Riv. Mat. Univ. Parma (7) **5** (2006), 69-79

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On radial continuity of farthest point map (**)

1 - Introduction

Let V be a non-empty bounded subset of a metric space (X, d). The anti metric projection (farthest point map) is the mapping Q_V, which takes each element x of X into the set $Q_V(x) = \{v \in V : d(x, v) = \delta(x, V) \equiv \sup_{y \in V} d(x, y)\}$ of all the farthest points to x in V. V is called remotal (uniquely remotal) if $Q_V(x) \neq \emptyset$ ($Q_V(x)$ contains exactly one point) for each $x \in X$. Q_V is called lower semi-continuous (upper semi-continuous) at x if for each open set W with $Q_V(x) \cap W \neq \emptyset$ ($Q_V(x) \subset W$) there exists a neighbourhood U of x such that $Q_V(y) \cap W \neq \emptyset$ ($Q_V(y) \subset W$) for all $y \in U$. If V is non-empty subset of a metric space (X, d) then nearest points and metric projections are defined similarly (see e.g. [**3**]). Many results on the lower (upper) semi-continuity of the farthest point map are available in the literature (see e.g. [**1**], [**8**], [**9**]). Concerning metric projections (nearest point maps) and anti metric projections (farthest point maps) Brosowski and Deutsch [**2**], [**3**] and, Panda and Kapoor [**12**] introduced and discussed in normed linear spaces some simple and more general "radial" continuity criteria (called Outer Radial Lower, Inner Radial Lower, Outer Radial Upper, Inner Radial Upper) which require that the restriction of the mapping to certain prescribed line segments be lower semi continuous

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(**) Received 3rd April 2006 and in revised form 24th July 2006. AMS classification 41 A 65, 54 C 60. This research was partially supported by University Grants Commission, India (F30-238/2004(SR)).

(lsc) or upper semi continuous (usc). In this paper, we discuss all these concepts for the farthest point map. The underlying spaces are convex metric spaces introduced by Takahashi [**15**].

2 - Notations, definitions and examples

In this section we give some notations, recall few definitions and give some examples.

Let (X, d) be a metric space and $x, y, z \in X$. We say that z is between x and y if $d(x, z) + d(z, y) = d(x, y)$. For any two points x, y of X, the set $\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ is called the metric segment and is denoted by $G[x, y]$. The set $G[x, y, -]$ denotes the ray starting from x and passing through y i.e. it is the largest line segment containing $G[x, y]$ for which x is an extreme point. The set $G_1(x, y, -) \equiv G[x, y, -] \setminus G[x, y]$ i.e. it is the set of all those points on the ray starting from x passing through y which do not lie between x and y.

A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all $u \in X$. A metric space (X, d) together with a convex structure is called a convex metric space [**15**].

Example [**15**]. Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i], I_j = [a_j, b_j]$ and $\lambda (0 \leq \lambda \leq 1)$, define a mapping W by $W(I_i, I_j, \lambda) = [\lambda a_i + (1 - \lambda) a_j, \lambda b_i + (1 - \lambda) b_j]$ and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i, I_j) = \sup_{a \in I} \left\{ \left| \inf_{b \in I_i} \{|a - b|\} - \inf_{c \in I_j} \{|a - c|\} \right| \right\}.$$

A subset K of a convex metric space (X, d) is said to be convex [**15**] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$ i.e. $G[x, y] \subset K$ for all $x, y \in K$.

Example [**15**]. The open spheres $S(x, r)$ and the closed spheres $S[x, r]$ in a convex metric space (X, d) are convex.

A convex metric space (X, d) is said to satisfy Property (I) [**5**] (respectively Property (I')) if for all $x, y \in X$ and $\lambda \in [0, 1]$, $d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y)$ (respectively $d(W(x, p, \lambda), W(y, p, \lambda)) = \lambda d(x, y)$).

Every normed linear space has Properties (I) and (I').

