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VALERIO TALAMANCA (*)

Prodromes for a theory of heights on non-commutative separable K-algebras (**)

Introduction

Let $\boldsymbol{X}$ be a finite dimensional vector space over a number field K. If we want to define an height function on $\boldsymbol{X}$ we need to endow $\boldsymbol{X}$ with some additional structure, a common choice is that of a basis for $\boldsymbol{X}$. If $\underline{b}$ is a basis for $\boldsymbol{X}$ then we get a height function $H_{\underline{b}}$ by composing the isomorphism of $\boldsymbol{X}$ to K^n associated to $\underline{b}$ with the standard Weil height on K^n. Since the Weil height is invariant under permutations of the coordinates, any permutation of the elements of the basis does not change $H_{\underline{b}}$. In fact $H_{\underline{b}}$ only depends on the algebra structure induced by the choice of the basis $\underline{b}$ on $\boldsymbol{X}$. Our original motivation for this research is a sort of converse to the above fact: is there a way to single out one height function on $\boldsymbol{X}$ having special properties with respect to the given K-algebra structure? To make our question and result precise we have to specify which class of height functions we are considering. We will work with heights defined by *adelic norms*. The precise definition of an adelic norm is given in section 1; for the moment we can briefly describe it as a family of local norms satisfying some compatibility condition. What is important is that we can recover all the heights frequently used in the literature as heights defined by an adelic norm, see the examples in section 1. Some

(*) Dipartimento di Matematica, Università «Roma Tre», Largo San Leonardo Murialdo 1, 00146 Roma, Italy; e-mail: valerio@matrm3.mat.uniroma3.it

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results regarding the above question for the case of separable commutative K-algebras were proven in [Ta1] and [Ta3]. One of the goals of this paper is to simplify and refine some of our previous arguments; the final result that we prove here, building upon our previous results, can be stated as follows:

Theorem A. *Let A be a commutative finite-dimensional separable K-algebra. Let $\mathcal{H}(A)$ denote the set of height functions on A that are defined by adelic norms. There exists a unique $H_{A/K} \in \mathcal{H}(A)$ such that:*

$$\textbf{(M1)} \qquad H_{A/K}(a^k) = H_{A/K}(a)^k$$

for all $a \in A$ and all $k \geqslant 1$. Moreover $H_{A/K}$ enjoys the following properties:

(M2) *Given $H \in \mathcal{H}(A)$ we have*

$$\lim_{n \to \infty} H(a^n)^{\frac{1}{n}} = H_{A/K}(a) \quad \textit{for all } a \in A .$$

(M3) $H_{A/K}$ is invariant under $\mathrm{Aut}_{K-alg}(A)$

(M4) *If $T : A \to A$ is a height preserving K-linear map such that $H(T(1_A)) = 1_A$. Then $T \in \mathrm{Aut}_{K-alg}(A)$.*

We call $H_{A/K}$ the **canonical height** on A. The construction of a height function satisfying **(M1)** and **(M3)** was carried out in [Ta1]. Properties **(M2)** and **(M4)** were established in [Ta3], but **(M2)** was proven only for those heights having an additional compatibility property with respect to the algebra structure of A (see section 2). In this paper we prove that **(M2)** holds for every $H \in \mathcal{H}(A)$ (see Theorem 2.1). Clearly **(M2)** implies the uniqueness of $H_{A/K}$.

Let us now turn to the case of non-commutative separable K-algebras which, after all, is our main concern in this work. Unfortunately the situation is not as good. In fact we show that an adelic norm whose associated height satisfies **(M1)** does not exists, see the remark after Proposition 3.1. But not all is lost, in fact we can still define a function $H_{A/K}$ for which **(M1)**, **(M2)** and **(M3)** holds. The definition of $H_{A/K}$ relies upon a spectral interpretation of the canonical height in the commutative case. The precise definition of $H_{A/K}$ and the proof of the desired properties are in section 3.

We would like to point out that a theory of heights in a non-commutative setting has also been recently developed by C. Liebendöerfer in [Li1] and [Li2]. The two theories are rather different in spirit and purpose. In this work we are concern on height theory on non-commutative algebras over a number field, while C. Liebendöerfer constructs and studies heights on vector space over positive definite rational quaternion algebras, her interest lying in the study of «small» solution of homogenous linear equation over such non-commutative division algebras.

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1 - Adelic norms and heights on vector spaces

This section contains definitions and results about heights defined by adelic norms on a finite dimensional K-vector space that are needed in the sequel. Let K be a number field of degree d over $\mathbb{Q}$.

