})clW_a$. Let $y \in C \in \mathcal{C}$ and $C \subset H_x = G_x \cap (X - F_x)$. Therefore $x \in (\mathcal{J}_{n_a})clW_a$, since otherwise $(\mathcal{J}_{n_a})clW_a \subset F_x$ and so $y \notin H_x$. Therefore $G_x \subset V_a \Rightarrow H_x \subset V_a \Rightarrow C \subset V_a \Rightarrow St(y, \mathcal{C}) \subset V_a$. It thus follows that $\mathcal{C}$ is a barycentric refinement of $\mathcal{V}$. □

In Theorem 3.2, we require the following condition: For each n, the topological space $(X, \mathcal{J}_n)$ is regular. In the next theorem, we use Theorem 3.2 and so we need the above condition which is stronger than (ω)regularity.

Theorem 3.5. *Suppose for each n, $(X, \mathcal{J}_n)$ is a regular topological space. If each (ω)open cover $\mathcal{U}$ of X has, for some n, a sequence $\{\mathcal{U}_k\}$ of $(\mathcal{J}_n)$open covers such that $\mathcal{U}_1$ is a barycentric refinement of $\mathcal{U}$ and for all $k > 1$, $\mathcal{U}_{k+1}$ is a barycentric refinement of $\mathcal{U}_k$, then the space X is (ω)paracompact.*

Proof. Let $\mathcal{U} = \{U_a \mid a \in A\}$ be a (ω)open cover of X satisfying the above condition. We denote by V_a the set of those points x of U_a for which $St(x, \mathcal{U}_k) \subset U_a$ for some k. Then the collection $\mathcal{V} = \{V_a \mid a \in A\}$ forms a refinement of $\mathcal{U}$: By definition

$V_a \subset U_a$. Also for each $x \in X$, $St(x, \mathcal{U}_1) \subset U_a$ for some a and so $x \in V_a$. Thus $\mathcal{V}$ is a cover of X.

We well order set A and let $A = \{1, 2, ..., a, ...\}$. For any $k \in N$, we define

$$F_{k1} = X - St(X - V_1, \mathcal{U}_k), \text{ and for } a > 1,$$

$$F_{ka} = X - St((X - V_a) \cup (\cup_{\beta < a} F_{k\beta}), \mathcal{U}_k).$$

Then F_{ka} are $(\mathcal{J}_n)$closed sets. Obviously for all a, $F_{ka} \cap St((X - V_a), \mathcal{U}_k) = \emptyset$ and so $(X - V_a) \cap St(F_{ka}, \mathcal{U}_k) = \emptyset$. Therefore

$$St(F_{ka}, \mathcal{U}_k) \subset V_a \text{ for all } a. \tag{1}$$

Also we have

$$St(F_{ka}, \mathcal{U}_k) \cap F_{k\beta} = \emptyset \text{ for all } \beta \neq a. \tag{2}$$

The collection $\{F_{ka} \mid k \in N, a \in A\}$ forms a cover of X : consider $x \in X$. Let a be the first index such that $x \in V_a$. Then $St(x, \mathcal{U}_k) \subset U_a$ for some k. Since $\mathcal{U}_{k+1}$ is a barycentric refinement of $\mathcal{U}_k$, it follows that $St(x, \mathcal{U}_{k+1}) \subset V_a$. Thus we can say

$$St(x, \mathcal{U}_l) \subset V_a \text{ for some } l. \tag{3}$$

If $x \in St((X - V_a) \cup (\cup_{\beta < a} F_{l\beta}), \mathcal{U}_l)$, then $St(x, \mathcal{U}_l)$ intersects $(X - V_a) \cup (\cup_{\beta < a} F_{l\beta})$ and so $St(x, \mathcal{U}_l)$ intersects $\cup_{\beta < a} F_{l\beta}$ (by (3)). Therefore $St(x, \mathcal{U}_l)$ intersects $F_{l\beta}$ for some $\beta < a$ and so $x \in St(F_{l\beta}, \mathcal{U}_l) \subset V_\beta$ (by (1)) which is impossible, since a is the first index such that $x \in V_a$. Therefore $x \notin St((X - V_a) \cup (\cup_{\beta < a} F_{l\beta}), \mathcal{U}_l)$ and so $x \in F_{la}$ (by the definition of F_{la}).

Now for each pair of k and a,

$$G_{ka} = St(F_{ka}, \mathcal{U}_{k+2}) \subset St(F_{ka}, \mathcal{U}_k) \subset V_a \quad \text{(by (1))}$$

and G_{ka} is a $(\mathcal{J}_n)$open. Also the collection $\mathcal{G} = \{G_{ka} \mid k \in N \text{ and } a \in A\}$ forms a cover of X. For if $x \in X$, then for some k and a, $x \in F_{ka} \subset St(F_{ka}, \mathcal{U}_{k+2}) = G_{ka}$.

There is no $B \in \mathcal{U}_{k+2}$ intersecting both G_{ka} and $G_{k\beta}$ for $a \neq \beta$. For if B intersects both G_{ka} and $G_{k\beta}$, there exist $B_1, B_2 \in \mathcal{U}_{k+2}$ such that B_1 intersects both F_{ka} and B, and B_2 intersects both $F_{k\beta}$ and B. Therefore $St(B, \mathcal{U}_{k+2})$ intersects both F_{ka} and $F_{k\beta}$. Since $\mathcal{U}_{k+2}$ is a star refinement of $\mathcal{U}_k$ (by 3.4, Dugundji [2], p. 167), some $D \in \mathcal{U}_k$ intersects both F_{ka} and $F_{k\beta}$, and hence $St(F_{ka}, \mathcal{U}_k) \cap F_{k\beta} \neq \emptyset$ which contradicts (2).

Therefore it follows that the collection $\mathcal{G}_k = \{G_{ka} \mid a \in A\}$ is $(\mathcal{J}_n)$locally finite. Thus $\mathcal{G} = \cup_{k=1}^{\infty} \mathcal{G}_k$ is a σ-$(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement of $\mathcal{V}$ and hence of $\mathcal{U}$. Therefore by Theorem 3.2, X is (ω)paracompact. □

Theorems 3.4 and 3.5 together give a (ω)topological version of Stone's theorem on paracompactness.

References

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Abstract

In this paper we prove some results on (ω)*topological spaces related to* (ω)*paracompactness.*

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