A convex metric space (X, d) is called strongly convex [**10**] or an M-space [**6**] if for every two points $x, y \in X$, and every $r \in [0,1]$ there exists unique point $z_r = W(x, y, r) \in X$ such that $d(x, z_r) = (1-r)d(x, y)$ and $d(y, z_r) = rd(x, y)$ i.e. unique point z_r of the metric segment $G[x, y]$.

Example [**6**]. Let (X, ρ) be a closed ball of the $S_{2,r}$ of radius ρ with $\pi r/4 < \rho < \pi r/2$. Then (X, ρ) is an M-space. Here $S_{2,r}$ is the 2-dimensional spherical space of radius r. Its elements are all the ordered 3-tuples $x = (x_1, x_2, x_3)$ of real numbers with $\sum_{i=1}^{3} x_i^2 = r^2$, distance is defined for each pair of elements x, y to be the smallest non-negative number xy such that $\cos\left(\frac{xy}{r}\right) = \frac{\left(\sum_{i=1}^{3} x_i y_i\right)}{r^2}$.

A bounded subset K of a convex metric space (X, d) is said to have Property (SF) [**4**] if $x_0 \in X$ and $k_0 \in Q_K(x_0)$ imply $k_0 \in Q_K(W(x_0, k_0, \lambda)), 0 < \lambda < 1$. Let K be a bounded set in a metric space (X, d). A point $k_0 \in K$ satisfying $d(x, k_0) = \delta(x, K) \equiv \sup\{d(x, k) : k \in K\}$ is called a farthest point to $x \in X$ and the mapping $Q_K : X \to 2^K$, the collection of all subsets of K, defined by $Q_K(x) = \{k_0 \in K : d(x, k_0) = \delta(x, K)\}$ is called the farthest point map or anti metric projection. The set K is said to be

(a) remotal if for each $x \in X$, the set $Q_K(x)$ is non-empty,
(b) uniquely remotal if for each $x \in X$, the set $Q_K(x)$ consists of exactly one point.

Examples [**11**] 1. Every compact set in a metric space is remotal. The set K consisting of open unit square together with its corners in the 2-dimensional Euclidean space R^2 is remotal although it is not compact.

2. Most natural examples of uniquely remotal sets are singletons. Let $X = R \setminus \{0\}$ with the usual metric $d(x, y) = |x - y|$ and $K = [0,1] \setminus \{0\}$. Then K is uniquely remotal but is not a singleton.

3. Let $(X, \|.\|)$ be the usual infinite dimensional real Hilbert space, $K = \{x \in X : \|x\| = 1\}$.

The farthest point map $Q_K : X \to 2^K$ is defined as

$$Q_K(x) = \left\{\frac{-x}{\|x\|}\right\}, x \neq 0$$

$$= K, x = 0.$$

It is well known (see e.g. [**12**]) that if K is the closed unit ball in a finite dimensional normed linear space or in a locally uniformly convex Banach space then the farthest point map is upper semi continuous. Blatter [**1**] showed that a bounded subset K of Banach space X is singleton if and only if K is remotal subset of X and Q_K is lower semi continuous. We now discuss some generalizations of upper and lower semi continuity for the farthest point map.

3 - Outer Radial Upper(ORU) continuity

In this section we consider a first generalization of upper semi continuity, called ORU-continuity.

Let G be a non-empty bounded subset of an M-space (X,d) and $x_0 \in X$. The farthest point map Q_G is called ORU-continuous at x_0 if for each $g_0 \in Q_G(x_0)$ and each open set $W \supset Q_G(x_0)$, there exists a neighbourhood U of x_0 such that $Q_G(x) \subset W$ for all $x \in U \cap G_1(g_0, x_0, -)$. Q_G is called ORU-continuous if it is ORU-continuous at each point of X .