Let $\mathcal{M}_K$ be the set of places of K. We denote by $\mathcal{M}_K^0$ (respectively $\mathcal{M}_K^\infty$) the subset of $\mathcal{M}_K$ consisting of non-archimedean (respectively archimedean) places. If $v \in \mathcal{M}_K^0$, $v|p$, we normalize $|\cdot|_v$ by $|p|_v = p^{-1}$; while if $v \in \mathcal{M}_K^\infty$ we normalize $|\cdot|_v$ by requiring that restricted to $\mathbb{Q}$ it coincides with the standard archimedean absolute value. Let K_v be the completion of K with respect to $|\cdot|_v$. We denote by n_v the local degree, and set $d_v = n_v/d$, where d is the degree of K over $\mathbb{Q}$. With this normalization the product formula reads $\prod_{v \in \mathcal{M}_K} |\lambda|_v^{n_v} = 1$. Finally $\mathcal{O}_K$ denotes the ring of integers of K and $\mathcal{O}_v$ the completion of $\mathcal{O}_K$ in K_v.

Let v be in $\mathcal{M}_K^0$ and let $\boldsymbol{Y}$ be a finite dimensional K_v-vector space. A subset $\Omega \subset \boldsymbol{Y}$ is called an $\mathcal{O}_v$ *– lattice* if Ω is a compact open $\mathcal{O}_v$-module. If $\Omega \subset \boldsymbol{Y}$ is an $\mathcal{O}_v$-lattice then the norm on Y associated to Ω is defined as

$$N_\Omega(\boldsymbol{x}) = \inf_{\gamma \in K_v^\times,\ \gamma \boldsymbol{x} \in \Omega} |\gamma|_v^{-1}.$$

Let $\boldsymbol{X}$ be a finite dimensional K-vector space and $\Lambda \subset \boldsymbol{X}$, an $\mathcal{O}_K$-module. We set $\boldsymbol{X}_v = \boldsymbol{X} \otimes_K K_v$ and $\Lambda_v = \Lambda \otimes_{\mathcal{O}_K} \mathcal{O}_v$. Finally Λ is called an $\mathcal{O}_K$-lattice if it is a finitely generated $\mathcal{O}_K$-module which contains a basis of $\boldsymbol{X}$ over K.

A family of norms $\mathcal{F} = \{N_v \colon \boldsymbol{X}_v \to \mathbb{R},\ v \in \mathcal{M}^K\}$ is called an *adelic norm* if it satisfies the following properties:

(a) N_v is a $|\cdot|_v$-norm, i.e., $N_v(\lambda \boldsymbol{x}) = |\lambda|_v N_v(\boldsymbol{x})$ for all $\lambda \in K_v$ and all $\boldsymbol{x} \in \boldsymbol{X}_v$.

(b) If $v \in \mathcal{M}_K^0$, then N_v is ultrametric, i.e., $N_v(\boldsymbol{x} + \boldsymbol{y}) \leq \max\{N_v(\boldsymbol{x}), N_v(\boldsymbol{y})\}$.

(c) There exists an $\mathcal{O}_K$-lattice $\Lambda \subset \boldsymbol{X}$, such that $N_v = N_{\Lambda_v}$ for all but finitely many $v \in \mathcal{M}_K^0$.

The height function $H_\mathcal{F}$ defined by $\mathcal{F}$ is:

$$H_\mathcal{F}(\boldsymbol{x}) = \prod_{v \in \mathcal{M}_K} N_v(\boldsymbol{x})^{d_v}$$

for all $\boldsymbol{x} \neq \boldsymbol{0}$ while we set $H_\mathcal{F}(\boldsymbol{0}) = 1$. The compatibility condition (c) ensure us that the infinite product appearing in the definition of $H_{\mathcal{F}}$ is in fact finite. It is evident from the product formula that $H_{\mathcal{F}}$ descend to a function on $\mathbb{P}(\boldsymbol{X})$ the projective space associated to $\boldsymbol{X}$.

We will denote by $\mathcal{H}(\boldsymbol{X}) = \{H_{\mathcal{F}}: \mathcal{F}$ is an adelic norm on $\boldsymbol{X}\}$, the set of height functions defined by adelic norms.

