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A variant of the pinwheel tiling (**)

0 - Introduction

Pinwheel tiling is the name that C. Radin [**8**] gave to a certain tesselation of the plane given by John Conway. This tiling has the particular feature that the tiles are rotated in an infinite number of ways. In [**3**] C. Bandt gave a general way to construct families of self-similar sets (usually fractal) which can be used to tile R^n, without the necessity of checking the so called open set condition. In some sense, the underlying structure of Bandt's construction is the existence of a periodic tiling. This was later extended by Gelbrich, see [**4**], [**5**]. We show in [**9**] a way to generate graph-directed sets, usually fractal, with tiling properties where the underlying structure may be a non-periodic tiling. The pinwheel tiling does not fulfill the hypotheses of the theorems of [**9**]. Nevertheless in the present paper we show that one may generate other nice tiles (using Conway's tesselation as the underlying structure) which tile the plane in the same sense as the pinwheel tiling does i.e. the tiles appear rotated in an infinite number of ways. Also our fractal tiles are generated as a graph-directed iterated function system but they seem to be not disk-like. It will be clear from the context and the conclusions that many other fractal tiles may be constructed by modifying our procedure.

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1 - Definitions and results

A **tiling** is a family of measurable sets T_i, $i = 1, 2, 3, ...$ of R^2 such that:

a) $\cup_i T_i = R^2$ and $Area(T_i \cap T_j) = 0$ if $i \neq j$ where $Area(S)$ is the Lebesgue measure of a set S of R^2,

b) each T_i coincides after a rigid motion (orientation-preserving euclidean isometry) of the plane with some K_i, $i = 1, ..., n$ (so we have only a finite family of different tiles).

The pinwheel tiling is generated by two tiles: the triangles K_i', $i = 1, 2$ with K_2' obtained from K_1' by reflection. Triangle K_1' is the right angle triangle ABC of figure 2 whose sides have lengh 1, $1/\sqrt{5}$, $2/\sqrt{5}$. We recall Conway's tesselation and refer to [8] for details. Take the triangle ABC and dissect into five equal triangles K_τ, K_υ, $K_\varphi = DEF$, K_χ, K_ψ as it is shown in figure 2 (call this the dissection procedure) and send the points D, E, F to points A, B, C by an expansion, that is, a map $\varphi^{-1}(z) = \lambda exp(i\theta)z + z_0$ with $\lambda = \sqrt{5}$, θ real. For K_2' the dissection procedure is just the reflection of figure 2. Now apply the dissection procedure to the five remaining triangles and then apply again the same expansion. If one repeats this procedure ad infinitum, we obtain Conway's tiling. One must notice that there exist triangles rotated in an angle $n\theta$ for any n natural number. Because $\theta = 2\pi a$ where a is an irrational number, the triangles are rotated in an infinite number of ways, see [8].

By a contraction (respectively reversing contraction) we mean a map $\lambda exp(i\theta)z + z_0$ (respectively $\lambda exp(i\theta)\bar{z} + z_0$) with $0 \leqslant \lambda < 1$, θ real.

We want to give an ad-hoc construction of a non-periodic tiling of the plane by compact sets K_i with non void interior, $i = 1, ..., 40$. We recall that a periodic tiling is one which has in its symmetry group at least two translations in non-parallel directions. The sets K_i have the property of being generated uniquely by a transitive graph-directed iterated function system (see [**1**], [**7**]). More precisely

Theorem. *There exists a non-periodic tiling of the plane given by sets* K_i, $i = 1, ..., 40$ *where* K_i, $i = 1, ..., 20$ *are the unique compact sets determined by the equations*

$$K_1 = \tau(K_2) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_5)$$

$$K_2 = \omega(K_6) \cup \upsilon(K_7) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{10})$$

$$K_3 = \tau(K_{11}) \cup \upsilon(K_3) \cup \varphi(K_{12}) \cup \chi(K_{13}) \cup \psi(K_{14})$$

$$K_4 = \omega(K_{15}) \cup \upsilon(K_7) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{16})$$

$$K_5 = \omega(K_{17}) \cup \upsilon(K_7) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{16})$$

$$K_6 = \omega(K_{15}) \cup \upsilon(K_7) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{10})$$

$$K_7 = \tau(K_{10}) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{14})$$

$$K_8 = \tau(K_{11}) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_5)$$

$$K_9 = \omega(K_{18}) \cup \upsilon(K_7) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{16})$$

$$K_{10} = \omega(K_7) \cup \upsilon(K_7) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{10})$$

