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**A space-time energetic formulation  
for wave propagation analysis by BEMs (\*\*)**

**1 - Introduction**

The present paper summarizes and extends some of the recent works of the authors concerning the study of wave propagation by boundary integral equations (BIEs) and boundary element methods (BEMs).

Time dependence is an essential feature in many physical and engineering applications that are modelled by partial differential equations of parabolic or hyperbolic type and, eventually, by BIEs. BEMs have been successfully applied to many such problems like the heat transfer or diffusion problem, scattering of an acoustic wave, electromagnetic wave propagation, linear elastodynamics, fluid dynamics, etc. (see [5, 6, 8, 9, 23, 25, 31, 32, 33, 46] and other references cited therein). When we have as model a homogeneous partial differential equation with constant coefficients, the initial conditions vanish, the data are given only on the boundary of the domain and this does not depend on time, the transformation of the problem to BIEs follows the same well-known procedure adopted for elliptic boundary value problems. Frequently claimed advantages over domain approaches are the dimensionality

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reduction, the easy implicit enforcement for radiation conditions at infinity, reduction of an unbounded exterior domain to a bounded boundary, the achievable high accuracy and simple pre- and post-processing for input and output data.

Compared to the vast literature about the BIEs related to the elliptic case, covering both their theoretical properties and their numerical approximation ([2, 17, 24, 26, 34, 35, 39, 40, 47, 48, 52, 55, 57]), the mathematical treatment of BIEs for parabolic and hyperbolic problems is rather modest in volume. There have been some significant developments in this area in recent years (see for example [6, 8, 9, 10, 13, 15, 25, 36, 44, 51, 53, 54]).

One can distinguish three approaches to the application of BEMs on parabolic and hyperbolic initial boundary value problem: (i) space-time integral equations, (ii) Laplace transform methods, (iii) time-stepping methods.

Space-time integral equations use the fundamental solution of the parabolic or hyperbolic partial differential equations. The construction of the BIEs via representation formulas and jump relations, the use of single layer and double layer potentials, and the classification into first kind and second kind integral equations follow in a large part the formalism known for elliptic problems. Causality condition and time-invariance imply that the integral equations are of Volterra type in time variable and of convolution type in time, respectively. Numerical methods constructed from these space-time BIEs are global in time. The boundary is the lateral boundary of the space-time cylinder and therefore has one dimension more than that of the spatial domain boundary. This increase in dimension at first means an increase in complexity: the system matrix is much larger and the integrals are higher-dimensional; for a 3D problem, the matrix elements in a Galerkin method can require 6-dimensional integrals. Moreover one drawback of Galerkin BEM is just the necessity of calculating integrals as discretization matrix elements involving singular and hypersingular kernels. The complexity of the discretization of the integral equations is in part neutralized by special features of the problem, in fact the system matrix has special structure related to the Volterra equation (finite convolution time). When low order basis functions in time are used, the matrix has a block triangular Toeplitz form, and for its inversion only one block needs to be inverted.

Laplace transform methods solve frequency-domain problem, possibly for complex frequencies. For each fixed frequency, a standard BEM for an elliptic problem is applied, and then the transformation back to the time domain employs special methods for the inversion of Fourier or Laplace transforms ([7]).

At last, the time-stepping methods start from a time discretization of the original initial boundary value problem via an implicit scheme and then use BIEs to solve the resulting elliptic problems for each time step. Here a difficulty lies in the form of the problem for one time step which has non-zero initial data and thus is not in the ideal

form for an application of the BEM, namely vanishing initial conditions and volume forces, and non-homogeneous boundary data. The solution after a time step, which defines the initial condition for the next step, has no reason to vanish inside the domain. Various methods have been devised to overcome this problem (see [5, 21, 44]).

These three approaches for the construction of BEMs can not be separated completely. The space-time integral equation method leads, after discretization, to a system that has the same finite time convolution structure one also gets from time-stepping schemes. The main difference is that the former needs the knowledge of a space-time fundamental solution. Also the Laplace transform appears in several roles. It can be used to translate between the time domain and the frequency domain at the level of the formulation of the problem, but also at the level of the solution.

Usual numerical discretization methods include collocation methods ([15, 16, 27, 28, 29, 38]) and Laplace-Fourier methods coupled with Galerkin boundary elements in space ([12, 30, 36]). The convolution quadrature method for the time discretization has been developed in [42, 43, 44]. It provides a straightforward way to obtain a stable time stepping scheme using the Laplace transform of the kernel functions. The application of Galerkin boundary elements in both space and time has been implemented by several authors but in this direction only the weak formulation due to Bamberger and Ha-Duong ([11, 12, 37]) provides natural convergence results. They obtain the well-posedness of the retarded BIE and stability of the BEM approximations owing to a coerciveness property of a suitable quadratic form in the unknown density closely related to the energy functional of the wave equation. Their approach relies, via Laplace transform, on uniform estimates with respect to complex frequencies of corresponding Helmholtz problem.

Here, first, we consider a Dirichlet problem for a temporally homogeneous one-dimensional wave equation, reformulated as a BIE with retarded potential and written directly in space-time domain. Special attention is devoted to a natural energy identity that leads to a space-time weak problem with robust theoretical properties. Taking advantage of the simple structure of this BIE, we prove a precise coerciveness property of the quadratic form related to the energy of the wave equation, and consequently, unconditionally stable schemes with well-behaved stability constants even for large times. We believe that the investigation of the coerciveness property of the energy functional avoiding the analysis in the frequency domain is of great interest both from theoretical and numerical point of view. In fact, a formulation based on the direct analysis in time of the energy functional would hopefully provide, even for multidimensional problems, an effective alternative to that proposed in [11, 12], in which the passage to complex frequencies leads to stability constants that grow exponentially in time.

Moreover, Section 3 of this paper is devoted to the introduction of (extended) variational formulations in the spirit of [1, 18, 58], which allow the definition of suitable functionals whose stationary point is solution of the given BIEs. It is not always easy to give a variational formulation to a mathematical problem. Tonti [58] has proposed a general technique to obtain variational formulations associated to any nonlinear problem; a similar method with reference to linear non potential operators has been proposed by Magri [45] and by Ortiz [50]. Few years later, Auchmuty [1] developed Tonti's ideas, while independently of this last work, Carini [19] and Carini and Genna [20] improved the technique with reference to several specific applications in the field of continuum mechanics. The main difficulty in all the mentioned approaches is that of introducing a suitable integrating operator (in the Tonti's terminology) which symmetrizes the given equation in the appropriate functional spaces. Here we show that the energy approach, introduced in Section 2, furnishes a natural solution to this problem providing variational problems whose solutions are minimum or saddle points of the corresponding functionals.

Using some results of Section 2, the second part of this paper is devoted to the analysis of one-dimensional wave propagation in layered media, with mixed boundary conditions, reformulated in terms of BIEs written directly in space-time domain.

An extension of theoretical study and numerical experiments for two- or three-dimensional dynamic interaction problems is currently under study and will be discussed in a future work. Of course, in two- or three-dimensional problems, one will also have to carefully treat the integration of spatial weakly singular, strongly singular and hypersingular integrals, arising in BEM discretization phase, but one can refer to a nowadays wide literature on this topic.

Finally, in the last Section, numerical results for one-dimensional problems in layered media have been collected and discussed, pointing out analogies and differences with recent literature results ([32, 49]). Instabilities phenomena arising with one of the classical  $L^2$  technique can be prevented using suitable time steps in the discretization phase, of course, with a higher computational cost with respect to the energetic procedure, which appears to be unconditionally stable. Further, comparisons between condition number of discretization matrices of energetic and Bamberger and Ha-Duong techniques ([11, 12, 37]) are presented too, showing the dependence of the latter on a strictly positive parameter that has to be properly chosen.

## 2 - Basic model problems and energetic weak formulation

In this section we consider the retarded potential representation of solutions of the Dirichlet problem for a one-dimensional wave equation, from which we will

deduce a suitable indirect boundary integral reformulation of the differential problem.

Let  $T > 0$  be fixed. We denote by  $\Omega$  the finite interval  $(0, L) \subset \mathbb{R}$  and let  $u(x, t)$  be the solution to the wave problem:

$$\begin{aligned} (1) \quad & u_{tt} - u_{xx} = 0, & x \in \mathbb{R} \setminus \{0, L\}, \quad t \in (0, T), \\ (2) \quad & u(x, 0) = u_t(x, 0) = 0, & x \in \mathbb{R} \setminus \{0, L\}, \\ (3) \quad & u(x, t) = g(x, t), & (x, t) \in \Sigma_T := \{0, L\} \times [0, T], \end{aligned}$$

where  $g(x, t)$  is a given function and  $g(x, t) = 0$  for  $t \leq 0$ . Note that  $u$  is considered as the solution on the whole  $\mathbb{R}$ , not only in  $\Omega$ . Whenever necessary we shall distinguish the internal solution  $u^-$ , i.e. for  $x \in \Omega$ , from the external one  $u^+$ , i.e. for  $x \in \mathbb{R} \setminus \Omega$ .

In order to rewrite problem (1)-(3) as a BIE, we need to recall the expression of the forward fundamental solution  $\mathcal{G}(x, t)$  of the wave operator:

$$(4) \quad \mathcal{G}(x, t) = \frac{1}{2}H[t - |x|] = \frac{1}{2}H[t](H[x + t] - H[x - t]),$$

where  $H[t]$  is the Heaviside step function. Using the fundamental solution (4) we obtain the single layer representation formula for the solutions of the equation (1), in terms of the unknown discontinuities density vector valued function  $\varphi(t) = (\varphi(0, t), \varphi(L, t))^T =: (\varphi_0(t), \varphi_L(t))^T$  at the end-points of interval  $(0, L)$ , i.e.:

$$(5) \quad \varphi = \left[ \frac{\partial u}{\partial \nu_x} \right] := \frac{\partial u^-}{\partial \nu_x} - \frac{\partial u^+}{\partial \nu_x}, \quad \text{where} \quad \frac{\partial}{\partial \nu_x} = \begin{cases} -\frac{\partial}{\partial x}, & x = 0 \\ \frac{\partial}{\partial x}, & x = L \end{cases}.$$

In fact, for  $t \in \mathbb{R}$ ,  $x \in \mathbb{R} \setminus \{0, L\}$ , it holds:

$$\begin{aligned} (6) \quad u(x, t) &= (V\varphi)(x, t) = \int_{-\infty}^{+\infty} \mathcal{G}(x, t - \tau) \varphi_0(\tau) d\tau + \int_{-\infty}^{+\infty} \mathcal{G}(x - L, t - \tau) \varphi_L(\tau) d\tau \\ &= \frac{1}{2} \int_{-\infty}^t H[t - \tau - |x|] \varphi_0(\tau) d\tau + \frac{1}{2} \int_{-\infty}^t H[t - \tau - |x - L|] \varphi_L(\tau) d\tau \\ &= \frac{1}{2} \int_{-\infty}^{t-|x|} \varphi_0(\tau) d\tau + \frac{1}{2} \int_{-\infty}^{t-|x-L|} \varphi_L(\tau) d\tau. \end{aligned}$$

Since problem (1)-(3) is formulated on the time interval  $[0, T]$ , in order to keep our notations as simple as possible, hereafter we shall consider functions  $\varphi$  defined on the whole real line but having support only in the fixed time interval  $[0, T]$ . From formula

(6), taking the limits as  $x \rightarrow 0$  and  $x \rightarrow L$ , we obtain the following system of BIEs at the end-points of the interval  $(0, L)$ :

$$(7) \quad \begin{aligned} (V\varphi)(0, t) &= \frac{1}{2} \left[ \int_0^t \varphi_0(\tau) d\tau + \int_0^{t-L} \varphi_L(\tau) d\tau \right] = g_0(t) \\ (V\varphi)(L, t) &= \frac{1}{2} \left[ \int_0^{t-L} \varphi_0(\tau) d\tau + \int_0^t \varphi_L(\tau) d\tau \right] = g_L(t) \end{aligned} \quad t \in [0, T].$$

The system (7) can be written with the compact notation:

$$(8) \quad V\varphi(t) = g(t),$$

where  $g(t) = (g_0(t), g_L(t))^T$  and  $g_0(t), g_L(t)$  represent the given boundary function in  $x = 0$  and  $x = L$ , respectively.