The following lemma will be used in proving the ORU-continuity of Q_G:

Lemma 3.1. [**14**] *If G is a bounded subset of an M-space (X,d) and $g_0 \in Q_G(x_0)$ for $x_0 \in X$ then $g_0 \in Q_G(x_\lambda)$, where $x_\lambda \in G_1(g_0, x_0, -)$.*

Proof. Consider

$$\begin{aligned} d(x_\lambda, g_0) =& d(x_\lambda, x_0) + d(x_0, g_0) \\ \geq& d(x_\lambda, x_0) + d(x_0, g) \text{ for all } g \in G \\ \geq& d(x_\lambda, g) \text{ for all } g \in G. \end{aligned}$$

Therefore $g_0 \in Q_G(x_\lambda)$.

Concerning ORU-continuity of Q_G, we have

Theorem 3.2. *For a non-empty bounded subset G of an M-space (X,d), the farthest point map is ORU-continuous.*

Proof. Assume that Q_G is not ORU-continuous at some $x_0 \in X$. Then there exists an element $g_0 \in Q_G(x_0)$ and an open set $W \supset Q_G(x_0)$ such that for every neighbourhood U of x_0 there is an $x \in G_1(g_0, x_0, -)$ satisfying $Q_G(x) \not\subset W$. Thus for a neighbourhood U_1 of x_0 there exists $x_\lambda \in G_1(g_0, x_0, -)$ such that $Q_G(x_\lambda) \not\subset W$. As $g_0 \in Q_G(x_0)$, by the above lemma, $g_0 \in Q_G(x_\lambda)$. We claim that $Q_G(x_\lambda) \subset Q_G(x_0)$.

Let $y \in Q_G(x_\lambda)$ be arbitrary. Then $\delta(x_0, G) \geq d(x_0, y)$. If $\delta(x_0, G) > d(x_0, y)$ then

$$\begin{aligned} d(x_\lambda, y) &\leq d(x_\lambda, x_0) + d(x_0, y) \\ &< d(x_\lambda, x_0) + \delta(x_0, G) \\ &= d(x_\lambda, x_0) + d(x_0, g_0) \\ &= d(x_\lambda, g_0) \\ &= d(x_\lambda, G) \end{aligned}$$

i.e. $d(x_\lambda, y) < \delta(x_\lambda, G)$ and so $y \notin Q_G(x_\lambda)$, a contradiction. Therefore, $\delta(x_0, G) = d(x_0, y)$ i.e. $y \in Q_G(x_0)$. Consequently, $Q_G(x_\lambda) \subset Q_G(x_0) \subset W$, a contradiction and hence Q_G is ORU-continuous at x_0 and so on X.

Remark. For normed linear spaces, Theorem 3.2 was proved by Rao and Chandrasekaran [**13**].

4 - Inner Radial Upper (IRU) continuity

A second generalization of upper semi continuity, called Inner Radial Upper (IRU) continuity [**12**] is:

Let V be a bounded subset of a convex metric space (X, d) and $x_0 \in X$. Q_V is called IRU-continuous at x_0 if for each $v_0 \in Q_V(x_0)$ and each open set $W \supset Q_V(x_0)$, there exists a neighborhood U of x_0 such that $Q_V(x) \subset W$ for all $x \in U \cap W(v_0, x_0, \lambda)$. Q_V is IRU-continuous if it is IRU-continuous at each point of X.

Concerning IRU-continuity of Q_V, we have the following:

Theorem 4.1. *Let V be a non-empty bounded subset of a convex metric space (X, d) and $x_0 \in X$. Consider the following statements:*

(i) *Q_V is IRU-continuous at x_0.*

(ii) *For each $v_0 \in Q_V(x_0)$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{v \in Q_V(x)} d(v, Q_V(x_0)) < \varepsilon$$

for every x in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $d(x, x_0) < \delta$.