(1)

$$K_{11} = \omega(K_{17}) \cup \upsilon(K_7) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{10})$$

$$K_{12} = \tau(K_{10}) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_5)$$

$$K_{13} = \tau(K_{10}) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{19})$$

$$K_{14} = \omega(K_7) \cup \upsilon(K_7) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{16})$$

$$K_{15} = \omega(K_{20}) \cup \upsilon(K_7) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{10})$$

$$K_{16} = \tau(K_{16}) \cup \upsilon(K_3) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{14})$$

$$K_{17} = \omega(K_{18}) \cup \upsilon(K_7) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{10})$$

$$K_{18} = \tau(K_{11}) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{14})$$

$$K_{19} = \tau(K_{16}) \cup \upsilon(K_3) \cup \varphi(K_8) \cup \chi(K_9) \cup \psi(K_{19})$$

$$K_{20} = \tau(K_2) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_{14})$$

where $\tau, \upsilon, \varphi, \chi, \psi, \omega$ *are the contractions (or reversing contractions) that send the triangle ABC to the triangles* K_τ*,* K_υ*,* K_φ*,* K_χ*,* K_ψ*,* K_ω *shown in figure 2. The sets* K_i*,* $i = 21, ..., 40$ *are obtained reflecting* K_i*,* $i = 1, ..., 20$*.*

In the tiling the sets T_j *corresponding to any fixed tile* K_{i_0} *are rotated in an infinite number of ways.*

The sets K_i, $i = 1, ..., 20$ are shown in figure 1.

Proof. Conway's tiling has sixteen types of vertex neighbourhoods as shown in figure 3. This can be verified applying the dissection procedure to each triangle in a

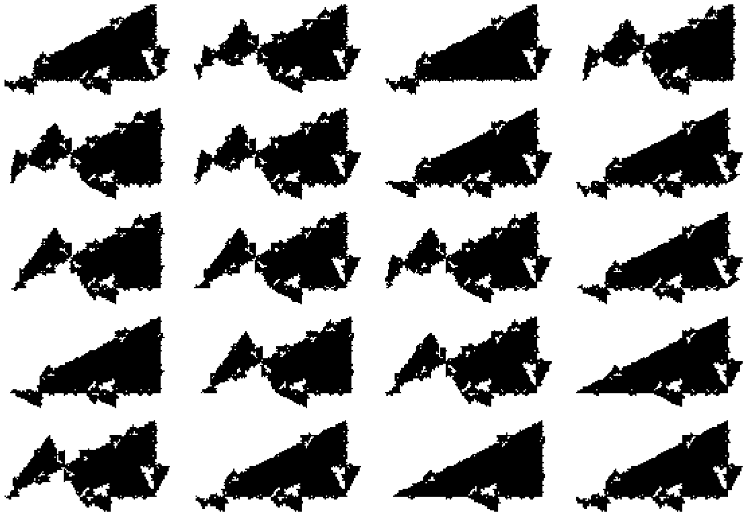

Fig. 1

vertex neighbourhood, obtaining a configuration of triangles which has as vertex neighbourhood some of the sixteen types of figure 3.

By construction the largest side of a triangle in Conway's tiling belongs to two triangles of the tiling and they are in contact in the ways a), b) shown in figure 3 or a reflection of a). We construct our tiling in the following way.

STEP 1. We 'paint' each triangle T_i' of Conway's tiling (which after a rigid motion coincides with K_j', $j = 1, 2$) with a color i.