In order to formulate the operator equation (8) in a suitable functional framework, we assume  $V$  as defined in  $L^2(\Sigma_T) := L^2(0, T) \times L^2(0, T)$ . With this choice, from (7) it is easy to verify that the range of  $V$  lies in  $H_{\{0\}}^1(\Sigma_T)$ , the space of  $H^1(0, T) \times H^1(0, T)$  functions vanishing for  $t = 0$ . Hence hereafter we shall consider  $V : L^2(\Sigma_T) \rightarrow H_{\{0\}}^1(\Sigma_T)$ .

### 2.1 - Energetic weak formulation related to the BIE $V\varphi = g$

A classical way to introduce a weak formulation for (8) is to project the BIE using  $L^2(\Sigma_T)$  scalar product. Now, considering the bilinear form  $a_{L^2}(\varphi, \psi) : L^2(\Sigma_T) \times L^2(\Sigma_T) \rightarrow \mathbb{R}$  defined by:

$$(9) \quad a_{L^2}(\varphi, \psi) := \langle V\varphi, \psi \rangle_{L^2(\Sigma_T)},$$

we can write the following weak problem:

given  $g \in H_{\{0\}}^1(\Sigma_T)$ , find  $\varphi \in L^2(\Sigma_T)$  such that

$$(10) \quad a_{L^2}(\varphi, \psi) = \langle g, \psi \rangle_{L^2(\Sigma_T)} \quad \forall \psi \in L^2(\Sigma_T).$$

There are two major drawbacks in the above formulation: the bilinear form  $a_{L^2}(\cdot, \cdot)$  is not coercive, in fact choosing  $\varphi \equiv \psi$ , the formula (10) does not give a positive definite expression; further, it is implicit in problem (10) that the equation (8) must be understood considering the compact operator  $V : L^2(\Sigma_T) \rightarrow L^2(\Sigma_T)$ , which obviously cannot have continuous inverse. As a consequence, it is not surprising that problem (10) gives rise to instability phenomena in the discretization phase, as it will be shown in Section 6.

An alternative approach is suggested by the well-known conservation law satisfied by the (real-valued) solutions to the d'Alembert equation:

$$0 = u_t(u_{tt} - u_{xx}) = \frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) - \frac{\partial}{\partial x} (u_t u_x).$$

Integrating with respect to space-time in  $\mathbb{R} \times (0, T)$  and taking into account that  $u$  and  $u_t$  vanish for  $t = 0$ , we get the energy identity:

$$(11) \quad \mathcal{E}(T) = \frac{1}{2} \int_{-\infty}^{+\infty} (u_t^2 + u_x^2) dx \Big|_{t=T} = \int_0^T u_t \cdot \left[ \frac{\partial u}{\partial v} \right] dt = \int_0^T (V\varphi)_t \cdot \varphi dt,$$

where the dot denotes the scalar product in  $\mathbb{R}^2$ .

The quadratic form appearing in the last term of (11) leads to a natural space-time weak formulation of the corresponding BIE (8) with robust theoretical properties. In fact, the main advantage of this approach is that the quadratic form given by the energy, i.e.

$$\mathcal{E}(T) = \langle (V\varphi)_t, \varphi \rangle_{L^2(\Sigma_T)},$$

is, at least in the one-dimensional case, both continuous and coercive in the appropriate spaces, i.e. exactly the functional spaces where the Dirichlet problem is well-posed [41].

In order to derive continuity and coerciveness properties of the total energy  $\mathcal{E}(T)$ , we concentrate our analysis on the operator  $\mathcal{A} : L^2(\Sigma_T) \rightarrow L^2(\Sigma_T)$ , defined for  $t \in [0, T]$  as:

$$(12) \quad \mathcal{A}\varphi(t) := (V\varphi)_t(t) = \frac{1}{2} \begin{bmatrix} \varphi_0(t) + H[t-L]\varphi_L(t-L) \\ \varphi_L(t) + H[t-L]\varphi_0(t-L) \end{bmatrix}.$$

By an application of the Cauchy-Schwarz inequality, we have immediately that  $\mathcal{A}$  is a continuous operator with norm  $\|\mathcal{A}\| \leq 1$ . More interesting is the positivity property of the quadratic form associated to the operator  $\mathcal{A}$ . In fact, having introduced the bilinear form  $a_{\mathcal{E}}(\varphi, \psi) : L^2(\Sigma_T) \times L^2(\Sigma_T) \rightarrow \mathbb{R}$  defined by:

$$(13) \quad a_{\mathcal{E}}(\varphi, \psi) := \langle \mathcal{A}\varphi, \psi \rangle_{L^2(\Sigma_T)},$$

we have the following result:

**Theorem 2.1 ([4]).** *For every  $T > 0$ , there exists a positive constant  $c(T)$  such that:*

$$(14) \quad a_{\mathcal{E}}(\varphi, \varphi) \geq c(T) \|\varphi\|_{L^2(\Sigma_T)}^2, \quad \varphi \in L^2(\Sigma_T).$$

Moreover, let  $m \in \mathbb{N}^+$  be such that  $T \leq mL$ ; then one has the explicit bound:

$$(15) \quad c(T) \geq \sin^2 \left( \frac{\pi}{2(m+1)} \right).$$

At this point we can write down the energetic weak problem related to the BIE (8), which admits a unique, stable solution:

given  $g \in H_{\{0\}}^1(\Sigma_T)$ , find  $\varphi \in L^2(\Sigma_T)$  such that

$$(16) \quad a_{\mathcal{E}}(\varphi, \psi) = \langle g_t, \psi \rangle_{L^2(\Sigma_T)}, \quad \forall \psi \in L^2(\Sigma_T).$$

## 2.2 - Neumann problem

Similar considerations as those developed in the subsection 2.1 can be done for the wave problem with Neumann boundary condition:

$$(17) \quad u_{tt} - u_{xx} = 0, \quad x \in \mathbb{R} \setminus \{0, L\}, \quad t \in (0, T)$$

$$(18) \quad u(x, 0) = u_t(x, 0) = 0, \quad x \in \mathbb{R} \setminus \{0, L\}$$

$$(19) \quad \frac{\partial u}{\partial \nu}(x, t) = f(x, t), \quad (x, t) \in \Sigma_T := \{0, L\} \times [0, T]$$

where  $f(x, t)$  is a given function. In this case we have the double layer representation of the solution  $u(x, t)$  through the unknown retarded potential  $\psi$ :

$$(20) \quad \begin{aligned} u(x, t) &= (\mathcal{K}\psi)(x, t) \\ &= \sum_{y=0, L} \int_{-\infty}^{+\infty} \frac{\partial}{\partial \nu_y} \mathcal{G}(x - y; t - \tau) \psi(y, \tau) d\tau \end{aligned}$$

where  $\mathcal{G}(\cdot, \cdot)$  is given by (4). After a straightforward calculation we obtain the more explicit formula:

$$u(x, t) = \frac{1}{2} \frac{x}{|x|} \psi_0(t - |x|) - \frac{1}{2} \frac{x - L}{|x - L|} \psi_L(t - |x|)$$

from which:

$$[u](0, t) := u^-(0, t) - u^+(0, t) = \psi_0(t), \quad [u](L, t) := u^-(L, t) - u^+(L, t) = \psi_L(t).$$

By deriving formula (20), taking the limits as  $x \rightarrow 0$  and  $x \rightarrow L$  and using Neumann data  $f(x, t)$ , we obtain the boundary equations for the unknown potential  $\psi$  at the end-



points of the interval  $(0, L)$ :

$$(21) \quad \mathcal{D}\psi := \frac{\partial}{\partial \mathbf{v}} \mathcal{K}\psi = f,$$

that is

$$(22) \quad \begin{aligned} (\mathcal{D}\psi)(0, t) &= \frac{1}{2} \left[ \psi_{0,t}(t) - \psi_{L,t}(t - L) \right] = f_0(t), \\ (\mathcal{D}\psi)(L, t) &= -\frac{1}{2} \left[ \psi_{0,t}(t - L) - \psi_{L,t}(t) \right] = f_L(t). \end{aligned}$$

In order to derive a weak formulation of the equation (21), our starting point is again the energy identity, which for the Neumann boundary condition assumes the form:

$$\mathcal{E}(T) = \int_0^T \frac{\partial u}{\partial \mathbf{v}_x} \cdot [u]_t dt = \int_0^T \mathcal{D}\psi \cdot \psi_t dt.$$

Thus, having defined the bilinear form:

$$\tilde{a}_{\mathcal{E}} : H_{\{0\}}^1(\Sigma_T) \times H_{\{0\}}^1(\Sigma_T) \rightarrow \mathbb{R}, \quad \tilde{a}_{\mathcal{E}}(\psi, \phi) := \langle \mathcal{D}\psi, \dot{\phi}_t \rangle_{L^2(\Sigma_T)},$$

the coerciveness of  $\tilde{a}_{\mathcal{E}}(\cdot, \cdot)$  follows at once from the observation that:

$$\tilde{a}_{\mathcal{E}}(\psi, \psi) = \int_0^T \mathcal{A} \tilde{\psi}_t \cdot \tilde{\psi}_t dt,$$

where  $\tilde{\psi}(t) = (\psi_0(t), -\psi_L(t))^\top$ , and  $\mathcal{A}$  is the operator defined in (12). Then, from Theorem 2.1 we have the following result.

**Theorem 2.2.** *For every  $T > 0$ , there exists a positive constant  $c(T)$  such that*

$$\tilde{a}_{\mathcal{E}}(\psi, \psi) \geq c(T) \|\psi_t\|_{L^2(\Sigma_T)}^2, \quad \psi \in H_{\{0\}}^1(\Sigma_T).$$

*The constant  $c(T)$  is bounded from below as in (15).*

Therefore, Dirichlet or Neumann boundary conditions are similar for what concerns the analysis of the energetic weak formulations related to the corresponding BIEs.

### 2.3 - Remarks on energy coerciveness

The coerciveness of the quadratic form  $a_{\mathcal{E}}(\varphi, \varphi)$ , defined in (13), asserts a coerciveness property of the total energy of the solution  $u$  to problem (1)-(3). This follows

at once from the equality:

$$a_{\mathcal{E}}(\varphi, \varphi) = \mathcal{E}(T).$$

Thus, remembering (5), Theorem 2.1 assures that:

$$(23) \quad \mathcal{E}(T) \geq c(T) \left\| \left[ \frac{\partial u}{\partial v_x} \right] \right\|_{L^2(\Sigma_T)}^2.$$

The purpose of this section is to point out some interesting facts about the different contributions to inequality (23) of the external and internal energies, defined respectively as:

$$\mathcal{E}_+(t) := \frac{1}{2} \int_{\mathbb{R} \setminus (0, L)} (u_t^2(x, t) + u_x^2(x, t)) dx, \quad \mathcal{E}_-(t) := \frac{1}{2} \int_0^L (u_t^2(x, t) + u_x^2(x, t)) dx.$$

We shall see that the main contribution to inequality (23) is provided by the external energy. Indeed, for any given time  $T$ , one may replace in (23) the global energy  $\mathcal{E}(T)$  with  $\mathcal{E}_+(T)$ , provided a slightly smaller coerciveness constant  $\tilde{c}(T)$  takes the place of  $c(T)$ . On the contrary, interactions of reflected waves make the contribution of the internal energy  $\mathcal{E}_-(T)$  almost negligible at least for large times  $T \gg L$ . More precisely, for any  $T$  greater than  $L$ , we have:

$$(24) \quad \mathcal{E}_-(T) = \frac{1}{4} \int_{T-L}^T |\varphi(t)|^2 dt,$$

thus  $\mathcal{E}_-(T)$  vanishes provided  $\varphi(t) = 0$  in the “small” interval  $(T - L, T)$ .

Although our arguments rely upon particular features of the one-dimensional d’Alembert equation, they could suggest a possible approach even for the much more difficult cases of the 2 or 3 dimensional wave equation. In fact also in the  $n$ -dimensional case, at least for convex domains, the external energy at a given time is strictly positive and thus may be viewed as a possible coercive quadratic form in the single layer potential variable  $\varphi$  with respect to a suitable norm  $|\cdot|_W$ . Of course, in the  $n$ -dimensional case the main open problem is the identification of the functional space  $W$ .

