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Asymptotic behaviours of hydromagnetic boundary layer flows past a semi infinite flat plate (**)

1 - Introduction

Hydromagnetic flows have been receiving considerable attention due to their applications in astrophysics because much of the universe is filled with widely spaced charged particles permeated by magnetic fields. Due to this reason, noticeable work has been done on the hydromagnetic flows by Shercliff [**1**], Rossow [**2**], dealing with the various phenomena in hydromagnetic boundary layers. Cobble [**3**] showed the conditions under which a similarity solution exists to hydromagnetic flow over a semi-infinite flat plat in presence of magnetic field and a pressure gradient with or without suction and injection. The heat transfer aspect of hydromagnetic boundary layer flows has been studied by Soundalgekar and Ramanamurthy [**4**].

The study of the asymptotic behaviours of the solutions of equations governing problems of physical significance in boundary layer theory is an interesting aspect of discussion in fluid mechanics. One of the most important problems in the study of differential equations and their applications is that of describing the nature of the solutions for large positive values of the independent variables and this purpose is completely served by the study of the asymptotic behaviours. Thus the asymptotic behaviour pays particular attention towards a desired problem for finding conditions

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under which a solution approaches zero as the independent variable tends to infinity, or is very small for all the independent variables, or is bounded as the independent variable tends to infinity.

As a matter of afore-mentioned facts, the study of the asymptotic nature of the solutions of the Falkner-Skan [**5**] equations governing a steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge was initiated by Hartman [**6**]. Serrin [**7**] also studied the asymptotic behaviours of the velocity profiles in the Prandtl boundary layer equation for the steady two-dimensional laminar flow of an incompressible viscous fluid past a rigid wall. Later, the study of the asymptotic behaviours of the differential equations governing the various flow fields was carried out by Singh [**8**], [**9**], [**19**], Singh and Singh [**10**], [**14**], Singh and Kumar [**15**], Lu et al. [**16**], Harris and Pucci [**17**], Parhi and Das [**18**], [**20**], Singh and Verma [**21**], Tiryaki and Yaman [**22**], etc.

The objective of the present paper is to study the asymptotic behaviours of the solutions, as the similarity independent variable $\eta \to \infty$, of the equations governing the flow of an electrically conducting incompressible viscous fluid in presence of a transverse magnetic field past a semi-infinite flat plate. The asymptotic behaviours of the corresponding heat transfer equation have also been studied. The mathematical analysis of the problem is based on an important method of asymptotic integration of second order linear differential equations. The asymptotic behaviours of the principal and linearly independent solutions have been studied for accelerated ($\beta > 0$) and decelerated flows ($\beta < 0$). The results pertaining to the asymptotic behaviours have been expressed in terms of theorems.

2 - Mathematical analysis

The similarity equations governing the low-speed laminar boundary layer flow of an electrically conducting incompressible viscous fluid with density ρ, specific heat at constant pressure C_p, kinematic viscosity υ in the presence of a transverse magnetic field H_0 past a semi-infinite flat plate are (Soundalgekar and Ramanmurthy [4])

(1) $$f''' + ff'' + \beta(1 - f'^2) + M(1 - f') = 0$$

(2) $$g'' + Pfg' = 0$$

under the boundary conditions

(3) $$f(0) = f'(0) = 0, f'(\infty) = 1$$

(4) $$g(0) = 1, g(\infty) = 0$$

together with the side condition

$$0 < f' < 1 \quad \text{on} \ [0, \infty). \tag{5}$$

Here β, M and P are respectively longitudinal acceleration, magnetic parameter and Prandtl number. Also f' and g are the dimensionless velocity and temperature profiles.

Now, for getting the asymptotic formulae of (1) - (5), we shall base ourselves on the following arguments given in the form of theorems :

Theorem 1. *The autonomous, third order, non-linear differential equation* (1), (3), (5) *will have one and only one solution for* $\beta \geq 0$ *such that* $f'' > 0$ *on* $[0, \infty)$.

Proof. From (1), $f''' + ff'' + [\beta(1 + f') + M](1 - f') = 0$. Since $f' > 0$ and $\beta \geq 0$, hence $\beta(1 + f') + M > M + \beta > 0$. Therefore, the proof of above theorem is similar to the proof of Theorem 6.1 (Hartman [**23**], p. 520).