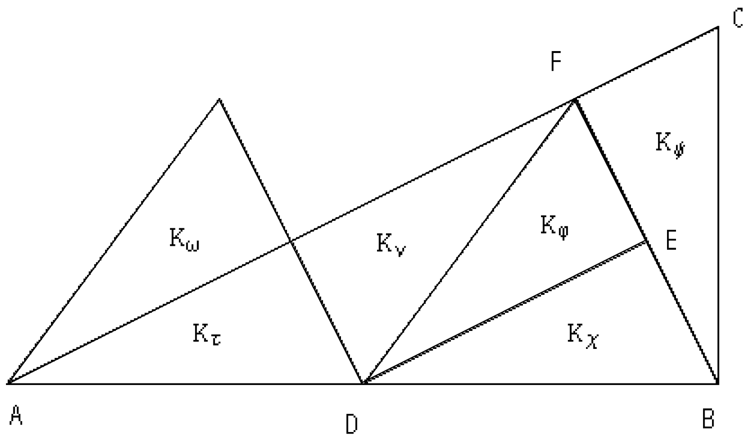

Fig. 2

STEP 2. To each triangle T'_i we apply the dissection procedure obtaining five equal triangles $T'_{i,j}$ $j = 1, ..., 5$, where $T'_{i,1}$ is the triangle touching the vertex with smallest angle of T'_i. We paint these triangles with the following rule: if in the tiling of the plane by the triangles T'_i, $i = 1, 2, ...$ the triangle T'_{i_0} touches T'_{i_1} in the way shown in figure 3 a) (or a reflection of figure 3 a) then we assign to $T'_{i_0,j}$ $j = 1, 2, 3, 4, 5$ the color i_0 and to $T'_{i_1,j}, j = 1, 2, 3, 4, 5$ the color i_1 (that is we keep the color); otherwise if T'_{i_0} touches T'_{i_1} as in figure 3 b) then we assign to the triangle $T'_{i_0,1}$ the color i_1, to $T'_{i_1,1}$ the color i_0, to the triangles $T'_{i_0,j}, j = 2, 3, 4, 5$ the color i_0 and to the triangles $T'_{i_1,j}, j = 2, 3, 4, 5$ the color i_1.

It is easy to see that this rule assigns to each triangle $T'_{i,j}$ one and only one color.

STEP n. In step $n - 1$ we have triangles $T'_{i_1,...,i_{n-1}}$ painted with some color. We subdivide this triangle into five smaller triangles $T'_{i_1,...,i_{n-1},j}$ $j = 1, ..., 5$ and use the procedure of step 2 to paint them.

The reader should notice that if we define $C_i(j)$ as the set painted with a color i in STEP j then $Area(C_i(j)) = Area(C_i(k)) = Area(C_k(j))$ for any i, j, k.

Also it is easy to see that for fixed i the sets $C_i(j)$ tend in the Hausdorff metric to a compact set T_i as j tends to infinity.

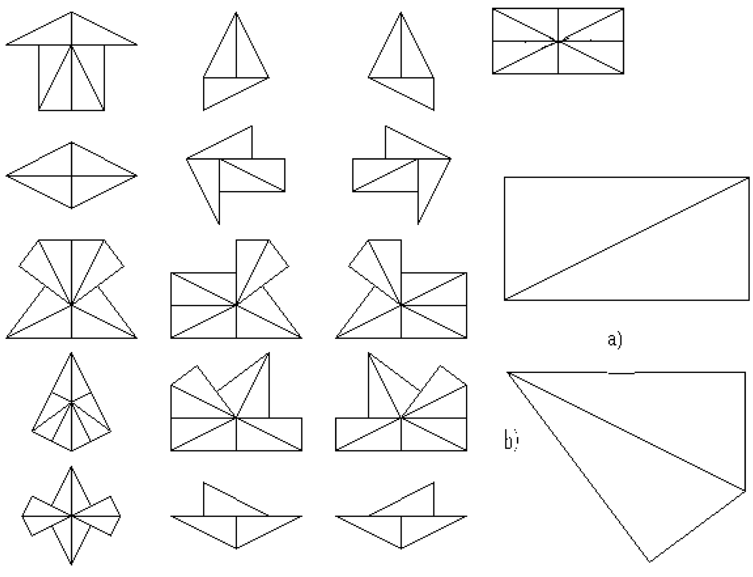

Fig. 3

To prove that T_i is a tiling of the plane we first prove $\cup_i\ T_i = R^2$.