Examples. (a) Let $\boldsymbol{X} = K^n$ and fix $1 \leq q \leq \infty$. For each $v \in \mathcal{M}_K$ we define $N_v^\infty(\boldsymbol{x}) = \max_{1 \leq i \leq n} |x_i|_v$. Thus if $v \in \mathcal{M}_K^0$ then N_v^∞ is the norm associated to the $\mathcal{O}_v$-lattice $\mathcal{O}_v^n$ of K_v^n. Moreover for $v \in \mathcal{M}_K^\infty$ let N_v^q be the ℓ^q-norm on $\boldsymbol{X}_v$, i.e. $N_v^q(\boldsymbol{x}) = \left(\sum_{i=1}^n |x_i|_v^q\right)^{\frac{1}{q}}$. Set $\mathcal{F}_q = \{N_v^\infty;\, v \in \mathcal{M}_K^0\} \cup \{N_v^q;\, v \in \mathcal{M}_K^\infty\}$, then $\mathcal{F}_q$ is an adelic norm. Let H_q denote the associated height, i.e.:

$$H_q(\boldsymbol{x}) = \prod_{v \in \mathcal{M}_K^0} \max_{1 \leq i \leq n} |x_i|_v \cdot \prod_{v \in \mathcal{M}_K^0} \left(\sum_{i=1}^n |x_i|_v^q\right)^{\frac{1}{q}}.$$

For $q = 1$, 2 or ∞ we recover the classical heights as defined by Northcott ($q = 1$) (see [No]), Weil ($q = \infty$) (see [We1]) and Schmidt ($q = 2$) (see [Sc]).

(b) Let $\overline{E}$ be an Hemitian vector bundle over $\operatorname{Spec}(\mathcal{O}_K)$, the precise definition of which can be found below. As a by-product of Arakelov theory (1) one gets a (logarithmic) height $h_{ar}^{\overline{E}}$ on the K-vector space $E_K = E \otimes_{\mathcal{O}_K} K$. The goal of this example is to show that there exists $H \in \mathcal{H}(E_K)$ such that $h_{ar}^{\overline{E}} = -[K:\mathbb{Q}] \log H$, the appearance of the factor $[K:\mathbb{Q}]$ is due to different normalizations.

First of all recall that $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \prod_\sigma K_\sigma$ where σ ranges over the embeddings of K into $\mathbb{C}$, and K_σ denotes the completion with respect to the absolute value $|\cdot|_\sigma$ associated to σ. In particular $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ is endowed with a canonical involution $a \to a^*$ which in terms of the isomorphism $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \prod_\sigma K_\sigma$ is simply described by $(a_\sigma)^* = (\overline{a_\sigma})$, here $\overline{a_\sigma}$ is the complex conjugate of a_σ. Next a Hermitian vector bundle $\overline{E} = (E, \langle\cdot,\cdot\rangle_{E_{\mathbb{R}}})$ over $\operatorname{Spec}(\mathcal{O}_K)$ is the datum of a locally free projective $\mathcal{O}_K$-module of finite rank E, and of an *Hermitian inner product* $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ on $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that a $\mathbb{R}$-bilinear, symmetric, positive definite pairing $\langle\cdot,\cdot\rangle$ on a $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R})$-module M is called Hermitian if $\langle am_1, m_2\rangle = \langle m_1, a^* m_2\rangle$ for all $m_1, m_2 \in M$ and all $a \in \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$. The decomposition of $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ induces a decomposition $E_{\mathbb{R}} \cong \prod_\sigma E_\sigma$ where $E_\sigma = E \otimes_{\mathcal{O}_K} K_\sigma$. Moreover we also get a decomposition of the in-

(1) For a comprehensive treatment of Arakelov theory see [GS]; for one of the relations between Arakelov theory and heights (of cycles) on projective varieties see [BGS].

ner product $\langle\cdot,\cdot\rangle_{E_{\mathbb{R}}}$ as a sum of the induced inner product of the various components, i.e. the components are perpendicular with respect to $\langle\cdot,\cdot\rangle_{E_{\mathbb{R}}}$. For more information about this approach see [Gr]. The Arakelov degree of Hermitian line bundle $\overline{L}=(L,\langle\cdot,\cdot\rangle_{L_{\mathbb{R}}})$ is defined as follows: let $s\in L$ be non-zero, then we set:

$$\widehat{\deg}(\overline{L}) = \#(L/sL) - \log\left(\sqrt{\langle s, s\rangle_{L_{\mathbb{R}}}}\right).$$