(iii) *For each $v_0 \in Q_V(x_0)$ and each sequence x_n in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$,*

$$\sup_{v \in Q_V(x_n)} d(v, Q_V(x_0)) \to 0.$$

(iv) *For each* $v_0 \in Q_V(x_0)$ *and each sequence* x_n *in* $W(v_0, x_0, \lambda)$ *with* $x_n \to x_0$ *and each sequence* v_n *with* $v_n \in Q_V(x_n), d(v_n, Q_V(x_0)) \to 0$,

(v) *For each* $v_0 \in Q_V(x_0)$ *and each sequence* x_n *in* $W(v_0, x_0, \lambda)$ *with* $x_n \to x_0$ *and each sequence* v_n *with* $v_n \in Q_V(x_n)$ *and* $v_n \to v,\ v \in \overline{Q_V(x_0)}$.

Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(v). *Moreover, if* $Q_V(x_0)$ *is compact, (iv)$\Rightarrow$(i) and hence the first four statements are equivalent. If* V *is compact then (v)$\Rightarrow$(i) and hence all the five statements are equivalent.*

Proof. (i)$\Rightarrow$(ii) Choose $v_0 \in Q_V$ and $W = \bigcup \left\{ B\left(v, \frac{\varepsilon}{2}\right) : v \in Q_V(x_0) \right\} \supset Q_V(x_0)$. By the IRU-continuity, there exists a neighborhood $B(x_0, \delta)$ of x_0 such that $Q_V(x) \subset W$ for every x in $B(x_0, \delta) \cap \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$.

Let $x \in B(x_0, \delta) \cap \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ and $v \in Q_V(x)$ then there exists $v' \in Q_V(x_0)$ such that $d(v, v') < \frac{\varepsilon}{2}$ and so $d(v, Q_V(x_0)) < \frac{\varepsilon}{2}$. It follows that $\sup\limits_{v \in Q_V(x)} d(v, Q_V(x_0)) \leq \frac{\varepsilon}{2} < \varepsilon$.

(ii)$\Rightarrow$(iii) Let $v_0 \in Q_V(x_0)$ and x_n be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$. Therefore, for $\delta > 0$ there exists a positive integer N such that $d(x_n, x_0) < \delta$ for all $n \geq N$.

If $n \geq N$, $x_n \in B(x_0, \delta) \cap W(v_0, x_0, \lambda)$ and so by the hypothesis,

$$\sup_{v \in Q_V(x_n)} d(v, Q_V(x_0)) < \varepsilon$$

for all $n \geq N$, which implies

$$\sup_{v \in Q_V(x_n)} d(v, Q_V(x_0)) \to 0.$$

(iii)$\Rightarrow$(ii) Suppose (ii) does not hold i.e. there exists $v_0 \in Q_V(x_0)$ and $\varepsilon > 0$ such that for every $\delta > 0$, $\sup\limits_{v \in Q_V(x)} d(v, Q_V(x_0)) \geq \varepsilon$ for all x in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $d(x, x_0) < \delta$. Let x_n be a sequence in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $x_n \to x_0$. Then $\sup\limits_{v \in Q_V(x_n)} d(v, Q_V(x_0)) \geq \varepsilon$ for all n and so $\sup\limits_{v \in Q_V(x_n)} d(v, Q_V(x_0))$ does not converge to 0, a contradiction.

(iii)$\Rightarrow$(iv) Let $v_0 \in Q_V(x_0)$ and x_n be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ and v_n be a sequence in $Q_V(x_n)$. By hypothesis, $\sup\limits_{v \in Q_V(x_n)} d(v, Q_V(x_0)) \to 0$ and so $d(v_n, Q_V(x_0)) \to 0$.

(iv)$\Rightarrow$(iii) Suppose (iii) does not hold i.e. for some $v_0 \in Q_V(x_0)$ there exists a sequence x_n in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ such that $\sup\limits_{v \in Q_V(x_n)} d(v, Q_V(x_0))$ does not converge to 0.

Let v_n be a sequence in $Q_V(x_n)$. So, we have $\sup\limits_{v_n \in Q_V(x_n)} d(v_n, Q_V(x_0))$ does not converge to 0. This implies $d(v_n, Q_V(x_0))$ does not converge to 0 for some $v_n \in Q_V(x_n)$, a contradiction.