In fact, for fixed j the sets $C_i(j)$, $i = 1, 2, 3, ...$ form a tiling of the plane and by a limit argument T_i, $i = 1, 2, ...$ cover the plane. By the Baire Category theorem at least one T_{i_0} has non void interior. Also by the area property of

$C_i(j)$, the T_i must have positive Lebesgue measure. In fact, using Fatou's lemma or the regularity of Lebesgue measure we conclude that $1/5 \leqslant C_i(j) \leqslant Area(T_i)$.

Next we shall see that, each T_i is, after a rigid motion, equal to some K_i, $i = 1, ..., 40$ where the K_i, $i = 1, ..., 20$ are generated by (1) and K_i, $i = 21, ..., 40$ are the reflections of K_i, $i = 1, ..., 20$. As (1) is a transitive graph-directed iterated function system it will follow that all the T_i have non void interior, see [**1**].

To prove the above assertion one notices that the 'shape' of T_i depends by construction on the triangles T'_j whose distance to T'_i is not more than a certain fixed number (a patch around T'_i). As we have only sixteen vertex neighbourhoods the number of such patches is finite and so the number of 'shapes' of the T_i is finite i.e. we have only a finite number of tiles K_i. Let P'_i be the patch corresponding to triangle T'_i. Reordering, if necessary, we may think that the P'_i, $i = 1, ..., n$ are different and any other patch P'_j is after rigid motion equal to some of them. We sketch the argument for T_1. Recall that the triangle T'_1 is painted with color 1. Take all indexes i, j (corresponding to STEP 2) such that $T'_{i,j}$ is painted with color 1. Each triangle $T'_{i,j}$ has a patch (in its own size) which is, after some expansion, equal to a patch P'_k corresponding to the triangle T'_k. This implies that $T_1 = \cup_k \psi_k(T_k)$, where $\psi_k(z)$ is the unique contraction such that $\psi_k(T'_k) = T'_{i,j}$. Basically, it is in this way that system (1) is obtained.

To carry out this general idea in order to obtain (1) we proceed as follows.

Fix a triangle as ABC. It is not difficult to prove that its patch contains only those triangles (of the same size as ABC) touching vertex A (the most acute angle) and B (right angle).

From figure 3 it is seen that there are sixteen types of vertex neighbourhoods for the vertex of type A (the most acute angle) which we call a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p and seven types of vertex neighbourhoods for the vertex of type B (right angle), called a, b, c, d, e, f, g. See figure 4.

Some observations are in order. First recall that the hypotenuse of a triangle in Conway's tiling always touches another hypotenuse i.e. there are no vertexes of triangles on it.

Second after subdividing the triangle ABC of figure 2 the only triangle which may not keep its color is the triangle K_τ. Also for such a subtriangle not to keep its color it is necessary that the adyacent angle of the triangle touching the hypotenuse should be also an acute angle as A. In figure 4 we have painted in dark the triangles which constitute a 'barrier' for the change of color i.e. the patch may not contain the triangles beyond them.