Given a point $s\in E_K$ the Arakelov height (2) $h_{ar}^{\overline{E}}(s)$ of s can be computed as follows (see [Ga], section 2): let L_s be the Hermitian line bundle $(Ks)\cap E$ (we are regarding E as contained in E_K, via $\boldsymbol{x}\mapsto\boldsymbol{x}\otimes 1$) with the induced metric, then $h_{ar}^{\overline{E}}(s) = -\widehat{\deg}(L_s)$. On the other hand we can associate to $\overline{E}$ an adelic norm on E_K, as follows: since E is a a projective $\mathcal{O}_K$-module of finite rank, for all $v\in\mathcal{M}_K^0$ the $\mathcal{O}_v$-lattice $E_v = E\otimes_{\mathcal{O}_K}\mathcal{O}_v$ defines a v-adic norm on $(E_K)_v = E\otimes_{\mathcal{O}_K}K_v$ that we denote by $N_v^{\overline{E}}$. If $v\in\mathcal{M}_K^\infty$ and v corresponds to one real embedding σ we set

$$N_v^{\overline{E}}(x) = \sqrt{\langle x, x\rangle_\sigma} \quad \text{for all } x\in(E_K)_v$$

where $\langle\cdot,\cdot\rangle_\sigma$ denotes the restriction of $\langle\cdot,\cdot\rangle$ to the component $E_\sigma\cong(E_K)_v$; if v is complex then it corresponds to a pair of complex conjugate embendings and we proceed as above the only difference being that we have to choose one of the two embeddings. For our purpose the choice is irrelevant because both give rise to the same inner product on $(E_K)_v$ by Lemma 3 of [Gr]. We then set $\mathcal{F}_{\overline{E}} = \{N_v^{\overline{E}}, v\in\mathcal{M}_K\}$. The computation of Appendix A of [Vi], yields:

$$h_{ar}(s) = -[K:\mathbb{Q}]\cdot\log\left(H_{\mathcal{F}_{\overline{E}}}(s)\right).$$

(c) Let $\mathcal{T}=(T_v)$ be an element of $\mathrm{GL}_n(\mathbb{A}_K)$, the adele group of $\mathrm{GL}_n(K)$. Let $\Lambda_{\mathcal{T}}$ be the $\mathcal{O}_K$-lattice defined by requiring that $(\Lambda_{\mathcal{T}})_v = T_v(\mathcal{O}_v^n)$. Set $N_v^{\mathcal{T}} = N_{(\Lambda_{\mathcal{T}})_v}$ for all $v\in\mathcal{M}_K^0$, and consider the adelic norm $\mathcal{F}_{\mathcal{T}} = \{N_v^{\mathcal{T}}, v\in\mathcal{M}_K^0\}\cup\{N_{\underline{e},v}^{\mathcal{T}}, v\in\mathcal{M}_K^\infty\}$, where $N_v^{\mathcal{T}}(\boldsymbol{x}) = N_{\underline{e},v}^2(T_v(\boldsymbol{x}))$. The height $H_{\mathcal{T}} = H_{\mathcal{F}_{\mathcal{T}}}$ associated to this adelic norm was introduced by D. Roy and J. Thunder in [RT] and referred to as the twisted height associated to $\mathcal{T}$.

Proposition 1.1. *Let $\boldsymbol{X}$ be a finite dimensional K-vector space. Let $H\in\mathcal{H}(\boldsymbol{X})$, then*

(2) This is a small fragment of a much more general procedure which yields height functions not only on points of E_K but on cycles of arithmetic varieties, again the main references are [BGS] and the literature cited therein.

(H1) *$H(\lambda \boldsymbol{x}) = H(\boldsymbol{x})$ for all $\lambda \in K^{\times}$.*
(H2) *Given $H' \in \mathcal{H}(\boldsymbol{X})$ there exists a constant $C = C(H, H') > 1$ such that for all $\boldsymbol{x} \in \boldsymbol{X}$ we have:*

$$C^{-1} H'(\boldsymbol{x}) \leqslant H_{\mathcal{F}}(\boldsymbol{x}) \leqslant CH'(\boldsymbol{x}).$$

(H3) *For all $C > 0$ the set $\{l \in \mathbb{P}(\boldsymbol{X}) \mid H_{\mathcal{F}}(l) \leqslant C\}$ is finite.*

Proof. **(H1)** follows directly from the product formula while **(H3)** follows from **(H2)** and the classical Northcott's theorem. Finally **(H2)** is a consequence of the fact that all norms on a finite dimensional vector space over a complete field are equivalent and that any two adelic norms differ only for finitely many norms; for more details see [Ta2] Lemma 2.1.

