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Some remarks on a conjecture of Dwork (**)

0 - Introduction

The aim of this note is to present some recent results related to the following problem proposed by Dwork [8] («Accessory parameter problem»):

Let S be the ring of all second order elements of $\mathbb{C}(X)\left[\frac{d}{dx}\right]$ *(the ring of differential operators with coefficients in* $\mathbb{C}(X)$*) having given Riemann data. S is parameterized by a certain number of accessory parameters. Let* $S' \subset S$ *be the subset corresponding to equations with a full set of algebraic solutions. Then S' corresponds to an algebraic subset of S.*

An element of $\mathbb{C}(X)\left[\frac{d}{dx}\right]$ is a differential operator

$$L = \left(\frac{d}{dx}\right)^2 + A(x)\frac{d}{dx} + B(x)$$

where $A, B \in \mathbb{C}(X)$. We see it as an operator on the x-Riemann sphere. The set of singular points of L, S, contains the poles of A and B and ∞.

More generally, if K is the function field of an algebraic curve C over $\mathbb{C}$ and D is a nontrivial derivation of $K/\mathbb{C}$, one can ask about the shape of the set of second

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order differential operators (or operators of order n) in $K[D]$ with a full set of solutions algebraic over K.

We suppose that all the singular points of the operators we work with are regular with rational exponents and without logarithmic solutions. Let $\Delta_{P,L} = |\alpha_1 - \alpha_2|$ be the exponent difference at the point $P \in \mathbb{P}^1$ and

$$\Delta_L = \sum_{P \in \mathbb{P}^1} (\Delta_{P,L} - 1).$$

If at a singular point P the exponent difference $\Delta_{P,L}$ is an integer, P is called an apparent singularity.

The monodromy group of a second order differential operator L is the image of the representation $\pi_1(\mathbb{P}^1 \setminus S) \to GL(2, \mathbb{C})$ given by the analytic continuation of two solutions in a basis along the closed paths representing the elements of $\pi_1(\mathbb{P}^1 \setminus S)$. The projective monodromy group is the image of the representation $\pi_1(\mathbb{P}^1 \setminus S) \to PGL(1, \mathbb{C})$ given by such a continuation of the ratio of the solutions in a basis.

The operators L_1 and L_2 are projectively equivalent if the ratio of two independent solutions of L_1 is also the ratio of two independent solutions of L_2. In this case they have the same exponent differences and their projective monodromy groups are isomorphic. Any second order differential operator is projectively equivalent to one in normalized form: $L = \left(\frac{d}{dx}\right)^2 + Q,\ Q \in \mathbb{C}(X)$.

If $f: \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function, an operator L_1 is the weak pull-back of an operator L_2 if, for any ratio of independent solutions of L_2, τ_2, $\tau_1 = \tau_2 \circ f$ is the ratio of two independent solutions of L_1. In this case $\Delta_{P,L_1} = e_P \cdot \Delta_{f(P), L_2}$ for all $P \in \mathbb{P}^1$ (where e_P is the ramification index of f in P) and $\Delta_{L_1} - 2 = \deg f (\Delta_{L_2} - 2)$.

We are interested in operators which have a full set of algebraic solutions. This happens if and only if the monodromy group of the operator is finite, or, equivalently, the projective monodromy group is finite and the Wronskian is algebraic over K. Let us also notice that, if τ is the ratio of the solutions in a basis, the projective monodromy group is the Galois group of the extension $\mathbb{C}(X)(\tau)/\mathbb{C}(X)$.

1 - Finite monodromy and Belyi functions

In 1872 Schwarz [**16**] determined the complete list of the second order differential operators with three singular points and finite monodromy. They are all hypergeometric

$$H_{\lambda,\mu,\nu} = \left(\frac{d}{dx}\right)^2 + \frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(x-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4x(x-1)}$$

with the following possible values for the parameters λ, μ, ν and the following projective monodromy groups («the basic Schwarz list», see for example [**1**]):

(λ, μ, ν)	$G_{H_{\lambda,\mu,\nu}}$
$(1/n, 1, 1/n)$	C_n, cyclic of order n
$(1/2, 1/n, 1/2)$	D_n, dihedral of order $2n$
$(1/2, 1/3, 1/3)$	$\mathcal{A}_4$, tetrahedral
$(1/2, 1/3, 1/4)$	$\mathcal{S}_4$, octahedral
$(1/2, 1/3, 1/5)$	$\mathcal{A}_5$, icosahedral