We conclude this short section with other two remarks. Even though the internal energy is in general only nonnegative, it still enjoys some coerciveness property if we add to  $\mathcal{E}_-(T)$  its integral with respect to time as in the following

**Proposition 2.1 ([4]).** *For every  $T > 0$ , one has*

$$(25) \quad |\varphi|_{L^2(\Sigma_T)}^2 \leq 4\mathcal{E}_-(T) + \frac{8}{L} \int_0^T \mathcal{E}_-(t) dt.$$

Finally, denoted with  $\mathcal{A}^*$  the adjoint of the operator  $\mathcal{A}$ , by combining the energy identity on the external domain

$$(26) \quad \begin{aligned} \mathcal{E}_+(T) &= - \int_0^T \frac{\partial u^+}{\partial \nu_x} \cdot g_t dt \\ &= \int_0^T |(V\varphi)_t|^2 dt = \langle \mathcal{A}\varphi, \mathcal{A}\varphi \rangle_{L^2(\Sigma_T)} = \langle \mathcal{A}^* \mathcal{A}\varphi, \varphi \rangle_{L^2(\Sigma_T)} \end{aligned}$$

with the energy identity (11) and the inequality (25), we can obtain an alternative proof of the coerciveness estimate (14). In fact, by an application of the Cauchy-Schwarz inequality in (11), we have:

$$\mathcal{E}(t) \leq \left( \int_0^t |\varphi(\tau)|^2 d\tau \right)^{1/2} \left( \int_0^t |g_t(\tau)|^2 d\tau \right)^{1/2} \leq |\varphi|_{L^2(\Sigma_T)} |g_t|_{L^2(\Sigma_T)}.$$

Therefore, since  $\mathcal{E}_-(t) \leq \mathcal{E}(t)$ , we get from (25) and (26):

$$|\varphi|_{L^2(\Sigma_T)}^2 \leq 4\mathcal{E}_-(T) + \frac{8T}{L} |\varphi|_{L^2(\Sigma_T)} \sqrt{\mathcal{E}_+(T)},$$

which yields the following inequality where the internal and external energies play a distinguished role:

$$|\varphi|_{L^2(\Sigma_T)}^2 \leq 8\mathcal{E}_-(T) + \frac{64T^2}{L^2} \mathcal{E}_+(T).$$

Note that the constant  $(8 + 64T^2/L^2)^{-1}$  is not optimal, nevertheless as function of the ratio  $T/L$  has the same asymptotic behavior of  $c(T)$  in (15).

### 3 - Variational formulations for the BIE $V\varphi = g$

A deeper understanding of the physical phenomenon modelled by differential or integral equations might be obtained by a variational formulation. Nevertheless, it is not always simple to give a variational formulation to a mathematical problem. In fact, in absence of symmetry of the problem governing operator, with respect to a given scalar product, it is impossible to construct a relevant variational formulation. This is the case of the weak problems (10) and (16), related to the BIE  $V\varphi = g$ , which cannot be rewritten as variational problems, since the operators  $V$  and  $\mathcal{A} = V_t$ , respectively, are not self-adjoint with respect to the classical  $L^2$  scalar product.

This difficulty can be overcome in two different directions: the first possibility is

to change the scalar product choosing a suitable bilinear form in such a way that the given BIE operator  $V$  is self-adjoint with respect to the new one. This way of reasoning was followed in [46], where a time-convolutive bilinear form was introduced. A second different strategy retains the  $L^2$  scalar product and suitably changes the given problem  $V\varphi = g$ . Following [58], we have to consider an invertible symmetric operator  $K$  in order to solve the equivalent problem:

$$(27) \quad V^*KV\varphi = V^*Kg,$$

and such that the new operator  $V^*KV$  is self-adjoint and possibly coercive with respect to  $L^2$  scalar product.

As operator  $K$ , one can certainly choose the identity operator, but in this case the discretization of the weak reformulation of (27) suffers of the same instability phenomena already cited. In fact, with this choice, we obviously have to consider the operator  $V : L^2(\Sigma_T) \rightarrow L^2(\Sigma_T)$  and of course the adjoint operator  $V^* : L^2(\Sigma_T) \rightarrow L^2(\Sigma_T)$ . The bilinear form:

$$(28) \quad \langle V^*V\varphi, \varphi \rangle_{L^2(\Sigma_T)} = |V\varphi|_{L^2(\Sigma_T)}^2$$

is certainly positive, but it is not coercive in  $L^2(\Sigma_T)$ . In fact, being  $V$  a compact operator, the right-hand side of (28) defines a weaker norm with respect to  $|\cdot|_{L^2(\Sigma_T)}$ . To conclude, we can say that the approximations converge, but with a derivative lost in the Sobolev scale (see [24]). If we instead consider the operator  $V : L^2(\Sigma_T) \rightarrow H_{\{0\}}^1(\Sigma_T)$ , i.e. as an isomorphism between Hilbert spaces, and consequently  $V^* : [H_{\{0\}}^1(\Sigma_T)]' \rightarrow L^2(\Sigma_T)$ , where  $[H_{\{0\}}^1(\Sigma_T)]'$  is the dual space of  $H_{\{0\}}^1(\Sigma_T)$ , a considerably better choice is:

$$K : H_{\{0\}}^1(\Sigma_T) \rightarrow [H_{\{0\}}^1(\Sigma_T)]', \quad K = \left( \frac{\partial}{\partial t} \right)^* \frac{\partial}{\partial t},$$

that is

$$\langle Kf, g \rangle := \langle f_t, g_t \rangle_{L^2(\Sigma_T)},$$

where  $\langle \cdot, \cdot \rangle$  is the duality product between  $H_{\{0\}}^1(\Sigma_T)$  and  $[H_{\{0\}}^1(\Sigma_T)]'$ ; hence, having introduced the symmetric bilinear form:

$$a_K(\varphi, \psi) = \langle V^*KV\varphi, \psi \rangle_{L^2(\Sigma_T)}$$

and remembering the definition (12) of the operator  $\mathcal{A}$ , it holds:

$$\begin{aligned} a_K(\psi, \psi) &= \langle KV\psi, V\psi \rangle = \langle (V\psi)_t, (V\psi)_t \rangle_{L^2(\Sigma_T)} \\ &= \langle \mathcal{A}^*\mathcal{A}\psi, \psi \rangle_{L^2(\Sigma_T)} = |\mathcal{A}\psi|_{L^2(\Sigma_T)}^2 \geq \frac{1}{\|\mathcal{A}^{-1}\|^2} |\psi|_{L^2(\Sigma_T)}^2, \end{aligned}$$

and the coerciveness of the bilinear form is verified. Therefore we have the following classical variational results.

**Proposition 3.1.** *A function  $\varphi \in L^2(\Sigma_T)$  is the unique solution of the BIE  $V\varphi = g$  if and only if it is the global minimum point for the quadratic functional:*

$$F_K(\psi) = \frac{1}{2} a_K(\psi, \psi) - \langle V^* K g, \psi \rangle_{L^2(\Sigma_T)}, \quad \psi \in L^2(\Sigma_T).$$

On the other side, the general theory in [58] allows to write the following:

**Proposition 3.2.** *A function  $\varphi \in L^2(\Sigma_T)$  is the unique solution of the BIE  $V\varphi = g$  if and only if it is the global minimum point for the quadratic functional:*

$$\tilde{F}_K(\psi) = \frac{1}{2} \langle K(V\psi - g), V\psi - g \rangle = F_K(\psi) + \frac{1}{2} \langle K g, g \rangle, \quad \psi \in L^2(\Sigma_T).$$

Other characterizations of the solution of the BIE  $V\varphi = g$  can be obtained considering the associated problem  $\mathcal{A}\varphi = g_t$ , and noting that the operator  $\mathcal{A}$  can be split as the sum of its symmetric and skew-symmetric parts:

$$\mathcal{A} = \mathcal{A}_s + \mathcal{A}_{ss},$$

where  $\mathcal{A}_s = (\mathcal{A} + \mathcal{A}^*)/2$  and  $\mathcal{A}_{ss} = (\mathcal{A} - \mathcal{A}^*)/2$ . The operator  $\mathcal{A}_s$  is continuous, self-adjoint with respect to the classical  $L^2$  scalar product and also coercive as proved in the Theorem 2.1 Then, following [1], we have:

**Proposition 3.3.** *A function  $\varphi \in L^2(\Sigma_T)$  is the unique solution of  $\mathcal{A}^* \mathcal{A}_s^{-1} \mathcal{A}\varphi = \mathcal{A}^* \mathcal{A}_s^{-1} g_t$  (or equivalently of  $\mathcal{A}\varphi = g_t$ ) if and only if it is the global minimum point for the quadratic functional:*

$$\begin{aligned} F_s(\psi) &= \frac{1}{2} \langle \mathcal{A}_s^{-1}(\mathcal{A}\psi - g_t), \mathcal{A}\psi - g_t \rangle_{L^2(\Sigma_T)} = \frac{1}{2} \langle \mathcal{A}_s \psi, \psi \rangle_{L^2(\Sigma_T)} \\ &+ \frac{1}{2} \langle \mathcal{A}_s^{-1}(g_t - \mathcal{A}_{ss}\psi), g_t - \mathcal{A}_{ss}\psi \rangle_{L^2(\Sigma_T)} - \langle \psi, g_t \rangle_{L^2(\Sigma_T)}. \end{aligned}$$

**Remark 1.** The functional  $F_s(\psi)$  can be written starting directly from the BIE  $V\varphi = g$  and following the strategy proposed in [58], choosing the operator  $K$  as:

$$K = K_s := \left[ \frac{\partial}{\partial t} \right]^* \mathcal{A}_s^{-1} \frac{\partial}{\partial t} = \left[ \frac{\partial}{\partial t} \right]^* \left[ \frac{1}{2} \left( \frac{\partial}{\partial t} V + V^* \left( \frac{\partial}{\partial t} \right)^* \right) \right]^{-1} \frac{\partial}{\partial t}.$$

#### 4 - One-dimensional elastodynamic analysis

##### 4.1 - Basic Dirichlet-Neumann problem

In the following, we will consider the wave propagation problem for a rod of finite length  $L$ , with vanishing external forces, homogeneous initial data and mixed (Dirichlet-Neumann) boundary conditions on the bounded time interval  $(0, T)$ . The rod under consideration can be represented by a structured three-dimensional body, with a higher dimension with respect to the remaining ones: the length  $L$  along the  $x$ -direction. For this reason its geometry is describable by a line, the axial line of the rod which passes through the barycenters of all transversal sections. The displacements and tractions are referred to axial coordinates of the bar. Having denoted with  $u(x, t)$  the longitudinal displacement of the rod and with  $p(x, t) := EA \frac{\partial u}{\partial \mathbf{n}_x}(x, t)$  the traction, depending on a unitary (outward) normal vector with respect to the transversal section of the rod, therefore direct along the  $x$ -axis, i.e.  $\mathbf{n}_x = (n_{1,x}, 0)^\top$  and on  $EA$ , which represents the axial stiffness of the rod, the differential model problem reads:

$$(29) \quad c^2 u_{xx} - u_{tt} = 0, \quad x \in (0, L), t \in (0, T),$$

$$(30) \quad u(x, 0) = u_t(x, 0) = 0, \quad x \in (0, L),$$

$$(31) \quad u(0, t) = \bar{u}(t), \quad p(L, t) = \bar{p}(t), \quad t \in (0, T),$$

where  $\bar{u}(t)$  and  $\bar{p}(t)$  are given functions. The scalar wave velocity  $c$  is given by  $c = \sqrt{E/\rho}$  with  $E$  and  $\rho$  denoting Young's modulus and mass density of the material, respectively.

In order to obtain a boundary integral formulation of the problem (29)-(31), we have to use the following Love's representation formula (the analogue in the dynamical case of the Somigliana identity of linear elastostatic) for  $x \in (0, L)$  and  $t \in (0, T)$ :

$$(32) \quad u(x, t) = \sum_{\xi=0,L} \mathcal{G}(x, \xi; t) * p(\xi, t) - \sum_{\xi=0,L} \mathcal{K}(x, \xi; t) * u(\xi, t),$$

where the asterisk denotes the time convolution product. Having set  $r = |x - \xi|$ , the function:

$$(33) \quad \mathcal{K}(x, \xi; t) = EA \frac{\partial \mathcal{G}}{\partial \mathbf{n}_\xi}(x, \xi; t) = \frac{c}{2} H[t] \frac{\partial H}{\partial \xi} [ct - r] n_{1,\xi},$$

is the so called Gebbia first fundamental solution and represents the displacement in point  $x$  and instant  $t$  due to a unit displacement concentrated in space (point  $\xi$ ) and time (instant  $\tau$ ) and  $\mathcal{G}(x, \xi; t)$  (Kelvin solution) is the fundamental solution for the wave problem (29) applied to the rod (see also (4)).