Next we introduce the height functions on $\mathrm{End}_K(\boldsymbol{X})$, the K-algebra of K-endomorphisms of $\boldsymbol{X}$, which are needed in the sequel. Let $\mathcal{F}$ be an adelic norm on $\boldsymbol{X}$. We define $H_{\mathcal{F}}^{op}$ the *operator height* associated to $H_{\mathcal{F}}$ (or to $\mathcal{F}$), by setting

$$H_{\mathcal{F}}^{op}(T) = \sup_{\boldsymbol{x} \in \boldsymbol{X} \setminus \{0\}} \frac{H_{\mathcal{F}}(T(\boldsymbol{x}))}{H_{\mathcal{F}}(\boldsymbol{x})}.$$

Last but not least we define the *spectral height*. Let us recall the definition of the local spectral radii. Let F be a complete local field and Y a finite dimensional F-vector space. The *spectral radius* of $T \in \mathrm{End}(Y)$ is defined as:

$$\varrho_F(T) = \sup_{\lambda \in \mathrm{sp}(T)} |\lambda|_{F(\lambda)},$$

where $\mathrm{sp}(T) \subset \overline{F}$ is the set of characteristic roots of T and $|\cdot|_{F(\lambda)}$ is the unique extension of $|\cdot|_F$ to $F(\lambda)$.

Let us go back to our settings: given a non-nilpotent $T \in \mathrm{End}_K(\boldsymbol{X})$ we set

$$H_s(T) = \prod_{v \in \mathcal{M}_K} \varrho_{K_v}(T_v)^{d_v}.$$

We also set $H_s(T) = 1$ for all nilpotent $T \in \mathrm{End}_K(\boldsymbol{X})$. The function thereby defined is called the *spectral height* and enjoys the following properties:

(S1) *$H_s(\lambda T) = H_s(T)$ for all $\lambda \in K^{\times}$*
(S2) *$H_s(T) \geqslant 1$.*
(S3) *$H_s(T^k) = H_s(T)^k$ for $k \geqslant 1$.*
(S4) *H_s is invariant under conjugation.*
(S5) *If $T, T' \in \mathrm{End}(V)$ commute, $H_s(TT') \leqslant H_s(T)\, H_s(T')$.*

(S6) *If W is another finite dimensional K-vector space and $S \in \mathrm{End}_K(W)$ then* $H_s(T \otimes S) = H_s(T)\, H_s(S)$.

(S7) H_s *is invariant under field extension.*

Properties **(S3)-(S6)** are direct consequences of the behavior of the spectrum under the various operations considered. Property **(S1)** follows from the product formula while **(S7)** is derived in a standard way from the formula for local degrees (see [La] chapter 3 section 1). Finally **(S2)** follows from **(S1)** and **(S7)**.

The fundamental relation between these height functions is the following:

Gelfand-Beurling formula for heights. *Let $\mathcal{F}$ be any adelic norm on the finite dimensional K-vector space $\boldsymbol{X}$. Then for all $T \in \mathrm{End}_K(\boldsymbol{X})$ we have:*

$$\text{(GBF)} \qquad \lim_{n \to \infty} H_{\mathcal{F}}^{op}(T^n)^{\frac{1}{n}} = H_s(T).$$

For a proof see [Ta2], Theorem A.

2 - The canonical height on a commutative separable K-algebra

In this section we recall the construction and the main properties on the canonical height on a commutative separable K-algebra. Since $\mathrm{char}(K) = 0$ a K-algebra A (commutative or not) is *separable* [3] if and only if it is semisimple (i.e. isomorphic to a product of field extensions of K) and finite dimensional. We employ the following notations:

A	a commutative separable finite dimensional K-algebra
$(S, \mathcal{O}_S)$	the affine K-scheme associated to A
$(S_v, \mathcal{O}_{S_v})$	the affine K_v-scheme associated to $A_v = A \otimes_K K_v$
$A \xrightarrow{\wedge} \Gamma(S, \mathcal{O}_S)$, $a \mapsto \widehat{a}$	the canonical isomorphism
$K_v(s)$	the residue field at $s \in S$.
$\lvert \cdot \rvert_s$	the unique extension of $\lvert \cdot \rvert_v$ to $K_v(s)$, for $s \in S_v$.

Note that under the assumption of separability the residue field at $s \in S$ is isomorphic to the stalk of $\mathcal{O}_S$.