(iv)⇒(v) Let $v_0 \in Q_V(x_0)$ and x_n be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \to x_0$ and v_n be a sequence in $Q_V(x_n)$. By hypothesis, $d(v_n, Q_V(x_0)) \to 0$ i.e. $\lim d(v_n, Q_V(x_0)) = 0$. This implies $d(v, Q_V(x_0)) = 0 \Rightarrow v \in \overline{Q_v(x_0)}$.

Suppose $Q_V(x_0)$ is compact. To prove (iv)⇒(i).

Suppose (i) does not hold i.e. there exists $v_0 \in Q_V(x_0)$ and some open set $W \supset Q_V(x_0)$ such that for every n there is an x_n in $\{W(v_0, x_0, \lambda) : 1 - \frac{1}{n} < \lambda \le 1\}$ such that $Q_V(x_n) \cap W^c \neq \phi$. Choose $v_n \in Q_V(x_n) \cap W^c$. Then $x_n \to x_0$ and so $d(v_n, Q_V(x_0)) \to 0$. Choose $y_n \in Q_V(x_0)$ such that $d(v_n, y_n) \to 0$. As $Q_V(x_0)$ is compact, there exists a subsequence $y_n \to y_0 \in Q_V(x_0)$. Consider

$$\begin{aligned} d(v_n, y_0) &\le d(v_n, y_n) + d(y_n, y_0) \\ &\to 0. \end{aligned}$$

This implies $v_n \to y_0$. Since $y_0 \in Q_V(x_0) \subset W,\ y_0 \in W$ and W is open, $v_n \in W$ for large n, a contradiction.

Suppose V is compact. To prove (v)⇒(i). We show that (v)⇒(iv)⇒(i).

Let $v_0 \in Q_V(x_0)$ and x_n be a sequence in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $x_n \to x_0$ and v_n be a sequence with $v_n \in Q_V(x_n)$. To prove $d(v_n, Q_V(x_0)) \to 0$. Since $Q_V(x_n)$ is a closed subset of the compact set V, it is compact and so v_n has a subsequence $v_{n_i} \to v$. By hypothesis, $v \in \overline{Q_V(x_0)}$. Consider

$$\begin{aligned} d(v_{n_i}, Q_V(x_0)) &\le d(v_{n_i}, v) + d(v, Q_V(x_0)) \\ &\to 0. \end{aligned}$$

i.e. (iv) is true. Since V is compact $Q_V(x_0)$ is compact and so (iv)→(i). Hence (v)→(i).

For uniquely remotal sets, we have:

Theorem 4.2. *For a uniquely remotal set V in a convex metric space (X, d) satisfying Property (SF), the farthest point map Q_V is IRU-continuous.*

Proof. Let $x_0 \in X$ be arbitrary and $v_0 \in Q_V(x_0)$. Let G be an open set with $G \supset Q_V(x_0) = \{v_0\}$. Let $x \in \{W(x_0, v_0, \lambda) : 0 \le \lambda \le 1\}$. Since V has Property (SF), $v_0 \in Q_V(x)$. As V is a uniquely remotal, $Q_V(x) = \{v_0\}$. Let U be any neighbourhood of x_0 then for all $x \in U \cap \{W(x_0, v_0, \lambda) : 0 \le \lambda \le 1\},\ Q_V(x) = \{v_0\} \subset G$. Hence Q_V is IRU-continuous at x_0 and so on X.

Remark 4.3. Since a bounded subset V satisfying property (SF) of a convex metric space is singleton [**14**], the farthest point map in the above case becomes constant and so it is not only IRU-continuous but also continuous.

5 - Outer Radial Lower (ORL) continuity

First generalization of lower semi-continuity (lsc) is the following:

Let V be a non-empty bounded subset of an M-space (X, d) and $x_0 \in X$. The map Q_V is said to be ORL-continuous at x_0 if for each $v_0 \in Q_V(x_0)$ and each open set W for which $Q_V(x_0) \cap W \neq \emptyset$ there exists a neighbourhood U of x_0 such that $Q_V(x) \cap W \neq \emptyset$ for all $x \in U \cap G_1(v_0, x_0, -)$. Q_V is called ORL-continuous if it is ORL-continuous at each point of X.