With a limiting procedure in (32), for  $x$  tending to the end-points of the rod, we obtain for  $t \in (0, T)$  a first BIE of the form:

$$(34) \quad u(x, t) = \sum_{\xi=0, L} \mathcal{G}(x, \xi; t) * p(\xi, t) - \sum_{\xi=0, L} \mathcal{K}(x, \xi; t) * u(\xi, t), \quad x \in \{0, L\}.$$

Remembering the definition of  $p(x, t)$ , from (34) one can obtain for  $t \in (0, T)$  a second BIE (see [3] for details), of the form:

$$(35) \quad p(x, t) = \sum_{\xi=0, L} \mathcal{K}'(x, \xi; t) * p(\xi, t) - \sum_{\xi=0, L} \mathcal{S}(x, \xi; t) * u(\xi, t), \quad x \in \{0, L\}.$$

In the BIE (35) we find the second Kelvin fundamental solution:

$$(36) \quad \mathcal{K}'(x, \xi; t) = EA \frac{\partial \mathcal{G}}{\partial \mathbf{n}_x}(x, \xi; t) = \frac{c}{2} H[t] \frac{\partial H}{\partial x} [ct - r] n_{1,x},$$

which represents the stress field in point  $x$  and instant  $t$  due to a unit force concentrated in point  $\xi$  and time instant  $\tau$ ; moreover in BIE (35) we have the derivative of Gebbia fundamental solution, i.e.:

$$(37) \quad \mathcal{S}(x, \xi; t) = EA \frac{\partial \mathcal{K}}{\partial \mathbf{n}_x}(x, \xi; t) = \frac{EA c}{2} H[t] \frac{\partial H}{\partial x \partial \xi} [ct - r] n_{1,\xi} n_{1,x}.$$

The function (37) describes tractions  $p(x, t)$  due to unit relative displacement concentrated in space (point  $\xi$ ) and time instant  $\tau$ . Of course, derivatives in (33), (36) and (37) have to be understood in a distributional sense.

Using equations (34), (35) in  $x = 0$  and  $x = L$ , respectively, the explicit expression of the kernels  $\mathcal{G}$ ,  $\mathcal{K}$ ,  $\mathcal{K}'$ ,  $\mathcal{S}$  and the boundary data, one obtains, after an integration by parts involving the kernel  $\mathcal{S}$  and some straightforward calculations, the following integro-differential equation system for  $t \in (0, T)$ :

$$(38) \quad \begin{cases} \frac{c}{EA} \int_0^t p(0, \tau) d\tau + H[t - \frac{L}{c}] u(L, t - \frac{L}{c}) = f^u(t) \\ -H[t - \frac{L}{c}] p(0, t - \frac{L}{c}) + \frac{EA}{c} u_t(L, t) = f^p(t) \end{cases}$$

having set:

$$(39) \quad \begin{aligned} f^u(t) &= \bar{u}(t) - \frac{c}{EA} H[t - \frac{L}{c}] \int_0^{t-L/c} \bar{p}(\tau) d\tau, \\ f^p(t) &= \frac{EA}{c} H[t - \frac{L}{c}] \bar{u}_t(t - \frac{L}{c}) + \bar{p}(t). \end{aligned}$$

In (38) the unknowns are the time history of traction in  $x = 0$  and of displacement in  $x = L$ , i.e. the two time functions  $p(0, t)$  and  $u(L, t)$ .

Now, using the following operators:

$$(40) \quad (Vg)(t) = \frac{c}{EA} \int_0^t g(\tau) d\tau, \quad (Dg)(t) = \frac{EA}{c} \frac{\partial g(t)}{\partial t},$$

and the following anticipated and retarded shift operators:

$$(41) \quad (S^+g)(t) = H[T - t - \frac{L}{c}]g(t + \frac{L}{c}), \quad (S^-g)(t) = H[t - \frac{L}{c}]g(t - \frac{L}{c}),$$

the system (38) can be rewritten in compact form as:

$$(42) \quad \begin{bmatrix} V & S^- \\ -S^- & D \end{bmatrix} \begin{bmatrix} p^0(t) \\ u^L(t) \end{bmatrix} = \begin{bmatrix} f^u(t) \\ f^p(t) \end{bmatrix}, \quad t \in (0, T),$$

where  $p^0(t)$ ,  $u^L(t)$  represent the unknown time functions  $p(0, t)$  and  $u(L, t)$ , respectively.

#### 4.2 - Weak formulations of the integro-differential system (42)

An usual way to introduce a weak formulation of integro-differential system (42), taken in account, is based on the classical  $L^2(0, T)$  scalar product ( $L^2$ -weak formulation). Unfortunately, there is an important drawback in this formulation: the related bilinear form is not coercive, and consequently is not unexpected that corresponding problem gives rise to instability phenomena in the approximation phase.

An alternative approach is suggested by the fact that for hyperbolic wave problem (29)-(31) the positive energy at the final time instant  $T$  inside the rod can be expressed as:

$$(43) \quad \mathcal{E}_{(0,L)}(T) = \frac{1}{EA} \int_0^T \sum_{x=0,L} u_t(x, t) p(x, t) dt \geq 0.$$

Since the first equation in (42) comes from the representation formula (34), we have to differentiate it with respect to time before projecting onto the space  $L^2(0, T)$ , while the second equation, coming from the representation formula for the traction (35), has to be projected onto the space of time derivatives of functions belonging to  $H_{\{0\}}^1$ . Therefore, if we define the spaces  $W := L^2(0, T) \times H_{\{0\}}^1(0, T)$ ,  $\tilde{W} := H_{\{0\}}^1(0, T) \times L^2(0, T)$  and we consider the bilinear form:

$$(44) \quad a_{\mathcal{E}}(\cdot, \cdot) = \left\langle \begin{bmatrix} V_t & S_t^- \\ -S^- & D \end{bmatrix} \cdot, \cdot \right\rangle : W \times W \rightarrow \mathbb{R}$$



we obtain the following energetic weak formulation of the problem (42):

$$(45) \quad \text{given } (f_u, f_p)^\top \in \tilde{W}, \text{ find } (p^0, u^L)^\top \in W \text{ such that}$$

$$a_\varepsilon\left(\begin{pmatrix} p^0 \\ u^L \end{pmatrix}, \begin{pmatrix} \psi \\ \varphi_t \end{pmatrix}\right) = \langle \begin{pmatrix} f_{u,t} \\ f_p \end{pmatrix}, \begin{pmatrix} \psi \\ \varphi_t \end{pmatrix} \rangle, \quad \forall (\psi, \varphi)^\top \in W.$$

Under the assumptions made, the following result holds.

**Theorem 4.1 ([3]).** *The real bilinear form  $a_\varepsilon(\cdot, \cdot)$  of the energetic weak formulation is continuous and coercive on  $W \times W$ .*

This theorem assures the existence and uniqueness of the weak solution of the problem (45).

An alternative weak formulation, which has the same robust theoretical background as (45), is that one proposed in [11] and [12]. It comes from a direct Laplace-Fourier transform of the differential problem, which then is formulated and analyzed in terms of BIEs and treated in standard weak form. This problem is finally set back in space-time domain by means of inverse transform. Following this procedure we have to introduce the continuous and coercive bilinear form (see e.g. [11, 12]):

$$(46) \quad a_\sigma(\cdot, \cdot) = \left\langle \begin{bmatrix} V_t & S_t^- \\ -S^- & D \end{bmatrix} \cdot, \cdot \right\rangle_\sigma : W \times W \rightarrow \mathbb{R}$$

where the scalar product is defined as:

$$(47) \quad \langle f, g \rangle_\sigma := \int_{-\infty}^{+\infty} e^{-2\sigma t} f(t) g(t) dt,$$

where the real parameter  $\sigma$  has to be strictly positive and properly chosen. Hence, we can obtain the following weak formulation of (42):

$$(48) \quad \text{given } (f^u, f^p)^\top \in \tilde{W}, \text{ find } (p^0, u^L)^\top \in W \text{ such that}$$

$$a_\sigma\left(\begin{pmatrix} p^0 \\ u^L \end{pmatrix}, \begin{pmatrix} \psi \\ \varphi_t \end{pmatrix}\right) = \langle \begin{pmatrix} f_t^u \\ f^p \end{pmatrix}, \begin{pmatrix} \psi \\ \varphi_t \end{pmatrix} \rangle_\sigma, \quad \forall (\psi, \varphi)^\top \in W.$$

From now on, problem (48) will be referred to as  $\sigma$ -weak formulation of our wave problem.

### 4.3 - Time discretization

Now, we introduce a uniform decomposition of the time interval  $(0, T)$  with time step  $\Delta t = T/n$ ,  $n \in \mathbb{N}^+$ , generated by the  $n + 1$  instants:

$$(49) \quad t_k = k \Delta t, \quad k = 0, \dots, n.$$

Let  $\mathbb{P}_r$  be with  $r \geq 0$ , the space of polynomials of degree less than or equal to  $r$ , we consider the standard finite element spaces:

$$(50) \quad X_{\Delta t}^r = \{\psi_{\Delta t} \in L^2(0, T) : \psi_{\Delta t}|_{[t_k, t_{k+1}]} \in \mathbb{P}_r, r \geq 0, k = 0, \dots, n-1\}$$

and

$$(51) \quad X_{\Delta t}^{r+1,0} = \{\varphi_{\Delta t} \in C^0(0, T) : \varphi_{\Delta t}|_{[t_k, t_{k+1}]} \in \mathbb{P}_{r+1}, r \geq 0, k = 0, \dots, n-1\}.$$

Then, considering the finite dimensional space  $W_{\Delta t}^{r,0} = X_{\Delta t}^r \times X_{\Delta t}^{r+1,0}$ , we can write down the discrete form of  $L^2$ ,  $\sigma$  and energetic weak problems. For instance, referring to [3] we have:

$$(52) \quad \begin{aligned} &\text{given } (f^u, f^p)^\top \in \widetilde{W}, \text{ find } (p_{\Delta t}^0, u_{\Delta t}^L)^\top \in W_{\Delta t}^{r,0} \text{ such that} \\ &a_{\mathcal{E}}\left(\begin{pmatrix} p_{\Delta t}^0 \\ u_{\Delta t}^L \end{pmatrix}, \begin{pmatrix} \psi_{\Delta t} \\ \varphi_{\Delta t,t} \end{pmatrix}\right) = \left\langle \begin{pmatrix} f_t^u \\ f^p \end{pmatrix}, \begin{pmatrix} \psi_{\Delta t} \\ \varphi_{\Delta t,t} \end{pmatrix} \right\rangle, \quad \forall (\psi_{\Delta t}, \varphi_{\Delta t})^\top \in W_{\Delta t}^{r,0}, \end{aligned}$$

where  $\varphi_{\Delta t,t}$  denotes derivative of the function  $\varphi_{\Delta t}$  with respect to the time. Now, indicating with  $\{\psi_k\}$  a basis for  $X_{\Delta t}^r$  and with  $\{\varphi_k\}$  a basis for  $X_{\Delta t}^{r+1,0}$ , the unknown functions  $p_{\Delta t}^0(t)$ ,  $u_{\Delta t}^L(t)$  can be expressed as:

$$(53) \quad p_{\Delta t}^0(t) = \sum_k p_k^0 \psi_k(t), \quad u_{\Delta t}^L(t) = \sum_k u_k^L \varphi_k(t)$$

and the discrete problems can be equivalently written, respectively, as linear systems:

$$(54) \quad A_{L^2} x_{L^2} = b_{L^2}, \quad A_{\mathcal{E}} x_{\mathcal{E}} = b_{\mathcal{E}}, \quad A_{\sigma} x_{\sigma} = b_{\sigma},$$

in the unknowns the coefficients  $p_k^0$  and  $u_k^L$ . Matrices  $A_{L^2}$ ,  $A_{\mathcal{E}}$  and  $A_{\sigma}$  have a  $2 \times 2$  block structure and are not symmetric. In fact the corresponding bilinear forms  $a_{L^2}(\cdot, \cdot)$ ,  $a_{\mathcal{E}}(\cdot, \cdot)$  and  $a_{\sigma}(\cdot, \cdot)$  are not symmetric and the related weak formulations can not be written as equivalent variational problems. Anyway, owing to the coerciveness of  $a_{\mathcal{E}}(\cdot, \cdot)$  and  $a_{\sigma}(\cdot, \cdot)$ , the diagonal blocks of matrices  $A_{\mathcal{E}}$  and  $A_{\sigma}$  are positive definite.