([3]) A k-algebra A is separable if the as a $A \otimes_k A^\circ$-module is projective, where A° is the opposite ring of A. This is equivalent to require that A is finite-dimensional and semisimple and that the center of each simple component is a separable extension of k, see [Re], Theorem 7.20.

Let us start by defining the *canonical* adelic norm of A. Given $v \in \mathcal{M}_K$ we set

$$N_v^A : A_v \to \mathbb{R}$$
$$a \mapsto \sup_{s \in S_v} |\hat{a}(s)|_s.$$

The family of norms $\mathcal{F}_A = \{N_v^A, v \in \mathcal{M}_K\}$ is then an adelic norm, see [Ta3], Lemma 3.1 for a proof. The height function $H_{A/K}$ defined by $\mathcal{F}_A$ is the canonical height on A of Theorem A of the introduction by property **(C3)** and Theorem 2.1 below.

Being the height attached to an adelic norm $H_{A/K}$ enjoys properties **(H1)-(H3)** stated in the previous section, but due to its intrinsic nature it actually enjoys many more properties, namely: Let A and B be separable K-algebras, $a \in A$, $b \in B$, then

(C1) $H_{A/K}(a) \geq 1$.
(C2) $H_{A/K}(aa') \leq H_{A/K}(a) \cdot H_{A/K}(a')$.
(C3) $H_{A/K}(a^k) = H_{A/K}(a)^k$.
(C4) $H_{A/K}(a) = 1$ *if and only if the set* $\{[a^n] \in \mathbb{P}(A), n \in \mathbb{N}\}$ *is finite.*
(C5) $H_{(A \otimes_K B)/K}(a \otimes b) = H_{A/K}(a) \cdot H_{B/K}(b)$.
(C6) *Let L be any extension of K. Then* $H_{A/K}(a) = H_{(A \otimes_K L)/L}(a \otimes 1)$.
(C7) *If $\varphi : A \to B$ is an injective homomorphism of separable K-algebras, then $H_{A/K}(a) = H_{B/K}(\varphi(a))$. In particular $H_{A/K}$ is invariant under* $\mathrm{Aut}_{K\text{-Alg}}(A)$.

For a proof of these statements see [Ta1] Propositions 2.2 and 2.6, Corollaries 2.4 and 2.5. In (C4) $\mathbb{P}(A) = \mathbb{P}([A])$ denotes the projective space associated to $[A]$ the K-vector space underlying A.

Example. Let $A = L$ be a finite extension of K of degree n. If $w \in \mathcal{M}_K$ extend $v \in \mathcal{M}_K$ we write $w|v$ and we set $r_w = [L_w : K_v]/[L : K]$. Then

$$H_{L/K}(a) = \prod_{v \in \mathcal{M}_K} \max_{w|v} |a|_w^{r_v}.$$

In the case $K = \mathbb{Q}$ it is actually possible to be a bit more explicit. Given $a \in L$, let $a = ra'$, where $r \in \mathbb{Q}^*$ and $a' \in \mathcal{O}_L$ satisfies $a'/n \notin \mathcal{O}_L$ for $n \in \mathbb{N}$, $n > 1$. Then, if p is unramified in the extension L/K, we have $\max_{v|p} |a'|_v = 1$. Therefore,

$$H_{L/\mathbb{Q}}(a) = H_{L/\mathbb{Q}}(a') = \Big(\prod_{p|\Delta} \max_{v|p} |a'|_v\Big) \max_{v|\infty} |a'|_v,$$

where Δ is the discriminant of the extension L/K.

To complete the proof of Theorem A of the introduction we only have to show that **(M2)** holds for all $H \in \mathcal{H}(A)$. As we mentioned in the introduction, **(M2)** has been show to hold in [Ta3], Theorem 2.4 but only for a class of adelic norms: the *compatible* adelic norm. Recall that an adelic norm $\mathcal{F} = \{N_v, v \in \mathcal{M}_K\}$ is compatible (with the K-algebra structure of A) if (A_v, N_v) is a Banach algebra over K_v for all $v \in \mathcal{M}_K$. We now prove **(M2)** in full generality

Theorem 2.1. *Let A be a commutative separable K-algebra. Let $H \in \mathcal{H}(A)$, then:*

$$\lim_{n \to \infty} H(a^n)^{\frac{1}{n}} = H_{A/K}(a)$$

for all $a \in A$.