Theorem 2. *Let* β *be fixed and* $\beta < 0$ *such that* $M + \beta < 0$. *Then there exists a number* $A = A(\beta, \beta')$ *and a continuous increasing function* $\gamma(a)$ *defined for* $a \geq A$ *with the properties that* $\gamma(A) = 0$ *and that if* $f(\eta)$ *is the solution of* (1), (3), (5), *then* $0 \leq f''(0) \leq \gamma(a)$, *such that* $f''(\eta) > 0$ *on* $[0, \infty)$, *where* $f(0) = a = 0, f'(0) = \beta' = 0$.

Proof. From (1), $f''' + ff'' + [\beta(1 + f') + M](1 - f') = 0$. Since $f' > 0$, $\beta < 0$ and $M + \beta < 0$, hence $\beta(1 + f') + M < 0$. Therefore, the proof of this theorem is similar to the proof of Theorem 7.1 (Hartman [**23**], p. 525). Here it may be stressed that while Theorem 1 provides sufficient conditions for the existence and uniqueness of the solutions, Theorem 2 gives necessary conditions for its existence.

3 - Asymptotic behaviour

If $f(\eta)$ is the solution of (1), let us put

$$h(\eta) = 1 - f'(\eta). \tag{6}$$

Then $h(\eta)$ satisfies the differential equation

$$h'' + fh' - h[(1 + f')\beta + M] = 0. \tag{7}$$

In order to eliminate the middle term in (7), let us put

$$h = x \exp\left(-\frac{1}{2}\int_0^\eta f(s)ds\right), \tag{8}$$

so that $x(\eta)$ satisfies

(9) $$x'' - q(\eta)x = 0,$$

where

(10) $$q(\eta) = \beta + M + \left(\beta + \frac{1}{2}\right)f' + \frac{1}{4}f^2 = \frac{1}{4}f^2\left[1 + \frac{4(\beta + M)}{f^2} + \frac{2(2\beta+1)f'}{f^2}\right].$$

From (10),

$$q'(\eta) = \left(\beta + \frac{1}{2}\right)f'' + \frac{1}{2}ff';$$

$$q''(\eta) = -\left(\beta + \frac{1}{2}\right)(M + \beta) + M\left(\beta + \frac{1}{2}\right)f' + \left\{\beta\left(\beta + \frac{1}{2}\right) + \frac{1}{2}\right\}f'^2 - \beta ff''.$$

Since $0 < f' < 1, f'' > 0$ and $f' \sim 1, f \sim \eta$ as $\eta \to \infty$, hence for large η a constant K can be so chosen that

$$\frac{q'^2}{q^{5/2}} \leq K\left[\frac{f''^2}{\eta^5} + \frac{f''}{\eta^4} + \frac{1}{\eta^3}\right]$$

$$\frac{|q''|}{q^{3/2}} \leq K\left[\frac{f''}{\eta^2} + + \frac{1}{\eta^3}\right].$$

In addition, $\int^{\infty} f'' d\eta$ is absolutely convergent, since $f'(\eta) \to 1$ as $\eta \to \infty$, so that

(11) $$\int^{\infty} \frac{q'^2 d\eta}{q^{5/2}} < \infty \quad \text{and} \quad \int^{\infty} \frac{|q''| d\eta}{q^{3/2}} < \infty,$$

provided that

(12) $$\int^{\infty} \frac{f''^2 d\eta}{\eta^5} < \infty;$$

(12′) $$\int^{\infty} \frac{f'' d\eta}{\eta^4} < \infty.$$

The validity of (12) can be proved by integration by parts by taking $\frac{f''}{\eta^5}$ and f'' as the first and second functions, so that

$$\int \frac{f''^2 d\eta}{\eta^5} = \frac{f'f''}{\eta^5} + \int \frac{f'}{\eta^5}\left[\beta(1 - f'^2) + M(1 - f') + \frac{5f''}{\eta}\right] d\eta$$

using (1). The last integral is absolutely convergent and $\liminf f'' = 0$ as $\eta \to \infty$.

Thus, (12) holds. On similar lines, the integral (12') will be absolutely convergent and hence the validity of (12') can also be established. Consequently, (11) holds.

Moreover, $q(\eta)$ is a continuous and positive function on $0 \leq \eta < \infty$, hence (9) has the principal solution $x(\eta)$ satisfying, as $\eta \to \infty$,

$$x \sim K_1 q^{-1/4} \exp\left(-\int^{\eta} q^{1/2}(s)ds \right), \tag{13}$$

where $K_1 \neq 0$ is a constant, while linearly independent solutions satisfy

$$x \sim K_1 q^{-1/4} \exp\left(\int^{\eta} q^{1/2}(s)ds \right),\ K_1 \neq 0; \tag{14}$$

cf. Exercise XI 9.6 (Hartman [**23**], p. 382). From the last part of (10) and $f \sim \eta$;

$$q^{1/2}(\eta) = \frac{1}{2}f + \left(\beta + \frac{1}{2}\right)\frac{f'}{f} + \frac{M+\beta}{f} + O\left(\frac{1}{\eta^3}\right),\ q^{-1/4}(\eta) \sim \left\{\frac{1}{2}\eta\right\}^{-1/2};$$

hence $\int^{\eta} q^{1/2}(\eta)d\eta = \frac{1}{2}\int^{\eta} f d\eta + \left(\beta + \frac{1}{2}\right)\log f + (M+\beta)\int \frac{d\eta}{f} + c^0 + o(1)$, where c^0 is a constant.