**Remark 2.** When we consider only the BIE (34) in both the end-points of the rod and we use constant and linear shape functions for the unknown tractions and displacements, respectively, the final linear system of the energetic weak formulation is completely equivalent to that obtained using the standard collocation technique. In fact, the energetic weak formulation starts from (43), where the time integral can be equivalently rewritten, with an integration by parts, in the form:

$$(55) \quad \sum_{x=0,L} \int_0^T u_t(x, t) p(x, t) dt = \sum_{x=0,L} u(x, T) p(x, T) - \sum_{x=0,L} \int_0^T u(x, t) p_t(x, t) dt.$$

Hence, the BIE (34), instead of being differentiated with respect to time, can be alternatively projected onto the space of time distributional derivatives of functions belonging to  $L^2(0, T)$ . If we consider Heaviside step functions  $\psi_k(t)$ , related to a time mesh with nodes  $t_k$ , as test and trial functions for the discretization of tractions  $p$ , time integral of Dirac distributions  $\psi_{k,t}(t)$  in the right-hand side of (55) leads to a collocation procedure at time instants  $t_k$ .

## 5 - Layered media

### 5.1 - The case of a bi-material rod

Let us consider a rod of length  $L$ , constituted by two portions of length  $L_1, L_2 = L - L_1$ , respectively, made up by different materials, with constants  $\rho_1 \neq \rho_2, E_1 \neq E_2, A_1 \neq A_2$ . As a consequence, the wave propagation velocity in the two parts of the rod will be different, i.e.  $c_1 \neq c_2$ . We want to study the wave propagation problem with assigned condition of mixed type at the end-points of the rod. Hence, considering (29) written in the local reference system of each portion, we have to solve the differential problem:

$$(56) \quad c_i^2 u_{i,xx}(x, t) - u_{i,tt}(x, t) = 0, \quad x \in (0, L_i), t \in (0, T), \quad i = 1, 2,$$

$$(57) \quad u_i(x, 0) = u_{i,t}(x, 0) = 0, \quad x \in (0, L_i), \quad i = 1, 2,$$

$$(58) \quad u_1(0, t) = \bar{u}_1(t), \quad p_2(L_2, t) = \bar{p}_2(t), \quad t \in (0, T),$$

with further continuity and equilibrium condition for the solution at the interface between the two materials:

$$(59) \quad u_1(L_1, t) = u_2(0, t) =: u_I(t), \quad p_1(L_1, t) = -p_2(0, t) =: p_I(t).$$

For what concerns the boundary integral reformulation of the problem (56)-(58), we have to consider for the first portion of the rod, the BIE (34) in  $x = 0$  and  $x = L_1$ , and the second BIE (35) in  $x = L_1$ . For the remaining part of the rod, i.e. for  $x \in (0, L_2)$  in the local system, we have to consider the equation (34) in  $x = 0$ , and the equation (35) in  $x = 0$  and  $x = L_2$ . Imposing conditions (59), with a straightforward computation one gets for  $t \in (0, T)$ :

$$(60) \quad \begin{bmatrix} V_1 & S_1 V_1 & S_1 & 0 \\ S_1 V_1 & V_1 + V_2 & 0 & -S_2 \\ -S_1 & 0 & D_1 + D_2 & -S_2 D_2 \\ 0 & S_2 & -S_2 D_2 & D_2 \end{bmatrix} \begin{bmatrix} p_1^0(t) \\ p_I(t) \\ u_I(t) \\ u_2^{L_2}(t) \end{bmatrix} = \begin{bmatrix} f^{u_1}(t) \\ f^{u_I}(t) \\ f^{p_I}(t) \\ f^{p_2}(t) \end{bmatrix},$$

in the unknown functions  $p_1^0(t) = p_1(0, t), p_I(t), u_I(t), u_2^{L_2}(t) = u_2(L_2, t)$ . The subscript

$i$ ,  $i = 1, 2$ , of the operators in the left-hand side of (60) means that they have to be considered in the  $i$ -th part of the rod and to simplify the notation we have denoted the operator  $S_{-i}(\cdot)$  with  $S_i(\cdot)$ ,  $i = 1, 2$ , while functions in the right-hand side are of the following form:

$$\begin{aligned} f^{u_1}(t) &= \bar{u}_1(t), \\ f^{u_i}(t) &= -H[t - \frac{L_1}{c_1}] \bar{u}_1(t - \frac{L_1}{c_1}) + \frac{c_2}{E_2 A_2} H[t - \frac{L_2}{c_2}] \int_0^{t-L_2/c_2} \bar{p}_2(\tau) d\tau, \\ f^{p_1}(t) &= H[t - \frac{L_2}{c_2}] \bar{p}_2(t - \frac{L_2}{c_2}) + \frac{E_1 A_1}{c_1} H[t - \frac{L_1}{c_1}] \bar{u}_{1,t}(t - \frac{L_1}{c_1}), \\ f^{p_2}(t) &= \bar{p}_2(t). \end{aligned}$$

At this stage, one can proceed introducing a weak formulation of (60), which can be, as in the case of a single domain, of  $L^2$ ,  $\sigma$  or energetic type, and then operating a suitable discretization of the derived problem. The final linear systems (54) present a matrix with the same block structure as that of the operator in the left-hand side of (60).

## 5.2 - Generalization

Let us consider a rod of length  $L$ , constituted by  $n$  parts of length  $L_i$ ,  $i = 1, \dots, n$ , made up by different materials, with proper constants  $\rho_i$ ,  $E_i$ ,  $A_i$ ,  $i = 1, \dots, n$ . As a consequence, the wave propagation velocity in the  $i$ -th component of the rod will be in general different from each other and will be denoted by  $c_i$ . The previously given definitions can be straightforwardly extended to the wave propagation problem in this multi-material rod, with assigned condition of mixed type at the end-points of the rod, i.e.  $u_1(0, t) = \bar{u}_1(t)$ ,  $p_n(L_n, t) = \bar{p}_n(t)$ . The  $2n$  unknown functions are: traction  $p_1^0(t) := p_1(0, t)$  at the first end-point of the rod, traction and displacement  $p_{I_\ell}(t)$ ,  $u_{I_\ell}(t)$  at every interface  $I_\ell$ ,  $\ell = 1, \dots, n-1$ , between two consecutive portions of the rod, displacement  $u_n^L(t) := u_n(L_n, t)$  at the other end-point. In particular, we want to give the expression of the multi-domain integro-differential operator, which reads for  $t \in (0, T)$ :

$$(61) \quad \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_{I_1} \\ \vdots \\ \mathcal{L}_{I_k} \\ \vdots \\ \mathcal{L}_{I_{n-1}} \\ \mathcal{L}_n \end{bmatrix} \begin{bmatrix} p_1^0(t) \\ z_{I_1}(t) \\ \vdots \\ z_{I_k}(t) \\ \vdots \\ z_{I_{n-1}}(t) \\ u_n^L(t) \end{bmatrix} = \begin{bmatrix} f^{u_1}(t) \\ f_{I_1}(t) \\ \vdots \\ f_{I_k}(t) \\ \vdots \\ f_{I_{n-1}}(t) \\ f^{p_n}(t) \end{bmatrix}, \quad \begin{aligned} z_{I_k}(t) &= \begin{bmatrix} p_{I_k}(t) \\ u_{I_k}(t) \end{bmatrix}, \\ k &= 1, \dots, n-1, \end{aligned}$$

where:

$$\begin{aligned}
 \mathcal{L}_1 &= [V_1 \quad S_1 V_1 \quad S_1 \quad O \quad \cdots \quad O], \\
 \mathcal{L}_{I_1}^1 &= [S_1 V_1 \quad V_1 + V_2 \quad O \quad -S_2 V_2 \quad -S_2], \\
 \mathcal{L}_{I_1}^2 &= [-S_1 \quad O \quad D_1 + D_2 \quad -S_2 \quad -S_2 D_2], \\
 \mathcal{L}_{I_1} &= \begin{bmatrix} \mathcal{L}_{I_1}^1 & O & \cdots & O \\ \mathcal{L}_{I_1}^2 & \underbrace{O & \cdots & O}_{2n-5} \end{bmatrix}, \\
 \text{for } k &= 2, \dots, n-2, \\
 \mathcal{L}_{I_k}^1 &= [-S_k V_k \quad S_k \quad V_k + V_{k+1} \quad O \quad -S_{k+1} V_{k+1} \quad -S_{k+1}], \\
 \mathcal{L}_{I_k}^2 &= [S_k \quad -S_k D_k \quad O \quad D_k + D_{k+1} \quad -S_{k+1} \quad -S_{k+1} D_{k+1}], \\
 \mathcal{L}_{I_k} &= \begin{bmatrix} O & \cdots & O & \mathcal{L}_{I_k}^1 & O & \cdots & O \\ \underbrace{O & \cdots & O}_{2k-3} & \mathcal{L}_{I_k}^2 & \underbrace{O & \cdots & O}_{2(n-k)-3} \end{bmatrix}, \\
 \mathcal{L}_{I_{n-1}}^1 &= [-S_{n-1} V_{n-1} \quad S_{n-1} \quad V_{n-1} + V_n \quad O \quad -S_n], \\
 \mathcal{L}_{I_{n-1}}^2 &= [S_{n-1} \quad -S_{n-1} D_{n-1} \quad O \quad D_{n-1} + D_n \quad -S_n D_n], \\
 \mathcal{L}_{I_{n-1}} &= \begin{bmatrix} O & \cdots & O & \mathcal{L}_{I_{n-1}}^1 \\ \underbrace{O & \cdots & O}_{2n-5} & \mathcal{L}_{I_{n-1}}^2 \end{bmatrix}, \\
 \mathcal{L}_n &= [O \quad \cdots \quad O \quad S_n \quad -S_n D_n \quad D_n],
 \end{aligned}$$

and

$$\begin{aligned}
 f^{u_1}(t) &= \bar{u}_1(t), \\
 f_{I_1}(t) &= \begin{bmatrix} f^{u_{I_1}}(t) \\ f^{p_{I_1}}(t) \end{bmatrix} = \begin{bmatrix} -H[t - \frac{L_1}{c_1}] \bar{u}_1(t - \frac{L_1}{c_1}) \\ \frac{E_1 A_1}{c_1} H(t - \frac{L_1}{c_1}) \bar{u}_{1,t}(t - \frac{L_1}{c_1}) \end{bmatrix}, \\
 f_{I_k}(t) &= \begin{bmatrix} f^{u_{I_k}}(t) \\ f^{p_{I_k}}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{for } k = 2, \dots, n-2, \\
 f_{I_{n-1}}(t) &= \begin{bmatrix} f^{u_{I_{n-1}}}(t) \\ f^{p_{I_{n-1}}}(t) \end{bmatrix} = \begin{bmatrix} \frac{c_n}{E_n A_n} H[t - \frac{L_n}{c_n}] \int_0^{t-L_n/c_n} \bar{p}_n(\tau) d\tau \\ H[t - \frac{L_n}{c_n}] \bar{p}_n(t - \frac{L_n}{c_n}) \end{bmatrix}, \\
 f^{p_n}(t) &= \bar{p}_n(t).
 \end{aligned}$$

## 6 - Numerical results

In this section, a series of significative examples and several numerical results for one-dimensional wave propagation problems in layered media are reported in order to verify the efficiency of the new energetic formulation we have proposed.

- As one-domain test problem, taken from [32], let us consider a rod of unitary length  $L = 1$ , fixed in  $x = 0$  and subjected to a uniform traction at the other end-point. Hence, we introduce the following boundary conditions:  $u(0, t) = 0$ ,  $p(L, t) = H[t]$ . The wave velocity is set  $c = 1$ ; further  $A = 1$ ,  $E = 1$ . The observation time interval is  $(0, 20)$ . For the discretization, we have considered two time steps: the first,  $\Delta t = 0.1$ , is such that the time  $L/c$  required by the elastic wave to cover the distance between the two end-points of the rod is a multiple of it; the second,  $\Delta t = 0.08$ , has not this property. Tractions in  $x = 0$  are approximated by constant shape functions and displacements in  $x = L$  by linear shape functions. In Figure 1 we show the numerical solution obtained with  $\Delta t = 0.1$ , starting from  $L^2$  weak formulation. The same graph is obtainable starting from energetic and from  $\sigma$  weak formulations. The approximate solution overlaps the analytical one and it is in perfect agreement with that reported in [32], obtained with a collocation technique using the first BIE (34) alone.