Proof. Let $H \in \mathcal{H}(A)$, by **(H1)** there exists $C > 1$ such that

$$C^{-1} H_{A/K}(a) \leqslant H(a) \leqslant C H_{A/K}(a)$$

for all $a \in A$. Therefore for a given $a \in A$ we have

$$(C^{-1} H_{A/K}(a^n))^{\frac{1}{n}} \leqslant H(a^n)^{\frac{1}{n}} \leqslant (C H_{A/K}(a^n))^{\frac{1}{n}}$$

for all $n \geqslant 1$. By **(C3)** we have

$$\lim_{n \to \infty} (C^{-1} H_{A/K}(a^n))^{\frac{1}{n}} = H_{A/K}(a) = \lim_{n \to \infty} (C H_{A/K}(a^n))^{\frac{1}{n}},$$

yielding the theorem.

We would like to dwell for a moment on our previous proof of Theorem 2.1: it was based on the Gelfand-Beurling formula for operator heights and on the following two results ([4]) Let $\mathcal{F}$ be a compatible adelic norm on A and for $a \in A$ let m_a denote the *multiplication-by* a map.

(K1) *For all $a \in A$ we have: $H^{op}_{\mathcal{F}}(m_a) \leqslant H_{\mathcal{F}}(a) \leqslant H^{op}_{\mathcal{F}}(m_a)\, H_{\mathcal{F}}(1_A)$.*
(K2) *For all $a \in A$ we have: $H_{A/K}(a) = H_s(m_a)$.*

The Gelfand-Beurling formula combined with **(K1)** and **(K2)** yields a quick proof of Theorem 2.1 for a compatible adelic norm $\mathcal{F}$:

$$\begin{aligned}
\lim_{k \to \infty} H_{\mathcal{F}}(a^k)^{\frac{1}{k}} &= \lim_{k \to \infty} H^{op}_{\mathcal{F}}(m_a^k)^{\frac{1}{k}} && \text{by } \textbf{(K1)} \\
&= H_s(m_a) && \text{by } \textbf{(GBF)} \\
&= H_{A/K}(a) && \text{by } \textbf{(K2)}.
\end{aligned}$$

([4]) For a proof of (K1) and (K2), see [Ta3], Lemma 3.2 and Lemma 3.3 respectively.

The advantage of this proof is that it puts the spotlight on the spectral height and makes it possible to generalize both the definition of the canonical height and Theorem 2.1 to the non-commutative setting as we shall see in the next section.

3 - Heights on separable algebras over number fields

As we have seen in the previous section (precisely property **(K2)**) the canonical height of an element a of a commutative separable K-algebra can be interpreted as the spectral height of the linear transformation m_a. We will use this approach to define a height function on non-commutative separable K-algebras. Let A be such a K-algebra. Given $v \in \mathcal{M}_K$ we regard A as contained in A_v via $a \mapsto a \otimes 1$. To any element $a \in A$ we can associate two K_v-linear maps of A_v to itself, the left and right regular representation of a, namely: $(\ell_a)_v \colon A_v \to A_v, \mapsto ab$, and $(r_a)_v \colon A_v \to A_v,\ b \mapsto ba$. Since A is separable so is A_v, and on a separable algebra over a field the characteristic polynomials of the the left and right regular representation af any element coincide, see e.g. [Re], section 9b. In particular $\varrho_v(\ell)_a = \varrho_v(r_a)$ which leads us to put forward the following definition:

Definition. Let A be a non-commutative separable finite deimensional K-algebra. We define $H_{A/K} \colon A \to \mathbb{R}$, the *canonical height* on A, by setting:

$$H_{A/K}(a) = H_s(\ell_a) = H_s(r_a)$$

for all $0 \neq a \in A$, and we set $H_{A/K}(a) = 1$.

Proposition 3.1. *Let A and B be non-commutative separable K-algebras. Let $a \in A$, $b \in B$ and $\lambda \in K^{\times}$. Then*

(1) $H_{A/K}(\lambda a) = H_{A/K}(a)$ *for all* $\lambda K^{\times}$.
(2) $H_{A/K}(a) \geq 1$.
(3) $H_{A/K}(a^k) = H_{A/K}(a)^k$ *for all* $k \geq 1$.
(4) *If E is a finite extension of K, then* $H_{A/K}(a) = H_{(A \otimes_K E)/E}(a \otimes 1)$.
(5) $H_{(A \otimes_K B)/K}(a \otimes b) = H_{A/K}(a) \cdot H_{B/K}(b)$.
(6) *If $\varphi \colon A \to B$ is an injective homomorphism K-algebras, then* $H_{A/K}(a) = H_{B/K}(\varphi(a))$. *In particular $H_{A/K}$ is invariant under* $\mathrm{Aut}_{K\text{-Alg}}(A)$.
(7) *Assume now that A is a central simple K-algebra and that F is a finite extension of K. If $\varrho \colon A \to \boldsymbol{M}_m(F)$ is any representation of A, then* $H_s(\varrho(a)) = H_{A/K}(a)^h$, *where* $h\sqrt{\dim_K A} = m$.