Therefore (13) and (14) become

$$x \sim K_1 \eta^{-\beta-1} \exp\left(-\int^{\eta} \left[\frac{1}{2}f(s) + \frac{M+\beta}{f(s)}\right] ds \right); \tag{15}$$

$$x \sim K_1 \eta^{\beta} \exp\left(+\int^{\eta} \left[\frac{1}{2}f(s) + \frac{M+\beta}{f(s)}\right] ds \right). \tag{16}$$

In view of (8), the equation (7) has the principal solution satisfying

$$h \sim K_1 \eta^{-\beta-1} \exp\left(-\int^{\eta} \left[f(s) + \frac{M+\beta}{f(s)}\right] ds \right),\ K_1 \neq 0, \tag{17}$$

while the linearly independent solutions satisfy

$$h \sim K_1 \eta^{\beta} \exp\left(-\int^{\eta} \left[\frac{M+\beta}{f(s)}\right] ds \right),\ K_1 \neq 0, \tag{18}$$

as $\eta \to \infty$.

Differentiating (7), one gets

$$h''' + fh'' + [(1-2\beta)f' - M]\,h' = 0. \tag{19}$$

Treating (19) as the second order differential equation in h' in the same way that (7) was handled, (19) has the principal solution satisfying

$$h' = K_1'\eta^{-2\beta}\exp\left(-\int^{\eta}\left[f(s)+\frac{M}{f(s)}\right]ds\right),\ K_1' \neq 0, \tag{20}$$

and that the linearly independent solutions satisfy

$$h' = K_1'\eta^{2\beta-1}\exp\left(+\int^{\eta}\left[\frac{M}{f(s)}\right]ds\right),\ K_1' \neq 0, \tag{21}$$

as $\eta \to \infty$.

If (6) satisfies (17), then, since $f(\eta) \sim \eta$, it follows that

$$\int^{\infty}\eta h(\eta)d\eta < \infty;$$

thus

$$f(\eta) = \eta + K_2 + o(1),\ \int^{\eta} f d\eta = \frac{\eta^2}{2} + K_2\eta + K_3 + o(1)$$

as $\eta \to \infty$.

Substituting this in (17), (20) gives

$$1 - f' \sim K_0\eta^{-M-\beta-1}\exp\left(-\frac{\eta^2}{2} - K_2\eta\right); f'' \sim \eta(1-f'), \tag{22}$$

as $\eta \to \infty$, where $K_0 > 0, K_2$ are the constants.

If (6) satisfies (18), then $f \sim \eta$ implies that $h \equiv 1 - f' \sim K_1\eta^{M+2\beta+o(1)}$ as $\eta \to \infty$. Hence $f(\eta) = \eta + O(\eta^{M+2\beta-1+\in})$ as $\eta \to \infty$ for all $\in > 0$.
If this is substituted into (18), (21) and if it is supposed that $M + \beta < 0$ (and $M + 2\beta + \in < 0$), then

$$1 - f' \sim K_0\eta^{M+2\beta},\ f'' \sim -(M+2\beta)K_0\eta^{M+2\beta-1} \tag{23}$$

as $\eta \to \infty$, where $K_0 > 0$ is constant.

Finally, for the equations (2), (4), (5), the principal solutions satisfy

$$g \sim K_0'\exp\left(-P\left\{\frac{\eta^2}{2} + K_2\eta\right\}\right);\ g' \sim \eta^{3/2}g, \tag{24}$$

as $\eta \to \infty$, where $K_0' > 0$ is constant, while the linearly independent solutions satisfy

$$g \sim K_0' \eta^{1/2}; \; g' \sim \eta^{-3/2}, \tag{25}$$

as $\eta \to \infty$.