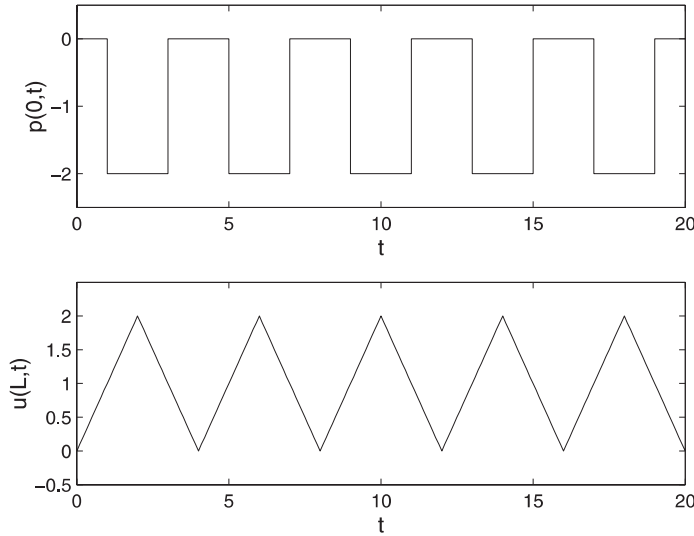


Fig. 1. – Numerical solution obtained with  $\Delta t = 0.1$ , starting from  $L^2$  weak formulation.

In Figures 2, 3 the sparse structure of matrices  $A_{L^2}$ ,  $A_{\mathcal{E}}$  and  $A_{\sigma}$  are reported. For what concerns their spectral condition number we have:  $\mu_2(A_{L^2}) = 3.46 \cdot 10^4$  and  $\mu_2(A_{\mathcal{E}}) = 6.82 \cdot 10^2$ .

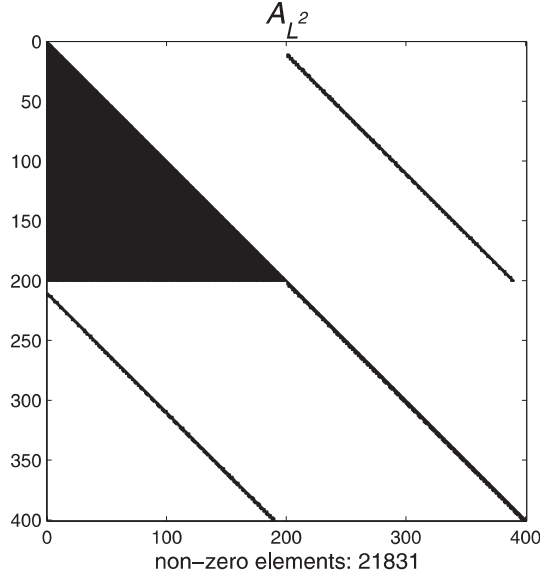


Fig. 2. – The sparse structure of matrix  $A_{L^2}$ .

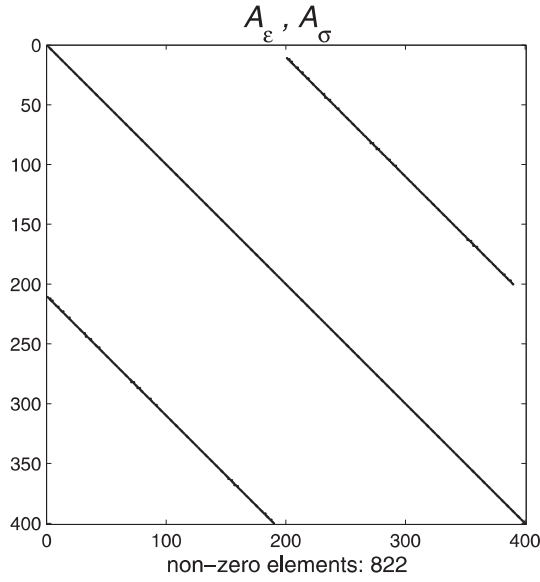


Fig. 3. – The highly sparse structure of matrices  $A_{\epsilon}$  and  $A_{\sigma}$ .

In Table 1 we show spectral condition numbers of  $A_{\sigma}$  for different values of the parameter  $\sigma$ , for two different time steps. As one can see, the condition number of  $A_{\sigma}$

rapidly increases even for small values of  $\sigma$ . In Figure 4 we show the numerical solution obtained with  $\Delta t = 0.08$ , starting from energetic weak formulation, compared with the analytical one. Also in this case the approximate solution is in perfect agreement with that reported in [32]. In Figure 5 we present numerical results obtained with  $\Delta t = 0.08$ , starting from  $L^2$  weak formulation. As one can see, they are affected by huge instability phenomena.

Table 1. – *Spectral condition numbers of matrix  $A_\sigma$  for different values of  $\sigma$  and  $\Delta t$ .*

$\sigma$	0.1	0.2	0.3	0.4	0.5
$\Delta t = 0.1$					
$\mu_2(A_\sigma)$	$8.41E+03$	$3.30E+05$	$1.62E+07$	$8.40E+08$	$4.44E+10$
$\Delta t = 0.08$					
$\mu_2(A_\sigma)$	$1.23E+03$	$4.91E+05$	$2.48E+07$	$1.31E+09$	$7.26E+10$

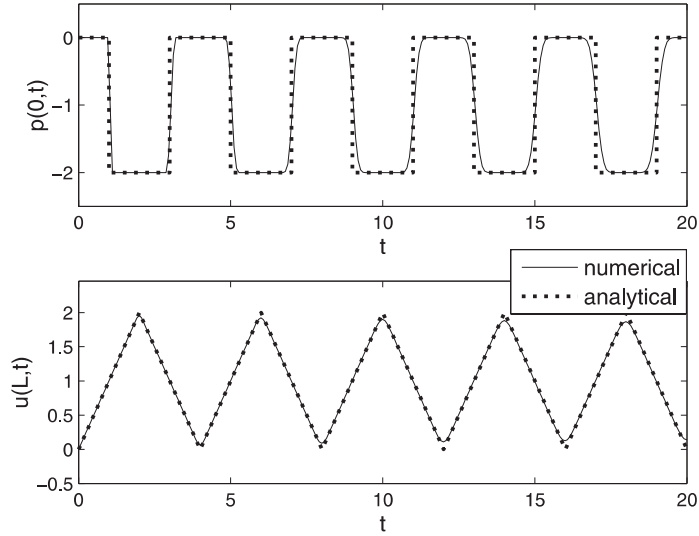


Fig. 4. – Numerical solution obtained with  $\Delta t = 0.08$ , starting from energetic weak formulation, compared with analytical one.

- Now, we analyze a rod of unitary length, fixed in  $x = 0$  and subjected to a uniform traction at the other end-point, as in the first test problem. We consider a subdivision of the rod in two equal parts having:  $A_1 = A_2$ ,  $E_1 = E_2$ , the same velocity of propagation:  $c_1 = c_2 = 1$  and, of course,  $L_1 = L_2 = 0.5$ . The observation



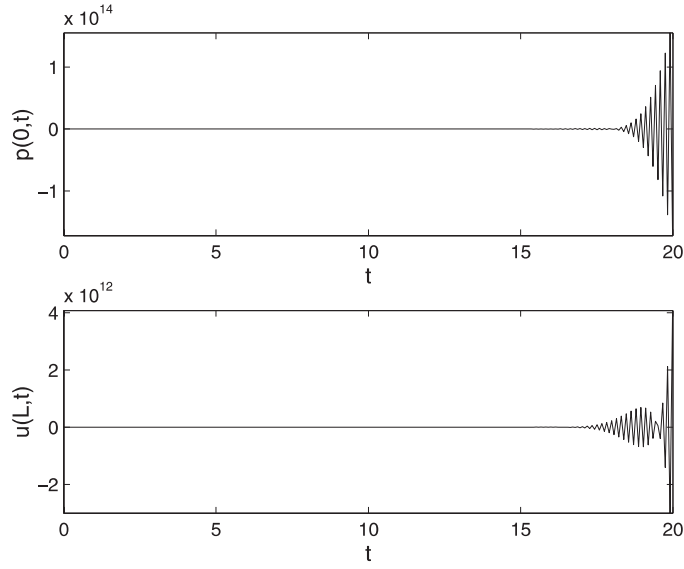


Fig. 5. – Instability phenomena arising with  $\Delta t = 0.08$ , using  $L^2$  weak formulation.

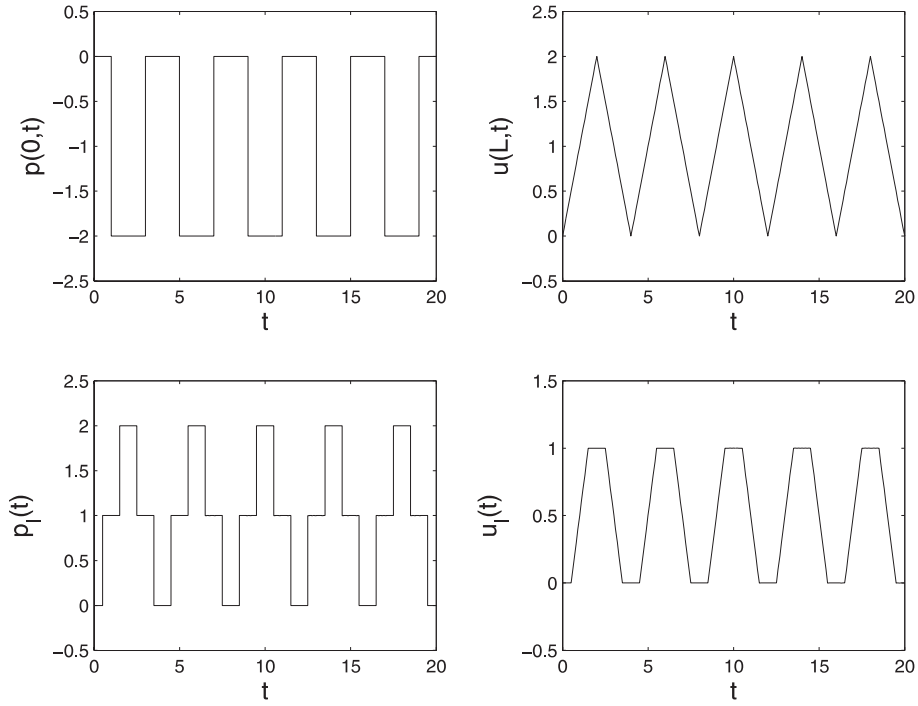


Fig. 6. – Numerical solution obtained with  $\Delta t = 0.1$ , starting from  $L^2$  weak formulation.

time interval is  $(0, 20)$ . For the discretization, we have used the same two time steps of the first test problem, i.e.  $\Delta t = 0.1$ , such that the time  $L_i/c_i$ ,  $i = 1, 2$  required by the elastic wave to cover the distance between the two end-points of each portion of the rod is a multiple of it; and  $\Delta t = 0.08$ , which has not this property. Traction in  $x = 0$  and at interface are approximated by constant shape functions, while displacements at the interface and in  $x = 1$  by linear shape functions. In Figure 6 we show the numerical solution obtained with  $\Delta t = 0.1$ , starting from  $L^2$  weak formulation. The same graph is obtainable starting from energetic and  $\sigma$  weak formulations. The approximate solution overlaps the ana-

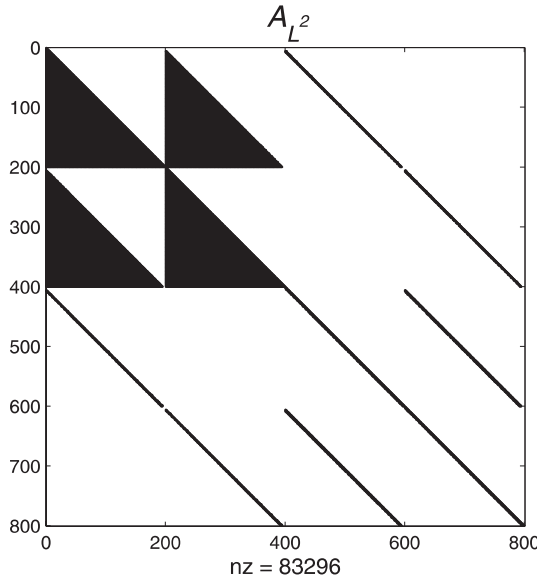


Fig. 7. – The sparse structure of matrix  $A_{L^2}$ .