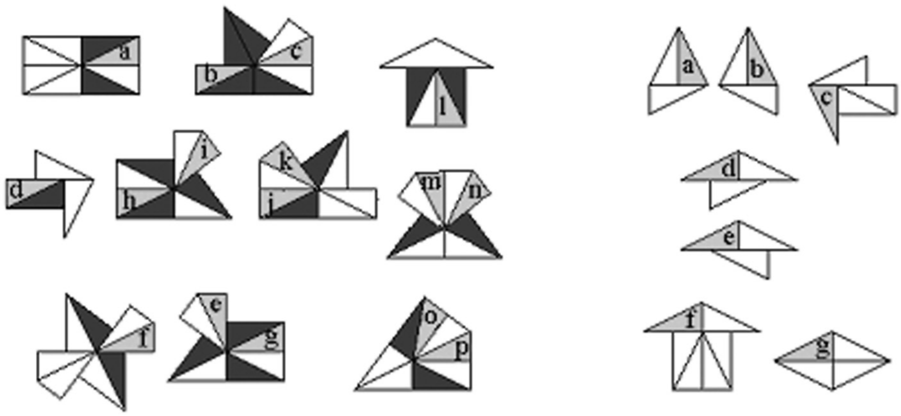

Fig. 4

Precisely, one can consider a, g, h, l as the same patch for the vertex A and also i, e, f as the same patch (again for the vertex A). So we are really left with 11 types of vertex neighbourhoods for the vertex of type A. Let us write ab for that configuration of triangles such that the acute vertex is of type a and its right angle vertex is of type b. Of the possible 11.7=77 combinations, many do not exist due to overlappings (as for example dg or ac) and others like ma do not exist because a right angle in a vertex would appear in a nonexisitng combination with other non-right angles. So an easy check shows that only the 25 types of patches shown in figure 5 are possible. By subdividing the patch *ke* one sees that at the boundary of the triangle this is as the patch *kf* and so generates the same tile. The same happens with the pairs *bf* and *be*, *ee* and *ef*, *nf* and *ne*. A detailed inspection shows that the patch nc does not exist. In fact, the colored triangle in nc could only be obtained in Conway's tiling as K_ψ by the dissection procedure of a 'big' triangle K a reflection of ABC of figure 2. But then the vertex neighbourhood at the right angle of K is none of those shown in figure 3. Now let us observe that the remaining 20 configurations of figure 5 (and their reflected) **do** really appear. Figure 5 also shows correspondence between patches P'_i and tiles K_i.

From this (1) is obtained. For example: figure 6 shows the configuration ab, where our main triangle is shadowed. We have applied the dissection and for $K_\tau, K_\upsilon, K_\varphi, K_\chi, K_\psi$ corresponds the patches $P'_{10}, P'_3, P'_1, P'_4, P'_5$ respectively, giving that

$$K_{12} = \tau(K_{10}) \cup \upsilon(K_3) \cup \varphi(K_1) \cup \chi(K_4) \cup \psi(K_5).$$

One easily checks that the graph directed system (1) is transitive (see [**1**]) i.e. iteration of (1) gives that any set K_i contains a contraction of **any** set $K_j, j = 1, ..., 40$. Therefore K_i has non void interior.