Proof. Properties (1)-(5) follow from the corresponding property of H_s. To see (6) it suffices to note that the local spectral radii only depend on the minimal polynomial which is clearly invariant under monomorphism. Finally (7) follows from the theory of the reduced characteristic polynomial for central simple algebras, see [Pi] chapter 16.

Remark. The main disavantage of $H_{A/K}$ is that it does not belong to $\mathcal{H}(A)$. To see this it suffices to note that $H_{A/K}$ is constant under conjugation and so Northcott's theorem does not hold for $H_{A/K}$. On the other hand by **(H2)** Northcott's theorem does hold for any height in $\mathcal{H}(A)$ thus preventing $H_{A/K}$ to be one of them.

The following theorem shows that $H_{A/K}$ enjoys the other main feature of its commutative counterpart:

Theorem 3.2. *Let A be a non-commutative separable K-algebra. Let $H \in \mathcal{H}(A)$, then*

$$\lim_{n \to \infty} H(a^n)^{\frac{1}{n}} = H_{A/K}(a),$$

for all $a \in A$.

Proof. We proceed in two steps: first we establish the theorem for heights defined by compatible adelic norms (5), then, using property **(H1)**, we extend the proof to the case of arbitrary adelic norms. Let $\mathcal{G}$ be a compatible adelic norm. As it is immediate to check the proof of **(K1)** and **(K2)** do not rely in any way on the commutativity of the algebra in question. Therefore the proof of Theorem 2.1, valid only for compatible adelic norms, presented at the end of the previous section carries over to the non-commutative case. Now let $\mathcal{F}$ be any adelic norm on A. Then by property **(H1)** there exist $C > 0$ such that

$$C^{-1} H_{\mathcal{G}}(a) \leqslant H_{\mathcal{F}}(a) \leqslant C H_{\mathcal{G}}(a)$$

for all $a \in A$. Therefore for a given $a \in A$ we have

$$\left(C^{-1} H_{\mathcal{G}}(a^n)\right)^{\frac{1}{n}} \leqslant H_{\mathcal{F}}(a^n)^{\frac{1}{n}} \leqslant \left(C H_{\mathcal{G}}(a^n)\right)^{\frac{1}{n}}$$

(5) As in the commutative case an adelic norm on A is compatible if (A_v, N_v) is a Banach algebra over K_v for all $v \in \mathcal{M}_K$.

for all $n \geq 1$. By the above we have

$$\lim_{n \to \infty} \left(C^{-1} H_{\mathcal{G}}(a^n)\right)^{\frac{1}{n}} = H_{A/K}(a) = \lim_{n \to \infty} \left(C H_{\mathcal{G}}(a^n)\right)^{\frac{1}{n}},$$

yielding $\lim_{n \to \infty} H_{\mathcal{F}}(a^n)^{\frac{1}{n}} = H_{A/K}(a)$. For the proof to be complete we also have to show that there exist one compatible adelic norm on A. Since A is separable there exists a (maximal) order Λ in A. For all $v \in \mathcal{M}_K^0$ let N_v be the norm associated to Λ_v. For $v \in \mathcal{M}_K^\infty$ let N_v be any norm that makes (A_v, N_v) a Banach algebra (since A_v is finite dimensional there are plenty of those). The adelic norm $\mathcal{G}_\Lambda = \{N_v;\ v \in \mathcal{M}_K\}$ so defined is a compatible adelic norm (cf. [Ta3], the example of section 3).

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Abstract

Let K be a number field. Based on our previous result on the construction of canonical heights on separable commutative finite dimensional K-algebras we propose a definition for the canonical height on non-commutative, finite dimensional, separable K-algebras. We prove that it satisfies an averaging property analogous to the one satisfied by the Néron-Tate height on abelian varieties and that is invariant under the group of K-algebra automorphisms of A.

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