lytical one. Note that this bi-domain formulation allows to know the time history of displacements and traction in an interior point of one-material rod. In Figures 7, 8 the sparse structure of matrices  $A_{L^2}$ ,  $A_{\mathcal{E}}$  and  $A_{\sigma}$  are reported. For what concerns their spectral condition number we have:  $\mu_2(A_{L^2}) = 9.12 \cdot 10^4$  and  $\mu_2(A_{\mathcal{E}}) = 2.60 \cdot 10^3$ . Table 2 contains spectral condition numbers  $\mu_2(A_{\sigma})$  for different values of the parameter  $\sigma$ , for two different time steps. As in the previous test, on a large observation time interval the condition number of  $A_{\sigma}$  rapidly increases even for small values of  $\sigma$ . In Figure 9 we show the numerical solution obtained with  $\Delta t = 0.08$ , starting from energetic weak formulation, compared with the

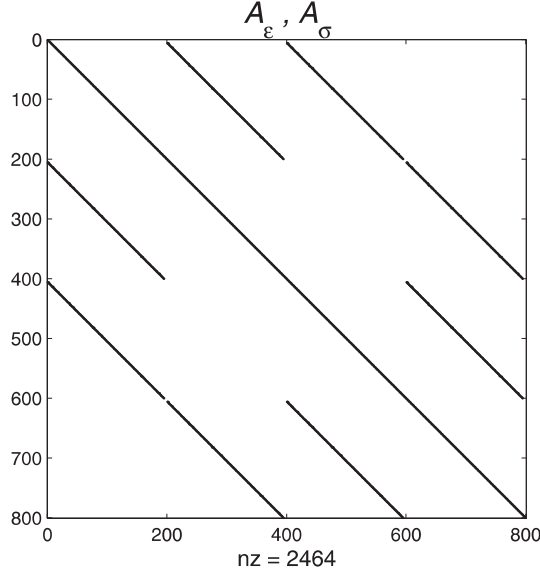


Fig. 8. – The highly sparse structure of matrices  $A_\epsilon$  and  $A_\sigma$ .

analytical one. If instead we start from  $L^2$  weak formulation, we observe high instability phenomena, as those presented in the one-domain numerical test.

Table 2. – Spectral condition numbers of matrix  $A_\sigma$  for different values of  $\sigma$  and  $\Delta t$ .

$\sigma$	0.1	0.2	0.3	0.4	0.5
$\Delta t = 0.1$					
$\mu_2(A_\sigma)$	$2.46E+04$	$8.43E+05$	$3.85E+07$	$1.90E+09$	$9.84E+10$
$\Delta t = 0.08$					
$\mu_2(A_\sigma)$	$3.51E+03$	$1.20E+06$	$5.62E+07$	$2.87E+09$	$1.50E+11$

• In this benchmark, taken from [49], we consider a rod of length  $L$  consisting of two parts having the same material properties, as in the previous test. But while till now we have treated dimensionless problems, here we consider the following constants: length of each portion:  $L_1 = L_2 = 5\text{ m}$ , transversal section area:  $A_1 = A_2 = 10^{-4}\text{ m}^2$ , Young modulus:  $E_1 = E_2 = 2.1 \cdot 10^8\text{ kN/m}^2$ , density:  $\rho_1 = \rho_2 = 7.85\text{ t/m}^3$  and therefore wave velocity:  $c_1 = c_2 = 5172\text{ m/s}$ . The analysis is carried out until  $T = 0.025\text{ s}$ . The rod is fixed in one end-point and subjected to a Heaviside load applied at the other end-point at the time instant  $t_0 = 0$ . Traction in  $x = 0$  and at interface are approximated by constant shape functions, while displacements at interface and in  $x = L$  by linear shape functions. In Figure 10 we show

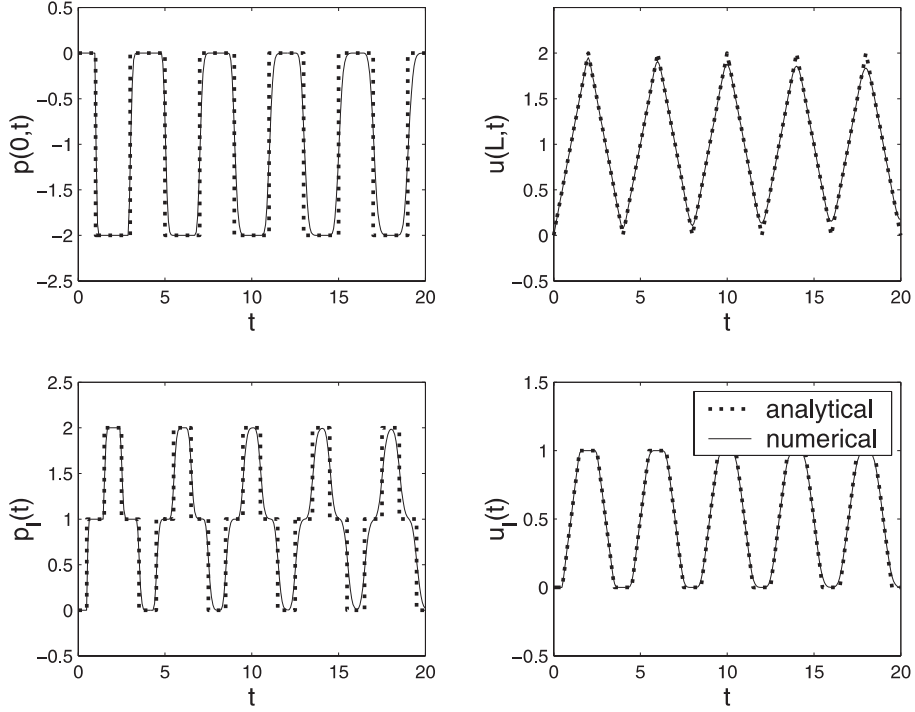


Fig. 9. – Numerical solution obtained with  $\Delta t = 0.08$ , starting from energetic weak formulation, compared with analytical one.

the numerical solutions obtained starting from energetic weak formulation and using two time steps: the same  $\Delta t = 5 \cdot 10^{-5}$  s used in [49] and  $\Delta t = \frac{1}{20} \frac{5}{5172}$  s, in such a way that time  $L_i/c_i$  is a multiple of it. The same graphs are obtainable starting from  $\sigma$  weak formulation. For what concerns the spectral condition number of matrices  $A_\varepsilon$  and  $A_\sigma$ , for the first time step we have:  $\mu_2(A_\varepsilon) = 1.17 \cdot 10^{11}$  that coincides with  $\mu_2(A_\sigma)$  for  $\sigma = 0$ . In this test, since we analyze the wave propagation on a small observation time interval, the spectral condition number of matrix  $A_\sigma$  increases very slowly for small values of parameter  $\sigma$ .

The approximate solution, for  $\Delta t = 5 \cdot 10^{-5}$  s, is better than that reported in [49], where a Duhamel integral based approach and a convolution quadrature method are used for the numerical solution of the benchmark. For the other time step, the numerical solution overlaps the analytical one and can be obtained starting from  $L^2$  weak formulation, too. On the contrary, for the former  $\Delta t$ , with this classical weak formulation we have observed huge instability phenomena.

- In this numerical example, a rod of length  $L$  consisting of three parts, two of

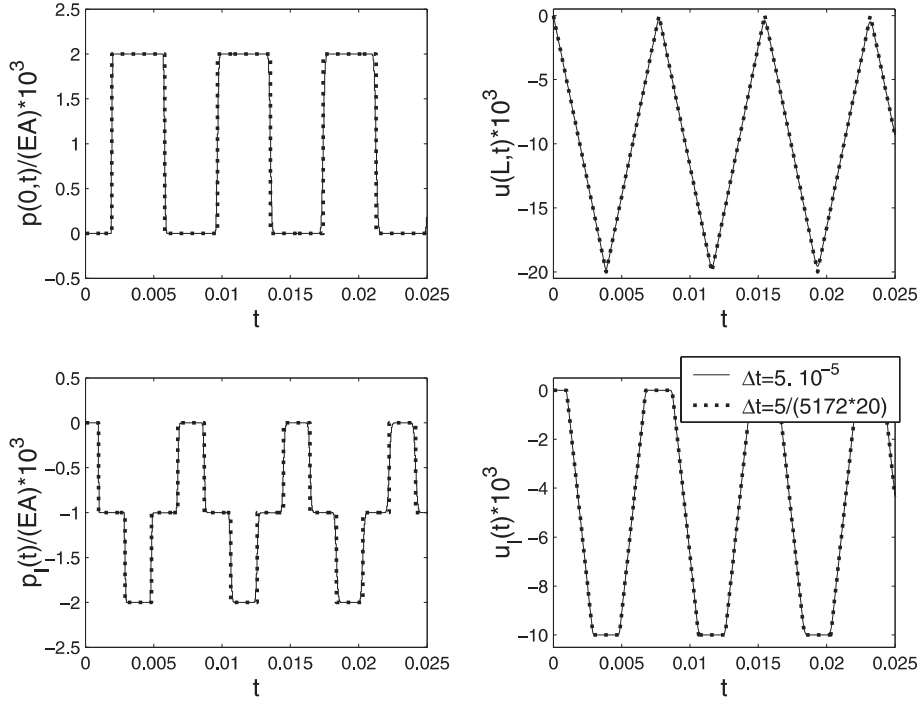


Fig. 10. – Numerical solutions obtained with two different time steps, starting from energetic weak formulation.

them, the first and the final, having the same material properties and lengths, is considered. The constants for this example are: length of each portion:  $L_1 = L_3 = 5\text{ m}$ ,  $L_2 = 10\text{ m}$ , transversal section area:  $A_1 = A_2 = A_3 = 10^{-4}\text{ m}^2$ , Young's modulus:  $E_1 = E_3 = 1.1 \cdot 10^8\text{ kN/m}^2$ ,  $E_2 = 2.1 \cdot 10^8\text{ kN/m}^2$  and density  $\rho_1 = \rho_3 = 8.96\text{ t/m}^3$ ,  $\rho_2 = 2.7\text{ t/m}^3$ ; the wave velocity are:  $c_1 = c_3 = 3.50383 \cdot 10^3\text{ m/s}$ ,  $c_2 = 8.8192 \cdot 10^3\text{ m/s}$ . The analysis is carried out until  $T = 0.05\text{ s}$ . The rod is fixed in one end-point and subjected to a Heaviside load at the other end-point. Tractions in  $x = 0$  and at interfaces  $I_1$  and  $I_2$  are approximated by constant shape functions, while displacements at interfaces and in  $x = L$  by linear shape functions. The problem has been studied for different time steps; in Figure 11 we show the numerical solutions obtained starting from energetic weak formulation and using  $\Delta t = 6.25 \cdot 10^{-5}\text{ s}$ . Since there is a discontinuity in the Young's modulus, the waves travelling downwards will be partly reflected and partly refracted when reaching the interfaces between the first and the second part, the second and the third part of the rod.