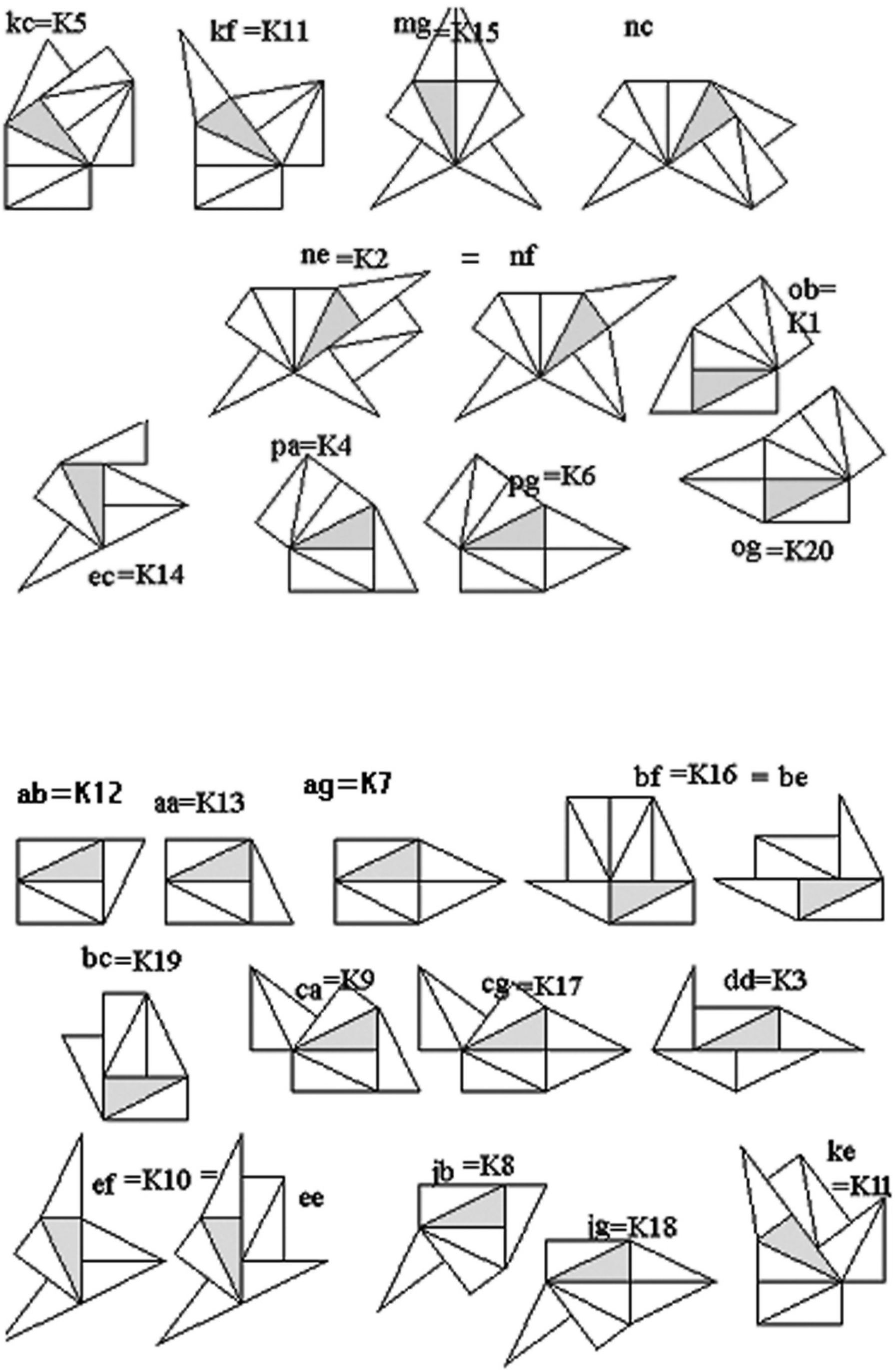
kc=K5
kf =K11
mg=K15
nc
ne =K2
= nf
ob=
K1
pa=K4
pg=K6
ec=K14
og=K20
ab=K12
aa=K13
ag=K7
bf =K16 = be
bc=K19
ca=K9
cg=K17
dd=K3
ef =K10 =
ee
jb =K8
jg=K18
ke
=K11

Fig. 5

Finally we prove that $Area(T_i \cap T_j) = 0$ if $i \neq j$.

If one writes (1) as $K_j = \bigcup_{i=1}^{5} \phi_i^j(K_{j_i}), j = 1, ..., 20$, where $\phi_i^j(z)$ is $\tau(z)$, $\omega(z)$, $\upsilon(z)$, $\varphi(z)$, $\chi(z)$ or $\psi(z)$ then

$$Area(K_j) \leqslant \sum_{i=1}^{5} Area(\phi_i^j(K_{j_i})) \tag{2}$$

for $j = 1, ..., 20$.

If $Area(K_j) = max_i\ Area(K_i)$, then because of (2) and the transitivity of (1), we have $Area(K_j) = Area(K_i)$, for all i. Therefore in (2) the equality sign holds and

$$Area(\phi_i^j(K_{j_i}) \cap \phi_m^j(K_{j_m})) = 0, \ \ for \ \ m \neq i. \tag{3}$$

In particular, the decomposition of the subtile generated by the triangle DEF in five contractions $K_{j_1}, ..., K_{j_5}$ verifies (3). If we continue this decomposition until step N, N great enough, and apply the similarity $\varphi^{-(N+1)}$ (φ the contraction in (1) sending ABC to DEF) we reach 5^{N+1} tiles T_h. For them it holds that $Area(T_i \cap T_j) = 0$. As $N \to \infty$ this holds for all i, j.