In the general case, we have of the following theorem ([**10**], [**1**]):

Theorem 1.1. *Let L be a second order linear differential operator in normalized form on the Riemann sphere, with finite projective monodromy group G. Then there exists an unique hypergeometric operator H in the Schwarz list, having the same projective monodromy group G, such that L is the weak pull-back of H via a rational function $f: \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, the function f is also unique, modulo the Möbius transformations leaving the operator H invariant and permuting its singular points.*

Baldassarri gave in [**2**] a generalization of this statement for the case of the operators on an arbitrary curve over $\mathbb{C}$.

The results that follow are based on the following simple observation:

Proposition 1.2 ([**12**], [**13**]). *Suppose that the operator L as in Theorem* 1.1 *has no apparent singularity. Then the morphism f is ramified at most over* $0, 1$ *and* ∞.

If C is an algebraic curve defined over the complex field, the existence of a morphism $f: C \to \mathbb{P}^1$ unramified over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ characterizes the fact that C can be defined over a number field (Belyi Theorem). Such a morphism is called a Belyi function. By the Grothendieck correspondence one associates to a Belyi function a bipartite graph on the topological model of C, graph that is called dessin d'enfant, and there is a nice dictionary between the ramification data of f and the combinatorial data of the dessin. We shall refer in this note to this correspondence in the case $C = \mathbb{P}^1$. Hereafter a $*$-function will be a Belyi function $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\{0, 1, \infty\} \subset f^{-1}(\{0, 1, \infty\})$.

We have the following finiteness result:

Theorem 1.3 ([**11**]). *Let $M > 1$ a fixed real number. The set of $*$-functions of degree at most M is finite.*

We obtain as an immediate consequence (see also [**5**])

Theorem 1.4. *The set of second order differential operators on the Riemann sphere, with finite monodromy and no apparent singularity, modulo homography and projective equivalence, is countable.*

Wolfart's theorem on the values of the hypergeometric functions ([**18**]) implies

Proposition 1.5. *If τ is the ratio of independent solutions of a second order differential operator L with finite projective monodromy and without apparent singularities, and $P \in \mathbb{P}^1(\overline{\mathbb{Q}})$, then $\tau(P) \in \overline{\mathbb{Q}}$.*

2 - Lamé operators with finite monodromy

We consider now Lamé differential operators

$$L_n = \left(\frac{d}{dx}\right)^2 + \frac{1}{2}\sum_{i=1}^{3}\frac{1}{x-e_i}\left(\frac{d}{dx}\right) - \frac{n(n+1)x+B}{4\prod_{i=1}^{3}(x-e_i)} \tag{2.1}$$

having in any point a basis of solutions which are algebraic over $\mathbb{C}(x)$. The parameter n is rational, and $e_i \in \mathbb{C}$ are distinct ($i = 1, 2, 3$). The complex number B is the accessory parameter. The singular points are e_i (with exponent differences $\frac{1}{2}$) and ∞ (with exponent difference $n + \frac{1}{2}$). The operator L_n has no apparent singularity if and only if $n \notin \mathbb{Z} + \frac{1}{2}$ (if $n \in \mathbb{Z} + \frac{1}{2}$ the only possible finite monodromy group is the Klein four group D_2, cf. for example [**3**]). We have the following result, firstly proved by Baldassarri ([**3**]):

Theorem 2.1. 1. *There is no Lamé operator with cyclic projective monodromy group.*

2. *There is no Lamé operator with tetrahedral projective monodromy group.*

3. *If the projective monodromy group of the Lamé operator* L_n *is octahedral, then* $n \in \frac{1}{2}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z} + \frac{1}{2})$.

4. *If the projective monodromy group of the Lamé operator* L_n *is icosahedral, then* $n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z} + \frac{1}{2})$.

5 *If the projective monodromy group of the Lamé operator* L_n *is dihedral, then* $n \in \mathbb{Z}$. *If* $n \in \mathbb{Z}$ *and the projective monodromy group is finite, then this group is dihedral of order at least* 6.