4 - Results

The results pertaining to the asymptotic behaviours of the principal and linearly independent solutions of (1) - (5) can be put in the form of following theorems:

Theorem 3. *For $\beta \geq 0$, there exist constants $K_0 > 0, K_2$ such that* (22) *holds as $\eta \to \infty$.*

Proof. For a given $f(\eta)$, it has to be decided whether $h = 1 - f'$ satisfies (17),(20) or (18),(21). If $\beta \geq 0$, (18) can not hold, for otherwise $h = 1 - f' \to 0$ as $\eta \to \infty$ fails to hold. Thus (17), (20) are valid as $\eta \to \infty$. As was seen, (17) and (20) give (22). Hence (22) holds as $\eta \to \infty$.

Theorem 4. *There exist constants $K_0' > 0, K_2$ such that* (24) *holds as $\eta \to \infty$.*

The proof is similar to that of Theorem 3.

Theorem 5. *Let $M + \beta < 0, a \geq A(\beta, \beta')$ where $A(\beta, \beta'), \gamma(a)$ are given by Theorem* 2. *Let $f(\eta)$ be a solution of* (1), (3), (5). *Then there exist constants $K_0 > 0, K_2$ such that* (22) *holds iff $f''(0) = \gamma(a)$; for other solutions $f(\eta)$ of* (1),(3),(5) *with $a > A(\beta, \beta')$ and $0 \leq f''(0) < \gamma(a)$, the asymptotic relations* (23) *hold with a suitable constant $K_0 > 0$.*

Proof. If $f^*(\eta)$ is the solution of (1), (3) and $f^{*''}(0) = \gamma(a)$, then (22) holds. Let $z^*(f) = f^2 > 0$ and let $\gamma(f)$ be a solution of Weber's equation $\upsilon + f\upsilon - 2c\upsilon = 0$ where $\dot{\upsilon} = \dfrac{d\upsilon}{df}$, satisfying $\dfrac{\upsilon}{\upsilon} \sim -f$ as $f \to \infty$, and $\gamma(f) > 0$ for large f.

Let $r^*(f) = -\dfrac{z^*}{1 - z^*}$, and $\tau(f) = \dfrac{\dot{\upsilon}}{\upsilon}$, where τ denotes the logarithmic derivative of a non-trivial solution $\tau = \dfrac{\dot{\upsilon}}{\upsilon}$; the corresponding Riccati equation being

$$\dot{\tau} = -\tau^2 + (2c - f\tau).$$

Then, for large $f, r^*(f) \leq \tau(f)$. For suppose that $r^*(f) > \tau(f)$ for some large $f = f_0$.

In this case, $r(f) > \tau(f)$ for $f = f_0$ if $z(f) = -\dfrac{\dot{z}}{1-z}$ belong to a solution of (1), (3) with $f''(0) = \gamma(a) + \varepsilon$ for small $|\varepsilon|$. But then, it follows that $r(f) > \tau(f)$ for all $f \geq f_0$ and that $f(\eta)$ satisfies (1),(3),(5). This contradicts the main property of $\gamma(a)$.

Hence $r^*(f) \leq \tau(f)$ for large f, and so $1 - z^*(f) \leq c^* \upsilon(f)$ for large f and some constant $c^* > 0$. Since $\upsilon(f) \sim -\dfrac{1}{2} f^2$ as $f \to \infty$, it follows that $h = 1 - f^{*'}$ can not satisfy (23) and therefore (18). This gives the conclusion that (22) will hold as $\eta \to \infty$, thereby proving the theorem.

5 - Concluding remarks

The asymptotic integration method to find out the solutions of non-linear boundary layer equations is the corner-stone of Applied Mathematics. This is a method to find the approximate solutions for velocity profiles for very large values of the independent variables. One of the other corner-stones of Applied Mathematics is scientific computing and it is interesting to note that these two subjects have grown together. However, this is not unexpected given their respective capabilities. By using computers, one is capable of solving problems that are non-linear, non-homogeneous and multi-dimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that the computer solutions do not provide much insight into the physics of the problem, particularly for those who do not have access to the appropriate software or computer and there is always a question as to whether or not the computed solution is correct. So, the main objective behind the use of the asymptotic integration method, at least as far as the author is concerned, is to provide reasonably accurate expression for the solution for large values of η. By doing this, one is able to derive an understanding of the physics of the problem. Also, one can use the result in conjunction with the original problem, to obtain the more efficient numerical procedures for computing the solution.

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Abstract

This paper deals with the asymptotic behaviours of the solutions, as the similarity independent variable tends to infinity, of equations governing the laminar flow of an incompressible viscous fluid in presence of a transverse magnetic field past a semi-infinite flat plate; the entire mathematical analysis being based on the asymptotic integrations of second order linear differential equations. The results pertaining to the asymptotic behaviours of the principal and linearly independent solutions have been expressed in terms of theorems.

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