The following lemma, which was proved by Brosowski and Deutsch ([3]-Lemma 2.2) in normed linear spaces for the metric projection P_V and which also holds in M-spaces for the mapping Q_V, will be used in the proof of our next theorem:

Lemma 5.1. *Let V be a non-empty bounded subset of an M-space (X, d) and $x_0 \in X$ then the following are equivalent:*

(i) *Q_V is ORL-continuous at x_0*

(ii) *for each $v_0, v_1 \in Q_V(x_0)$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that $Q_V(x) \cap B(v_1, \varepsilon) \neq \emptyset$ for all $x \in \{z : d(z, x_0) < \delta$ and $z \in G_1(v_0, x_0, -)\}$, where $B(v_1, \varepsilon)$ denotes open ball in X with centre v_1 and radius ε.*

(iii) *for each $v_0, v_1 \in Q_V(x_0)$ and every sequence z_n in $G_1(v_0, x_0, -)$ with $z_n \to x_0$ there exists $v_n \in Q_V(z_n)$ such that $v_n \to v_1$.*

Theorem 5.2. *If V is a non-empty bounded subset of an M-space (X, d) with Property* (I) *such that $Q_V(x)$ is convex for all x then Q_V is ORL-continuous.*

Proof. Let $x_0 \in X$ be arbitrary. We show that Q_V is ORL-continuous at x_0. If $Q_V(x_0) = \emptyset$ then result trivially holds. Let $v_0, v_1 \in Q_V(x_0)$ and $x_n \in G_1(v_0, x_0, -)$ with $d(x_n, v_0) = \lambda_n d(x_0, v_0)$ and $d(x_n, x_0) = (\lambda_n - 1) d(x_0, v_0)$ where $\lambda_n = 1 + \frac{1}{n}$, $n \in N$. Then $x_n \to x_0$. Take $v_n \in G[v_0, v_1]$, $v_n = W\big(v_1, v_0, \frac{1}{\lambda_n}\big)$. Then $v_n \to v_1$. We claim that $v_n \in Q_V(x_n)$. Since $Q_V(x_0)$ is convex, $v_n \in Q_V(x_0)$. Also by Lemma 3.1, $v_0 \in Q_V(x_n)$. Now $d(x_n, v_0) = \lambda_n d(x_0, v_0)$. This implies $\frac{1}{\lambda_n} d(x_n, v_0) = d(x_0, v_0)$ and

$$d(x_n, v_0) = d(x_n, x_0) + d(x_0, v_0) = d(x_n, x_0) + \frac{1}{\lambda_n} d(x_n, v_0) \Rightarrow d(x_n, x_0) = \big(1 - \frac{1}{\lambda_n}\big) d(x_n, v_0).$$

Therefore M-convexity of X gives $x_0 = W\left(x_n, v_0, \frac{1}{\lambda_n}\right)$.

For any $v \in V$, consider

$$\begin{aligned} d(x_n, v) &\leq d(x_n, v_0) \\ &= \lambda_n d(x_0, v_0) \\ &= \lambda_n d(x_0, v_n) \text{ as } v_n \in Q_V(x_0) \\ &= \lambda_n d\left(W\left(x_n, v_0, \frac{1}{\lambda_n}\right), W\left(v_1, v_0, \frac{1}{\lambda_n}\right)\right) \\ &\leq \lambda_n \left[\frac{1}{\lambda_n} d(x_n, v_1)\right] \text{ by Property (I)} \\ &= d(x_n, v_1) \end{aligned}$$

i.e. $d(x_n, v_1) \geq d(x_n, v)$ for all $v \in V$ and so $v_1 \in Q_V(x_n)$. Also $v_0 \in Q_V(x_n)$. As $Q_V(x_n)$ is convex, $v_n \in Q_V(x_n)$. Therefore by Lemma 5.1, Q_V is ORL-continuous at x_0 and so on X.