Proof. An operator L_n with finite projective monodromy is the pull-back of a hypergeometric operator in the basic Schwarz list via a rational function *f*. If $n \notin \mathbb{Z} + \frac{1}{2}$, *f* is a Belyi morphism. The assertions of the theorem follow from the combinatorial properties of the dessins d'enfants associated to the functions *f*. For details, see [**13**]. ■

In an independent work [**5**], Beukers and Van der Waall realize another study of Lamé differential operators with algebraic solutions, based mainly on the properties of the monodromy group. For another recent approach see also Maier [**14**].

The combinatorics associated to the Belyi covers also allows explicit computations - see [**12**], [**13**] where the number of Lamé operators L_1 and L_2 with finite monodromy is computed (see also [**6**]). Our method has been generalized recently by Dahmen [**7**].

Without any loss of generality, we can suppose $e_1 = 0$, $e_2 = 1$ and $e_3 = \lambda \in \mathbb{C}\setminus\{0, 1\}$. Let E_λ be the elliptic curve described by the equation $y^2 = x(x-1)\cdot(x-\lambda)$. The pull-back of L_n by the projection $(x, y) \mapsto x$ is the operator, denoted again L_n, $D^2 - [n(n+1) + B]$ where $D = y\frac{d}{dx}$. Alternatively, if $x = p(u)$ is the Weierstrass function, then

$$L_n = \left(\frac{d}{du}\right)^2 - [n(n+1)\, p(u) + B].$$

It has only one singular point (0_E), with exponents $-n$, $n+1$.

In [4], Baldassarri states the following conjecture, attributed to Dwork:

Fix an integer n and a dihedral group D_N. *The set of isomorphism classes of* *elliptic curves E on which there exists a Lamé operator L_n with projective monodromy D_N is finite.*

We prove the following more general result:

Theorem 2.2. *Fix $n \notin \mathbb{Z} + \frac{1}{2}$, G finite group. The set of isomorphism classes of elliptic curves E on which there exists a Lamé operator L_n with projective monodromy G is finite. Moreover, on each elliptic curve there are finitely many such operators.*

Proof. According to Theorem 2.1, we have the following possible cases:

– $n \in \mathbb{Z}$ and $G = D_N$; in this case the operator L_n (viewed on $\mathbb{P}^1$) is the pull-back of the hypergeometric operator $H_{1/2,\,1/N,\,1/2}$ via a rational function f, $\deg f = nN$;

– $n \in \frac{1}{2}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z} + \frac{1}{2})$ and $G = \mathcal{S}_4$; in this case the operator L_n is the pull-back of the hypergeometric operator $H_{1/2,\,1/3,\,1/4}$ via a rational function f, $\deg f = 12n$;

– $n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z} + \frac{1}{2})$ and $G = \mathcal{A}_5$; in this case the operator L_n is the pull-back of the hypergeometric operator $H_{1/2,\,1/3,\,1/5}$ via a rational function f, $\deg f = 30n$.

It follows then that if we fix n and the projective monodromy group, we fix the degree of the rational function f. As f is a $*$-function, Theorem 1.3 implies the finiteness of the set of such f, so the finiteness of the set of the possible values of λ and B. But the isomorphism class of an elliptic curve E associated to a Lamé operator is determined by λ, so we get finitely many isomorphism classes of elliptic curves on which there is an operator L_n with projective monodromy G. The finiteness of the set of the rational functions f, for each curve E fixed, implies the last statement of the theorem. ■

Remark 2.1. *According to Baldassarri* [**4**], *Dwork gave a proof of this problem in the case $n \in \mathbb{Z}$ in a private communication. To the knowledge of the author, this proof remains unpublished* (*see also* [**9**], [**15**], [**17**]).

Corollary 2.3. *For fixed $n \notin \frac{1}{2}\mathbb{Z}$, there are finitely many Lamé operators L_n with a full set of algebraic solutions.*

(Statement conjectured by Baldassarri in [**3**])

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Abstract

We present recent results concerning the shape of the set containing the second order differential operators on the projective line, having a full set of algebraic solutions. Using a finiteness property of the Belyi functions, we show that the set of Lamé operators L_n *with* n *fixed and fixed projective monodromy is finite.*

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