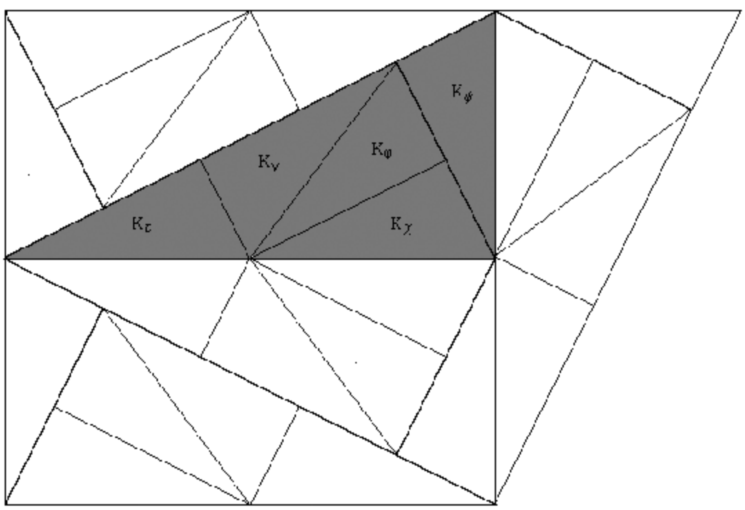

Fig. 6

R e m a r k 1. We have painted the triangles using a certain procedure. This, of course could be modified, and certainly other nice sets could be obtained. But other procedures seem to increase the number of tiles.

Remark 2. Some of the above tiles could be used to give periodic tilings of the plane as the figure 7 shows (this is the tile K_7 and its reflected tile). This is due to the fact that a periodic tiling of the plane is obtained with the patches ag and its reflected.

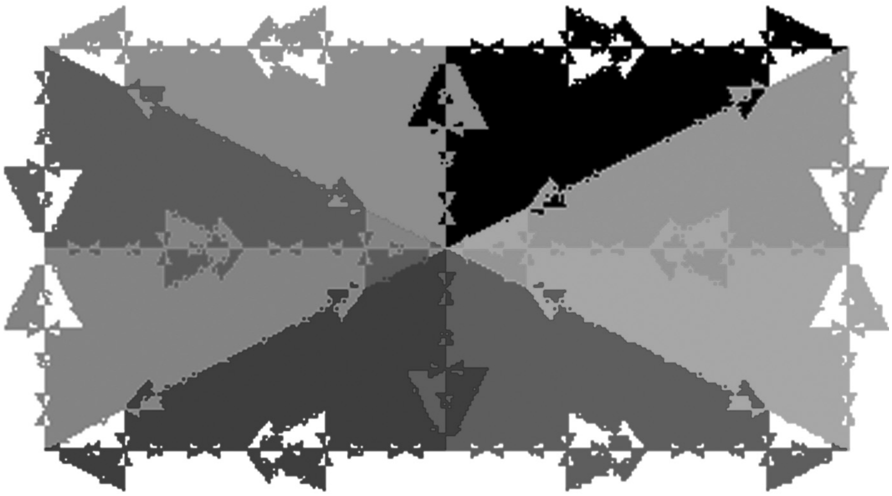

Fig. 7

C. Radin proved in [8] the following striking property: one can impose in the original triangles of the pinwheel tiling certain matching conditions so that any tiling with these tiles one must have the triangles rotated in an infinite number of ways (because of K_7 our tiles do not have this property). Radin needed a lot of different copies of the prototiles to implement his matching rules.

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Abstract

Using the structure of the pinwheel tiling we give an ad-hoc construction of a fractal finite family of tiles which tile the plane in a non-periodic way and appear rotated in an infinite number of ways, as the pinwheel does. Our tiles are generated by a graph-directed iterated function system. It will be clear from the context that many other constructions of such tilings are possible.

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