- The present benchmark, always taken from [49], is once again a rod consisting of two parts, one of them extended to infinity. These parts have different material properties and the corresponding constants are:  $L_1 = \infty$ ,  $L_2 = 5\text{ m}$ ,

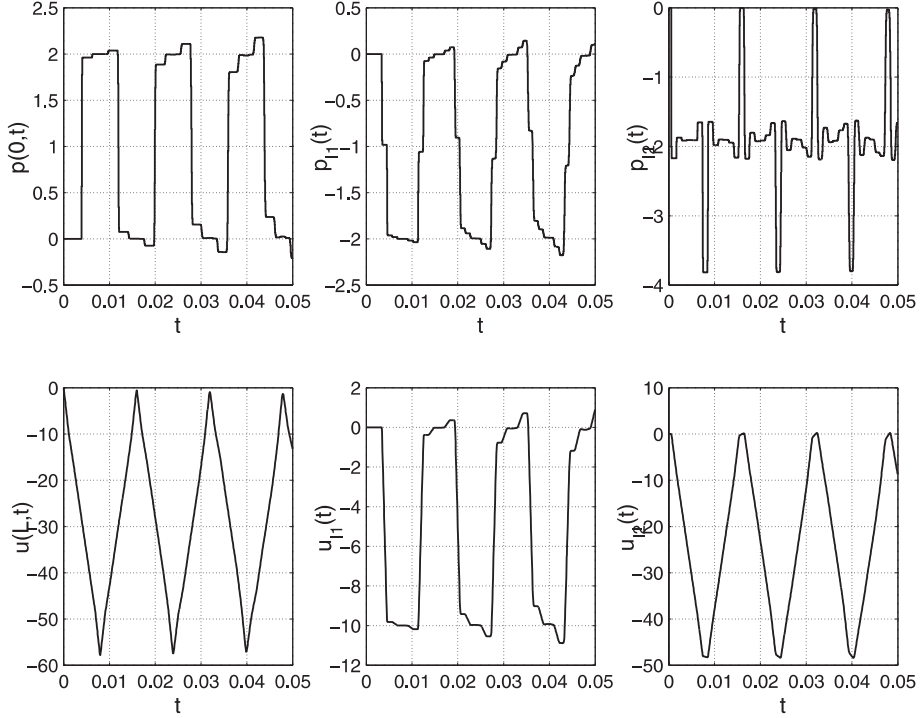


Fig. 11. – Three coupled finite rods under a Heaviside load: numerical solutions obtained starting from energetic weak formulation.

$A_1 = A_2 = 10^{-4} \text{ m}^2$ ,  $E_1 = 2.1 \cdot 10^8 \text{ kN/m}^2$ ,  $E_2 = 0.8 \cdot 10^8 \text{ kN/m}^2$ ,  $\rho_1 = \rho_2 = 7.85 \text{ t/m}^3$ ; as a consequence  $c_1 = 5172 \text{ m/s}$ ,  $c_2 = 3192 \text{ m/s}$ . The analysis is carried out until  $T = 0.015 \text{ s}$ . The rod is loaded at the end-point of the bounded portion with a rectangular impulse at time  $t_0 = 0.001 \text{ s}$  with a duration of  $t_1 - t_0 = 0.001 \text{ s}$ . A Sommerfeld radiation condition is satisfied by displacements at infinity. Traction at interface is approximated by constant shape functions, while displacements at the interface and at the end-point of the bounded portion are approximated by linear shape functions. For the discretization, we have used the largest time step used in [49] for the approximation of this benchmark, i.e.  $\Delta t = 5 \cdot 10^{-5} \text{ s}$ . In Figures 12, 13 we show respectively the interface traction and the loaded end-point displacement, obtained starting from energetic weak formulation: our numerical results perfectly agree with those reported in [49] for what concerns displacement and are substantially better for what concerns traction. Note that if we want to use  $L^2$  weak formulation preventing instability phenomena we would have to consider for this problem a time step  $\Delta t = 399^{-1} \cdot 10^{-3} \text{ s} \simeq 2.5 \cdot 10^{-6} \text{ s}$ , such that times  $L_2/c_2$  and  $t_0 = t_1 - t_0$  are multiples of it. This of course enlarges very much the computational

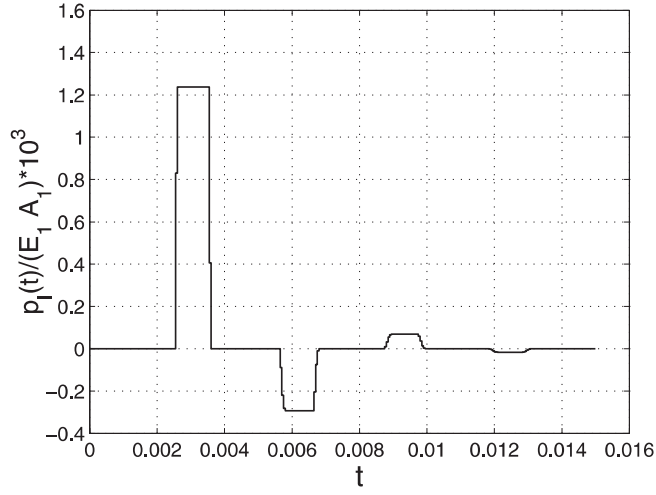


Fig. 12. – Approximated traction at interface.

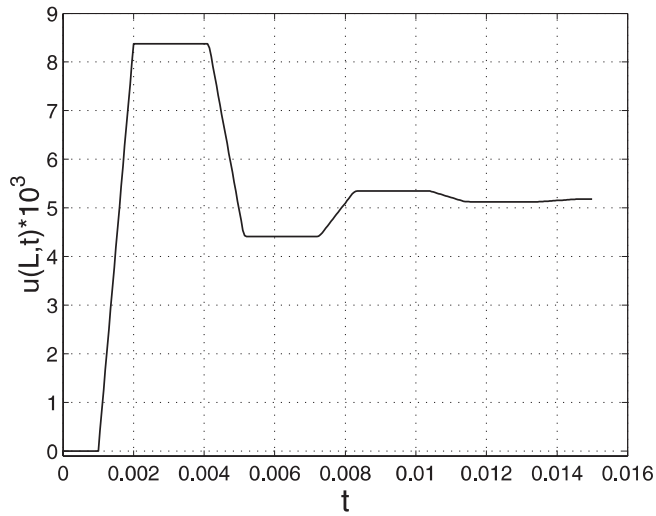


Fig. 13. – Approximated displacement at loaded end-point.

cost of the procedure, both for linear system generation time and for memory requirements.

- The final benchmark, which extends that one involving the above semi-infinite bi-material rod, is again a rod now consisting of three parts, one of them extending to infinity. These parts have different material properties and the

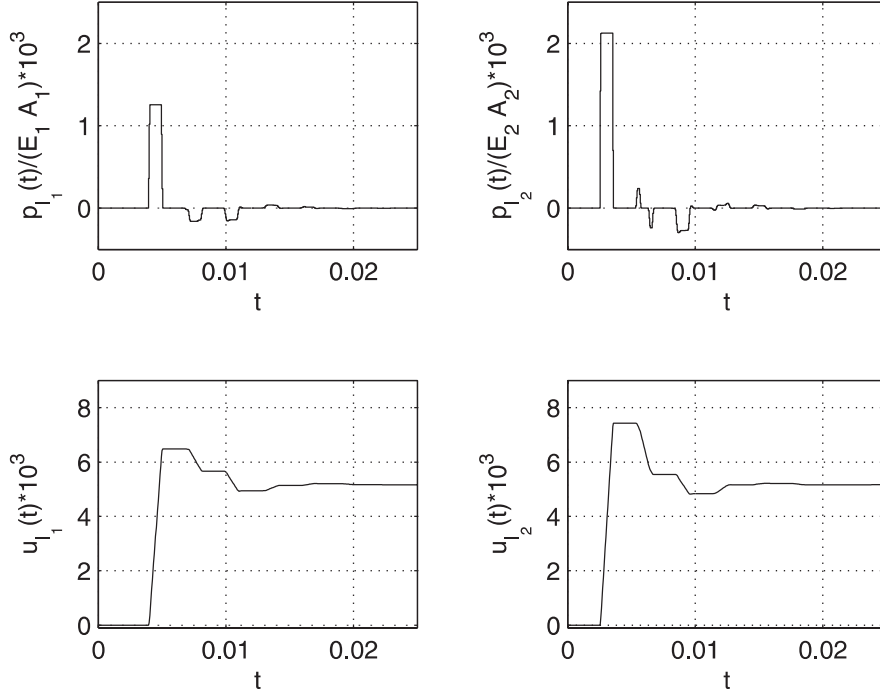


Fig. 14. – Numerical solution at both the interfaces in the semi-infinite three-material rod, starting from energetic weak formulation.

corresponding constants are:  $L_1 = \infty$ ,  $L_2 = L_3 = 5\text{ m}$ ,  $A_i = 10^{-4}\text{ m}^2$ ,  $i = 1, 2, 3$ ,  $E_1 = 2.1 \cdot 10^8\text{ kN/m}^2$ ,  $E_2 = 1.1 \cdot 10^8\text{ kN/m}^2$ ,  $E_3 = 0.8 \cdot 10^8\text{ kN/m}^2$ ,  $\rho_1 = \rho_3 = 7.85\text{ t/m}^3$ ,  $\rho_2 = 8.98\text{ t/m}^3$ ; as a consequence  $c_1 = 5172\text{ m/s}$ ,  $c_2 = 3500\text{ m/s}$ ,  $c_3 = 3192\text{ m/s}$ . The analysis is carried out until  $T = 0.025\text{ s}$ . The rod is loaded at the end-point of the bounded portion with a rectangular impulse at time  $t_0 = 0.001\text{ s}$  with a duration of  $t_1 - t_0 = 0.001\text{ s}$ . A Sommerfeld radiation condition is satisfied by displacements at infinity. Tractions at interfaces are approximated by constant shape functions, while displacements at interfaces and at the end-point of the bounded portion by linear shape functions. For the discretization of the energetic weak problem, we have used different time steps  $\Delta t$ . In Figure 14, we show tractions and displacements of the two interfaces, obtained with  $\Delta t = 5 \cdot 10^{-5}\text{ s}$ , while in Figure 15 we present the loaded end-point displacements obtained with  $\Delta t = 5 \cdot 10^{-4}$  (dotted line) and with  $\Delta t = 5 \cdot 10^{-5}\text{ s}$  (solid line), compared with the three analytical solutions related to semi-infinite homogeneous rods each constituted by one of the above three materials. Since there is a discontinuity in the Young's modulus, the wave which starts to travel from the loaded end-point will be partly reflected and partly refracted when reaching the two interfaces. After



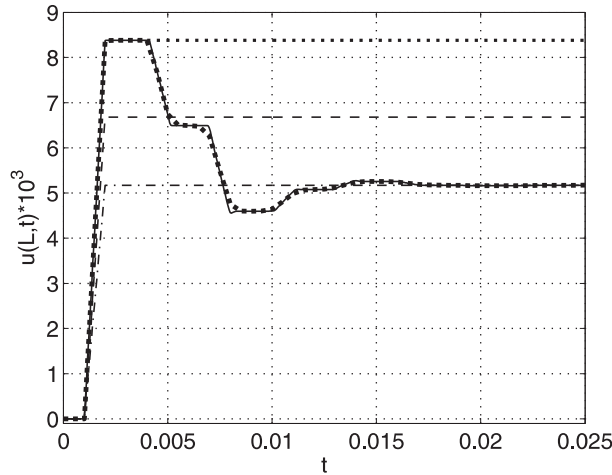


Fig. 15. – Approximated displacements at  $x = L$ , for  $\Delta t = 5 \cdot 10^{-4} ( \cdot )$  and  $\Delta t = 5 \cdot 10^{-5} ( - )$ , starting from energetic weak formulation, compared with semi-infinite single-material analytical solutions.

being reflected on the loaded end-point, the wave starts travelling again and refraction and reflection occur again. This happens repeatedly until the whole wave has left the two bounded portion of the rod and is travelling towards infinity. Before the reflected wave reaches the loaded end-point of the rod at time  $t_0 + 2 \frac{L_3}{c_3}$ , the behavior of this end-point must be equal to that of a homogeneous rod with  $E_3 = 0.8 \cdot 10^8 \text{ kN/m}^2$  subject to the same rectangular impulse. For larger values of  $t$ , the loaded end-point displacement must converge to the analytical solution for a semi-infinite homogeneous rod with  $E_1 = 2.1 \cdot 10^8 \text{ kN/m}^2$ .

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### Abstract

*In this paper we consider one-dimensional wave propagation problems, with suitable boundary conditions, reformulated using space-time boundary integral equations with retarded potential. In the first part, special attention is devoted to a formulation based on a natural energy identity that leads to a space-time weak formulation of the corresponding boundary integral equations with robust theoretical properties. Continuity and coerciveness of the bilinear form related to energetic formulation are proved.*

*Then we compare the new energetic weak formulation with different other time-domain boundary element method procedures applied to wave propagation analysis in layered media. The paper concludes with several numerical tests to demonstrate the effectiveness of the introduced technique in the numerical solution of Dirichlet-Neumann problems in their integral formulation, pointing out the numerical properties of the derived linear systems.*

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