6 - Inner Radial Lower (IRL) continuity

A second generalization of lower semi continuity is IRL-continuity defined as under.

Let V be a non-empty bounded subset of a convex metric space (X, d) and $x_0 \in X$. Q_V is said to be IRL-continuous at x_0 if for each $v_0 \in Q_V(x_0)$ and each open set W with $Q_V(x_0) \cap W \neq \emptyset$, there exists a neighbourhood U of x_0 such that $Q_V(x) \cap W \neq \emptyset$ for every $x \in U \cap G[v_0, x_0]$. Q_V is called IRL-continuous if it is IRL-continuous at each point of X.

The following lemma, the proof of which is similar to the one given in normed linear spaces for the metric projection P_V(see [3]) and holds for Q_V in convex metric spaces will be used in the proof of our next theorem.

Lemma 6.1. *Let V be a non-empty bounded subset of a convex metric space (X, d) and $x_0 \in X$ then the following are equivalent:*

(i) *Q_V is IRL-continuous at x_0.*

(ii) *for each $v_0, v_1 \in Q_V(x_0)$ and each $\varepsilon > 0$ there exists a $\delta > O$ such that $Q_V(x) \cap B(v_1, \varepsilon) \neq \emptyset$ for all x in $G[v_0, x_0]$ with $d(x, x_0) < \delta$.*

(iii) *for each $v_0, v_1 \in Q_V(x_0)$ and every sequence x_n in $G[v_0, x_0]$ with $x_n \to x_0$, $d(v_1, Q_V(x_n)) \to 0$ i.e. there exists $v_n \in Q_V(x_n)$ such that $v_n \to v_1$.*

Using Lemma 6.1, we prove

Theorem 6.2. *If V is a non-empty bounded subset satisfying property (SF) in a convex metric space (X, d) with Property* (I') *then the farthest point map is IRL-continuous at $x \in X$ if $Q_V(x)$ is convex.*

Proof. If $Q_V(x) = \emptyset$ then result is trivially true. Let $v_0, v_1 \in Q_V(x)$ and $x_n = W(x, v_0, 1 - \lambda_n)$ where $0 < \lambda_n \leq 1$ and $\lambda_n \to 0$. Then by the Property (SF), $v_0 \in Q_V(x_n)$. Let $v_n = W(v_1, v_0, 1 - \lambda_n)$ then $v_n \in Q_V(x)$ as $Q_V(x)$ is convex and $v_n \to v_1$. We claim that $v_n \in Q_V(x_n)$. Consider

$$\begin{aligned} d(x_n, v_n) &= d(W(x, v_0, 1 - \lambda_n), W(v_1, v_0, 1 - \lambda_n)) \\ &= (1 - \lambda_n) d(x, v_1) \text{ by Property } (\mathrm{I}') \\ &= d(x_n, v_0) \\ &= \delta(x_n, V). \end{aligned}$$

Therefore $v_n \in Q_V(x_n)$. Also $v_n \to v_1$ and so by Lemma 6.1, Q_V is IRL-continuous at x.

Remark 6.3. In view of Remark 4.3, the map Q_V is not only IRL-continuous but also continuous.

Acknowledgments. The authors are thankful to the referee for valuable suggestions leading to an improvement of the paper.

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Abstract

Some new continuity concepts, called Outer Radially Lower (ORL), Outer Radially Upper (ORU) and Inner Radially Lower (IRL) for metric projection (nearest point map) and Inner Radially Upper (IRU) for anti metric projection (farthest point map) are known in the theory of nearest and farthest points in normed linear spaces (see e.g. B. Brosowski and F. Deutsch [Bull. Amer. Math. Soc. 78 (1972), 974-978], B. B. Panda and O. P. Kapoor [J. Math. Anal. Appl. 62 (1978), 345-353]). *In this paper we discuss all these concepts for the farthest point map when the underlying spaces are convex metric spaces.*

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